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IMPLEMENTABLE QUADRATIC REGULARIZATION METHODS FOR SOLVING PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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Abstract

In this paper we use the quadratic regularization technique to develop two iterative algorithms for solving equilibrium problems. The first one is an extension of the extragradient algorithm to pseudomonotone equilibrium problem satisfying a certain Lipschitz condition. To avoid the Lipschitz condition we propose a line search technique to obtain a convergent algorithm for pseudomonotone equilibrium problems.

1 Introduction and the Problem Statement

Let K be a nonempty, closed, convex set in R^n and let $f : K \times K \to R \cup \{+\infty\}$. In this paper we consider the following problem

$$\textit{Find } x^* \in K \textit{ such that: } f(x^*, y) \geq 0 \quad \textit{for all } y \in K. \qquad (EP)$$

Key words: Equilibrium problem, quadratic regularization, extragradient method, line-search algorithm.
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Throughout the paper, as usual, we suppose that $f(x, x) = 0$ for every $x \in K$ and we call such a function equilibrium bifunction. Equilibrium problem (EP) dealing with monotone-type equilibrium bifunction f has been considered by a lot of authors (see e.g. [3, 8, 10, 11, 12, 14]). Conditions for existence of solutions of equilibrium problems can be found, for example, in [3] and recently in [4, 17]. Stability for equilibrium problem (EP) is studied in [10]. An application of equilibrium problem to a Nash market model having concave cost function recently is investigated in [15]. In [11, 12] a number of problems including mathematical programming problems, variational inequality, Kakutani fixed point as well as Nash equilibrium and minimax problems are formulated in the form of equilibrium problems of the form (EP).

It is well known that the regularization technique is a powerful tool for analyzing and for solving optimization problems and variational inequalities. Recently the regularization technique has been used to develop iterative algorithms for equilibrium problems [7, 9, 11, 14]. In [6, 14] the authors use auxiliary problem principle to obtain iterative algorithms for monotone equilibrium problem (EP) where the subproblems arised in each iteration are equilibrium problems having strongly monotone equilibrium bifunction. These algorithms are not easy to implement, since the latter problems are again difficult to solve. In [7] Konnov proposes to use a gap function technique to solve the strongly monotone equilibrium subproblems arised in the proximal point method for equilibrium problem (EP) dealing with weakly monotone equilibrium bifunction. This approach leads to the problem of minimizing a gap function which, in general, is not easy to solve, since this gap function may not be convex.

In this paper we will use the regularization technique to extend the well known extragradient method widely used in mathematical programming as well as in variational inequality to the equilibrium problem (EP). By this way we obtain a linearly convergent algorithm for solving (EP) when f is psedomonotone and Lipschitz on K. In order to avoid the Lipschitz condition we will use a line search to obtain a convergent algorithm for solving equilibrium problem (EP) with pseudomonotone equilibrium bifunction f. Computational results on a class of problem (EP) show the efficiency of the algorithm.

The paper is organized as follows. In next section we list some examples and present fixed point formulations for (EP). The third section is devoted to description of the extragradient algorithm and its convergence. In the fourth section we modify the extragradient algorithm by using a line search technique which allows avoiding the Lipschitz condition. We close the paper with some computational results and experiences on the proposed algorithms.

2 Examples and Fixed Point formulations

In this section we first recall some definitions for monotonicity and list a number of examples for equilibrium problem (EP). Then we give fixed point formulations to (EP).

2.1. Examples. First we recall some well known definitions on monotonicity (see e.g. [6]).

Definition 2.1. Let M and K be convex sets in R^n such that $M \subseteq K$, and let $\varphi : K \times K \to R \cup \{+\infty\}$. The bifunction φ is said to be

(a) *strongly monotone* on M with constant $\tau > 0$ if for each pair of points $x, y \in M$, one has

$$\varphi(x, y) + \varphi(y, x) \leq -\tau \|x - y\|^2;$$

(b) *strictly monotone* on M if for all distinct $x, y \in M$, we have

$$\varphi(x, y) + \varphi(y, x) < 0;$$

(c) *monotone* on M if for each pair of points $x, y \in M$, we have

$$\varphi(x, y) + \varphi(y, x) \leq 0;$$

(d) *pseudomonotone* on M if for each pair of points $x, y \in M$ it holds

$$\varphi(x, y) \geq 0 \quad \text{implies} \quad \varphi(y, x) \leq 0.$$

From the definitions above we obviously have the following implications:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d).$$

Specially, if $f(x, y) := \sup_{v \in T(x)} \langle v, y - x \rangle$ where $T : K \rightrightarrows 2^K$ then Definition 2.1 is equivalent to the following well known definitions of generalized variational inequality (see e.g. [5, 6]).

- T is said to be pseudomonotone on $M \subseteq K$ if for any pair $x, y \in M$ and $u \in T(x), v \in T(y)$ it holds

$$\langle u, x - y \rangle \geq 0 \quad \text{implies} \quad \langle v, y - x \rangle \leq 0.$$

- T is said to be monotone on $M \subseteq K$ if for any $x, y \in M, u \in T(x), v \in T(y)$ it holds

$$\langle u - v, x - y \rangle \geq 0.$$

- T is said to be strongly monotone on $M \subseteq K$ with modulus $\beta > 0$, if for any $x, y \in M, u \in T(x), v \in T(y)$ it holds

$$\langle u - v, x - y \rangle \geq \beta \|x - y\|^2.$$

Equilibrium problem (EP) contains the following problems as special cases.
1. Optimization Problem. Let K be as before and $\varphi : K \to I\!R$. Consider the optimization problem

$$\min \{\varphi(x) : x \in K\}. \tag{OP}$$

By setting $f(x,y) := \varphi(y) - \varphi(x)$ it is easy to see that (OP) is equivalent to (EP) in the sense that their solution-sets are coincided. Clearly, if φ is convex, then $f(x,\cdot)$ is convex for each $x \in K$
2. Mixed Multivalued Variational Inequality: Let $T : K \rightrightarrows 2^K$ be a multivalued mapping such that $T(x) \neq \emptyset$ for all $x \in K$. The following problem is called mixed variational inequality:

$$\begin{aligned} &\textit{Find } \ x^* \in K, v^* \in T(x^*) \textit{ such that:}\\ &\langle v^*, x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \textit{for all } x \in K. \end{aligned} \tag{MVIP}$$

By setting

$$f(x,y) := \sup_{v \in T(x)} \langle v, y - x\rangle + \varphi(y) - \varphi(x),$$

we can easily see that (MVIP) is equivalent to (EP).

In an important special case when T is single valued and $\varphi \equiv 0$, $f(x,y) := \langle T(x), y - x\rangle$. Problem (MVIP) is equivalent to the variational inequality:

$$\textit{Find } x^* \in K \textit{ such that:} \langle T(x^*), x - x^*\rangle \geq 0 \quad \textit{for all } x \in K. \tag{VIP}$$

By Definition 2.1, it is easy to check that T is pseudomonotone (resp. monotone, strongly monotone) on K if and only if f is pseudomonotone (resp. monotone, strongly monotone) on K.
3. Kakutani Fixed Point. Let $\phi : K \rightrightarrows 2^K$. The well known Kakutani fixed point Theorem says that if K is nonempty compact convex and ϕ is upper semicontinuous on K in the sense of Berge [2] with $\phi(x)$ is nonempty, compact, convex for every $x \in K$, then there exists $x^* \in K$ such that $x^* \in \phi(x^*)$.

Taking

$$f(x,y) := \max_{v \in T(x)} \langle x - v, y - x\rangle,$$

we can see that (EP) collapses into the fixed point problem

$$\textit{Find } \ x^* \in K \textit{ such that: } x^* \in \phi(x^*). \tag{FP}$$

4. Nash Equilibria Problem. Let $I := \{1, \cdots, p\}$ (the set of p players), $\emptyset \neq K_i \subseteq \mathbb{R}^{n_i}$ be the strategy set of player i ($i \in I$) and $l_i : K_1 \times K_2 \times \cdots \times K_p \to \mathbb{R}$ be the loss function of player $i \in I$.

By definition, $x^* = (x^{*1}, \cdots, x^{*p}) \in K_1 \times \cdots \times K_p$ is said to be Nash equilibrium point of $l := (l_1, \cdots, l_p)$ on K if and only if

$$l_i(x^*) \leq l_i(x[y^i]) \quad \forall y^i \in K_i, \forall i \in I,$$

where $x[y^i]$ stands for the vector obtained from $x = (x^1, \cdots, x^p) \in K_1 \times \cdots \times K_p$ by replacing x^i with y^i.

The problem of finding a Nash equilibrium point of l on K can be formulated equivalently (see [11, 12]) by setting for each $x, y \in K$:

$$f(x, y) := \sum_{i=1}^{p} \left\{ l_i(x[y^i]) - l_i(x) \right\}.$$

Note that in all problems mentioned above, $f(x, x) = 0$ for all $x \in K$.

2.2. Fixed Point Formulations. It is well known that (see e.g. [3, 10, 6]) if f is pseudomonotone and $f(x, \cdot)$ is convex on K, for each fixed $x \in K$, then the solution-set K^* of (EP) is closed and convex.

The following lemmas, can be found, for example in [9, 12].

Lemma 2.1 *Let $f : K \times K \to R \cup \{+\infty\}$ be an equilibrium bifunction. Then the following statements are equivalent:*

(i) x^* is a solution to (EP).

(ii) $x^* \in K$ is a solution to the problem:

$$\min_{y \in K} f(x^*, y). \tag{2.1}$$

Proof. See e.g. Lemma 1 in [12] □

In what follows we suppose that:

Hypothesis (C): *K is nonempty, closed, convex and $f(x, \cdot)$ convex and subdifferentiable on K for each $x \in K$.*

In some cases, it is very useful to use the following regularized auxiliary problem

$$\text{Find } x^* \in K \text{ such that: } \rho f(x^*, y) + \frac{1}{2}\|y - x^*\|^2 \geq 0 \quad \text{for all } y \in K, \tag{AuxEP}$$

where $\rho > 0$ is a regularization parameter.

The equivalence between (EP) and (AuxEP) is due to the following lemma.

Lemma 2.2 *Let $f : K \times K \to R \cup \{+\infty\}$ be an equilibrium bifunction and let $x^* \in K$. Suppose Hypothesis (C) is satisfied. Then $x^* \in K$ is a solution to (EP) if and only if x^* is a solution to (AuxEP).*

Proof. See e.g. Proposition 1 in [12] □

3 The Extragradient Algorithm for EP.

Lemma 2.2 suggests an iterative procedure $x^{k+1} = s(x^k)$, for solving the equilibrium problem (EP) where $s(x^k)$ is the solution of the strongly convex program

$$\min_{y\in K}\left\{\rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\right\}.$$

However it is well known that (see e.g. [5, 6]), for monotone variational inequality which is a special case of monotone equilibrium problem, the sequence $\{x^k\}$ does not converge to a solution. So to obtain a convergent algorithm, the extragradient (or double projection) algorithm has been introduced for monotone variational inequality (see e.g. [5, 12, 13]).

3.1. Description of the Algorithm. The algorithm we are going to describe is an extension of the extragradient algorithm to equilibrium problem (EP) where f is pseudomonotone on K.

ALGORITHM 1.

Step 0: Take $x^0 \in K$, set $k := 0$.

Step 1: Find $y^k \in K$ as the unique solution to the strongly convex program:

$$\min_{y\in K}\left\{\rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\right\}. \tag{3.1}$$

If $y^k = x^k$, then terminate: x^k is a solution to (EP).

Otherwise go to **Step 2**.

Step 2: Find $x^{k+1} \in K$ as the unique solution to strongly convex program:

$$\min_{y\in K}\left\{\rho f(y^k, y) + \frac{1}{2}\|y - x^k\|^2\right\}. \tag{3.2}$$

Step 3: Set $k := k + 1$, and go to **Step 1**.

For convergence of the algorithm we suppose, in addition to a Hypothesis (C) that the equilibrium bifunction f satisfies the following Lipschitz condition:

There exist two constants $c_1 > 0$ and $c_2 > 0$, such that:

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|y - x\|^2 - c_2\|z - y\|^2 \quad \forall x, y, z \in K. \tag{3.3}$$

We note that when $x = z$, since $f(x, x) = 0$, this condition reduces to

$$f(x, y) + f(y, x) \geq -(c_1 + c_2)\|y - x\|^2 \quad \forall x, y \in K.$$

Theorem 3.1 *Suppose, in addition to Assumption (C) that the equilibrium bifunction $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ is pseudomonotone on K and satisfies the Lipschitz condition* (3.3)*. Let f is lower semicontinuous on $K \times K$ and $f(\cdot, y)$ is upper semicontinuous on K. Then*

(i) If Algorithm 1 terminates at some iteration point x^k by Step 1, then x^k is a solution to (EP).

(ii) If the sequence $\left\{x^k\right\}_{k\geq 0}$ is infinite, then for all $x^* \in K^*$, it holds that

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2-(1-2\rho c_1)\|y^k-x^k\|^2-(1-2\rho c_2)\|x^{k+1}-y^k\|^2. \tag{3.4}$$

(iii) If $0 < \rho < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, the sequence $\left\{x^k\right\}$ Q-linearly converges to a solution $x^* \in K^*$.

Proof. First we prove part (i). Suppose that at Step 1, we have $y^k = x^k$. Then x^k is the solution of the problem

$$\min_{y\in K}\left\{\rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\right\}.$$

By Lemma 2.1 it is a solution to (EP).
Let $x^* \in K^*$, we have

$$\begin{aligned}\|x^k - x^*\|^2 &= \|x^k - x^{k+1} + x^{k+1} - x^*\|^2 \\ &= \|x^{k+1} - x^k\|^2 + \|x^{k+1} - x^*\|^2 + 2\left\langle x^{k+1} - x^k, x^* - x^{k+1}\right\rangle.\end{aligned} \tag{3.5}$$

Since x^{k+1} solves the convex problem

$$\min_{y\in K}\left\{\rho f(y^k, y) + \frac{1}{2}\|y - x^k\|^2\right\}.$$

By the well known necessary and sufficient condition for optimality of convex programming, we have

$$0 \in \partial_y \left\{\rho f(y^k, y) + \frac{1}{2}\|y - x^k\|^2\right\}(x^{k+1}) + N_K(x^{k+1}),$$

where $N_K(x)$ stands for the normal cone of K at x and $\partial_y \{f(x, y)\}$ is subgradient of f respect to second variable. Since $f(y^k, \cdot)$ is proper convex and subdifferentiable on K, using the Moreau-Rockafellar theorem, we see that

$$x^k - x^{k+1} - \rho\partial_y f(y^k, x^{k+1}) \in N_K(x^{k+1}).$$

Therefore

$$\left\langle x^k - x^{k+1} - \rho w, y - x^{k+1}\right\rangle \leq 0 \quad \forall y \in K, w \in \partial_y f(y^k, x^{k+1}),$$

or

$$\left\langle x^{k+1} - x^k, y - x^{k+1} \right\rangle \geq \rho \left\langle w, x^{k+1} - y \right\rangle \quad \forall y \in K, w \in \partial_y f(y^k, x^{k+1}).$$

By the definition of subgradient, we have from the last inequality that

$$\left\langle x^{k+1} - x^k, y - x^{k+1} \right\rangle \geq \rho f(y^k, x^{k+1}) - \rho f(y^k, y) \quad \forall y \in K.$$

If $y = x^* \in K$, it becomes

$$\left\langle x^{k+1} - x^k, x^* - x^{k+1} \right\rangle \geq \rho f(y^k, x^{k+1}) - \rho f(y^k, x^*).$$

We note that, since x^* is a solution to (EP), $f(x^*, y^k) \geq 0$. Then by pseudomonotonicity, it follows $f(y^k, x^*) \leq 0$. Thus

$$\left\langle x^{k+1} - x^k, x^* - x^{k+1} \right\rangle \geq \rho f(y^k, x^{k+1}). \tag{3.6}$$

Now applying the Lipschitz condition (3.3) with $x = x^k$, $y = y^k$ and $z = x^{k+1}$ we obtain

$$\begin{aligned} \left\langle x^{k+1} - x^k, x^* - x^{k+1} \right\rangle \geq\ & \rho f(x^k, x^{k+1}) - \rho f(x^k, y^k) \\ & - \rho c_1 \|y^k - x^k\|^2 - \rho c_2 \|x^{k+1} - y^k\|^2. \end{aligned} \tag{3.7}$$

On the other hand, since y^k is the solution to the convex problem

$$\min_{y \in K} \left\{ \rho f(x^k, y) + \frac{1}{2} \|y - x^k\|^2 \right\},$$

by the same way, we can show that

$$\rho f(x^k, y) - \rho f(x^k, y^k) \geq \left\langle y^k - x^k, y^k - y \right\rangle, \quad \forall y \in K.$$

If $y = x^{k+1} \in K$, it becomes

$$\rho f(x^k, x^{k+1}) - \rho f(x^k, y^k) \geq \left\langle y^k - x^k, y^k - x^{k+1} \right\rangle. \tag{3.8}$$

From (3.5), (3.7) and (3.8), it follows that

$$\begin{aligned} \|x^k - x^*\|^2 \geq\ & \|x^{k+1} - x^k\|^2 + \|x^{k+1} - x^*\|^2 + 2 \left\langle y^k - x^k, y^k - x^{k+1} \right\rangle \\ & - 2\rho c_2 \|x^{k+1} - y^k\|^2 - 2\rho c_1 \|y^k - x^k\|^2. \end{aligned} \tag{3.9}$$

Substituting

$$\|x^{k+1} - x^k\|^2 + 2 \left\langle y^k - x^k, y^k - x^{k+1} \right\rangle = \|x^{k+1} - y^k\|^2 + \|y^k - x^k\|^2 \tag{3.10}$$

into (3.9), we obtain the estimation

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - 2\rho c_1)\|y^k - x^k\|^2 - (1 - 2\rho c_2)\|x^{k+1} - y^k\|^2,$$

which proves the first part of (ii).

Now we prove (iii). The assumption $0 < \rho < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ implies

$$1 - 2\rho c_1 > 0 \quad \text{and} \quad 1 - 2\rho c_2 > 0. \tag{3.11}$$

Thus

$$\|x^{k+1} - x^*\| \le \|x^k - x^*\| \quad \forall k \ge 0, \tag{3.12}$$

and

$$(1 - 2\rho c_1)\|y^k - x^k\|^2 \le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2. \tag{3.13}$$

The inequality (3.12) implies that $\left\{\|x^k - x^*\|\right\}_{k\ge 0}$ is nonincreasing. Since it is bounded below by 0, it must be convergent. To show that this sequence converges to 0 it suffices to show that it has a subsequence converging to 0. Indeed, since the sequence $\left\{x^k\right\}$ is bounded it has a subsequence converges to a point in K. Let $\left\{x^{k_i}\right\}_{i\ge 0}$ be the subsequence converging to some point $\overline{x}$. Applying inequality (3.12) iteratively, we obtain

$$(1 - 2\rho c_1)\sum_{k=0}^{n}\|y^k - x^k\|^2 \le \|x^0 - x^*\|^2 - \|x^{n+1} - x^*\|^2 \quad \forall n \ge 0.$$

As the sequence $\left\{\|x^k - x^*\|\right\}_{k\ge 0}$ is convergent, passing $n \to \infty$ we have

$$\sum_{k=0}^{\infty}\|y^k - x^k\|^2 < \infty,$$

which implies

$$\lim_{k\to\infty}\|y^k - x^k\| = 0. \tag{3.14}$$

Since $x^{k_i} \to \overline{x}$, it follows that

$$\lim_{i\to\infty} y^{k_i} = \overline{x}. \tag{3.15}$$

We note that y^{k_i} is the solution to the problem (3.1) at Step 1 of Algorithm 1, we can write

$$\rho f(x^{k_i}, y^{k_i}) + \frac{1}{2}\|y^{k_i} - x^{k_i}\|^2 \le \rho f(x^{k_i}, y) + \frac{1}{2}\|y - x^{k_i}\|^2 \quad \forall y \in K.$$

By the lower semicontinuity of f and the upper semicontinuity of $f(\cdot, y)$, passing to the limit as $i \to \infty$ and using (3.15) we arrive at Because of lower semicontinuity of f and upper semicontinuity of $f(\cdot, y)$, passing to the limit as $i \to \infty$ and using (3.15) we arrive at

$$\rho f(\overline{x}, y) + \frac{1}{2}\|y - \overline{x}\|^2 \ge 0 \quad \forall y \in K,$$

which means that $\overline{x}$ is a solution (AuxEP). Then, by Lemma 2.2, $\overline{x}$ is solution to (EP). Thus we can apply (3.12) with $x^* = \overline{x}$, to obtain

$$\|x^{k+1} - \overline{x}\| \leq \|x^k - \overline{x}\| \quad \forall k \geq 0, \tag{3.16}$$

which implies that the sequence $\|x^k - \overline{x}\|$ is convergent. Noting that the sequence $\left\{x^k\right\}_{k\geq 0}$ has a subsequence converging to $\overline{x}$ we deduce that the whole sequence $\left\{x^k\right\}_{k\geq 0}$ converges to $\overline{x} \in K^*$ as desired.

Finally, from the inequality (3.12), we obtain:

$$\frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|} \leq 1.$$

Passing to the limit above inequality as $k \to \infty$, it implies

$$\lim_{k\to\infty} \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|} < +\infty,$$

which suffices to show that the sequence $\left\{x^k\right\}$ converges to x^* in Q-linear rate. $\square$

3.2. Variational Inequality Case. We consider the following variational inequality

$$\textit{Find } x^* \in K \textit{such that: } \langle T(x^*), x - x^* \rangle \geq 0 \quad \textit{for all } x \in K, \tag{VIP}$$

where $T : K \to K$ is continuous.

As we have above mentioned, by setting

$$f(x, y) := \langle T(x), y - x \rangle\,, \tag{3.17}$$

we can easy to check that x^* is a solution to (EP) if and only if it is a solution to (VIP). Clearly, $f(x, \cdot)$ is linear on K and $f(x, x) = 0$ for every $x \in K$.

It is easy to see that if T is pseudomonotone on K, that is

$$\langle T(y), y - x \rangle \leq 0 \quad \textit{implies} \quad \langle T(x), x - y \rangle \geq 0 \quad \forall x, y \in K,$$

then f defined by (3.17) is pseudomonotone on K.

We recall that the mapping T is said to be Lipschitz on K with constant L (shortly L-Lipschitz) if

$$\|T(x) - T(y)\| \leq L\|x - y\| \quad \forall x, y \in K.$$

A relationship between Lipschitz continuity of T and the Lipschitz continuity of f in the sense of (3.3) is due to the following lemma.

Lemma 3.1 *If T is Lipschitz on K and $f(x, y) := \langle T(x), y - x \rangle$, then*

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \ \forall x, y, z \in K.$$

where c_1 and c_2 may be any positive numbers satisfying $2\sqrt{c_1c_2} \geq L$.

Proof. We have

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) &= \langle T(y) - T(x), z - y\rangle \\ &\geq -\|T(y) - T(x)\|\|z - y\| \\ &\geq -L\|y - x\|\|z - y\| \\ &\geq -2\sqrt{c_1 c_2}\|y - x\|\|z - y\| \\ &= -2\sqrt{c_1}\|y - x\|\sqrt{c_2}\|z - y\| \\ &\geq -c_1\|y - x\|^2 - c_2\|z - y\|^2. \end{aligned}$$

Thus f satisfies Lipschitz condition (3.3) □

Remark 3.1 *When $f(x,y) := \langle T(x), y - x\rangle$, Algorithm 1 becomes the extragradient algorithm for (VIP) when T is pseudomonotone on K (see e.g. [5, 12]). By Theorem 3.1 and Lemma 3.1, the algorithm is convergent when T is pseudomonotone and Lipschitz on K whenever $0 < \rho < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ for any pair $c_1 > 0$ and $c_2 > 0$ satisfying $2\sqrt{c_1c_2} \geq L$. The last condition for the regularization ρ is somewhat weaker than that in the extragradient algorithm in [5, 12].*

Remark 3.2 *In practice we terminate the algorithm when $\|x^k - y^k\| \leq \epsilon$, where $\epsilon > 0$ is a given tolerance.*

4 A Line-Search Algorithm

Algorithm 1 requires that f satisfies the Lipschitz condition (3.3) which in some cases is not known. In this section, in order to avoid this requirement, we modify Algorithm 1 by using a line search. The line search technique has been used widely in descent methods for solving mathematical programming problems as well as variational inequalities [5, 6].

First, we begin with he following definition

Definition 4.1. Let K be a nonempty closed set in R^n. A mapping $P : R^n \to R^n$ is said to be

(i) *feasible with respect to* K, if

$$P(x) \in K \quad \forall x \in R^n,$$

(ii) *quasi-nonexpansive with respect to* K, if for every $x \in R^n$, we have

$$\|P(x) - y\| \leq \|x - y\| \quad \forall y \in K. \tag{4.1}$$

Noting that, if $\pi_K(\cdot)$ is the Euclidean projection on K, then $\pi_K(\cdot)$ is a feasible quasi-nonexpansive mappings. By $\mathcal{F}(K)$ we denote the class of feasible quasi-nonexpansive mappings with respect to K.

Next, we choose a sequence $\{\gamma_k\}_{k\geq 0}$ such that

$$\gamma_k \in (0,2); \quad k = 0,1,\ldots; \quad \liminf \gamma_k(2-\gamma_k) > 0.$$

The algorithm then can be described as follows.

ALGORITHM 2.

Data: $x^0 \in K, \alpha \in (0,1), \theta \in (0,1)$.

Step 0: *Set* $k = 0$.

Step 1: *Find* $y^k \in K$ *as a solution to the following optimization problem:*

$$\min_{y \in K} \left\{ \rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 \right\}. \tag{4.2}$$

If $y^k = x^k$ *stop.*

Step 2: *(Auxiliary procedure)*

Step 2.1: Find m *as the smallest number in* $\mathbb{N}$ *such that*

$$\begin{cases} z^{k,m} = (1-\theta^m)x^k + \theta^m y^k; \\ \rho f(z^{k,m}, y^k) + \frac{\alpha}{2}\|y^k - x^k\|^2 \leq 0. \end{cases} \tag{4.3}$$

Step 2.2: Set $\theta_k = \theta^m, z^k = z^{k,m}$. *Take any* $g^k \in \partial_y f(z^k, z^k)$. *If* $\|g^k\| = 0$ *stop. Otherwise, go to* **Step 3**.

Step 3: *(Main iterate) Set*

$$\sigma_k = \frac{-\theta_k f(z^k, y^k)}{(1-\theta_k)\|g^k\|^2} \quad \text{and} \quad x^{k+1} = P_k(x^k - \gamma_k \sigma_k g^k), \tag{4.4}$$

where $P_k \in \mathcal{F}(K)$.

Step 4: *Set* $k := k+1$, *and go to* **Step 1**.

The following lemma indicates that if Algorithm 2 terminates at Step 1 or Step 2.2, then indeed a solution of (EP) has been found.

Lemma 4.1 *Suppose Hypothesis (C) holds. Then if Algorithm 2 terminates at Step 1 (resp. Step 2.2),* x^k *(resp.* z^k*) is a solution to (EP).*

Proof. If Algorithm 2 terminates at Step 1, then $x^k = y^k$. Since y^k is a solution to the convex optimization problem (4.2), we have

$$\rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 \geq \rho f(x^k, y^k) + \frac{1}{2}\|y^k - x^k\|^2 = 0, \quad \forall y \in K.$$

By the same argument as in the proof of Lemma 2.2 we can show that x^k is a solution to (EP).

If Algorithm 2 terminates at Step 2.2, then $\|g^k\| = 0$, that means $0 \in \partial_y f(z^k, z^k)$. Since $f(x, \cdot)$ is convex subdifferentiable on K, z^k is a solution to the following convex problem:

$$\min_{y \in K} f(z^k, y)$$

Then by virtue of Lemma 2.1, z^k is a solution to (EP). $\square$

The next lemma shows that there always exists a number $m \in \mathbb{N}$ such that the condition (4.3) in Step 2.1 is satisfied.

Lemma 4.2 *Suppose that f is upper semicontinuous on K with respect to the first variable and that $x^k \in K$ is not a solution of problem (4.2). Then*

(i) there exist a finite integer $m \geq 0$ such that the inequality in (4.3) holds;

(ii) $f(z^k, y^k) < 0$.

Proof. We firstly prove the statement (i). Assume for contradiction that for every nonnegative integer m, we have

$$\begin{cases} z^{k,m} = (1 - \theta^m)x^k + \theta^m y^k; \\ f(z^{k,m}, y^k) + \frac{\alpha}{2\rho}\|y^k - x^k\|^2 > 0. \end{cases}$$

Passing to the limit above inequality (as $m \to \infty$), by upper semicontinuity of $f(\cdot, y)$, we obtain

$$\rho f(x^k, y^k) + \frac{\alpha}{2}\|y^k - x^k\|^2 \geq 0. \tag{4.5}$$

On the other hand, since y^k is a solution to the convex optimization problem (4.2), we have

$$\rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 \geq \rho f(x^k, y^k) + \frac{1}{2}\|y^k - x^k\|^2 \quad \forall y \in K.$$

With $y = x^k$ the last inequality implies

$$\rho f(x^k, y^k) + \frac{1}{2}\|y^k - x^k\|^2 \leq 0. \tag{4.6}$$

From (4.5) and (4.6) it follows that From (4.5) and (4.6) it follows that

$$\frac{\alpha}{2}\|y^k - x^k\|^2 \geq \frac{1}{2}\|y^k - x^k\|^2.$$

Hence it must be $x^k = y^k$ or else $\alpha \geq 1$. The first case contradicts to $x^k \neq y^k$ while the second one contradicts to the fact $0 < \alpha < 1$.

The statement (ii) is obvious by (4.3). $\square$

In order to prove the convergence of Algorithm 2, we give the following key property of the sequence $\{x^k\}_{k\geq 0}$ generated by the algorithm.

Lemma 4.3 *Suppose, in addition to (C), that f is pseudomonotone on K. Let $\{x^k\}$ be the sequence generated by Algorithm 2. Then the following statements hold true:*

(i) For every solution x^* of (EP) we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma_k(2-\gamma_k)(\sigma_k\|g^k\|)^2. \tag{4.7}$$

(ii) The sequence $\{\|x^k - x^*\|\}_{k\geq 0}$ is nonincreasing (therefore convergent).

(iii) The sequence $\{x^k\}_{k\geq 0}$ is bounded.

(iv) $\sum_{k=0}^{\infty}\gamma_k(2-\gamma_k)(\sigma_k\|g^k\|)^2 < +\infty$.

(v) If the sequence $\{x^k\}$ has a cluster point x^* such that x^* is a solution to (EP), then the whole sequence $\{x^k\}$ converges to x^*.

Proof. Take any $x^* \in K^*$. By the property (4.1) of P_k and (4.4), setting $w^k = z^k - \gamma_k\sigma_k g^k$, Take any $x^* \in K^*$. By the property (4.1) of P_k and (4.4), setting $w^k = z^k - \gamma_k\sigma_k g^k$, we have

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|P_k(w^k) - x^*\|^2 \\
&\leq \|w^k - x^*\|^2 &\quad (4.8)\\
&= \|x^k - \gamma_k\sigma_k g^k - x^*\|^2 \\
&= \|x^k - x^*\|^2 - 2\gamma_k\sigma_k\left\langle g^k, x^k - x^*\right\rangle + (\gamma_k\sigma_k\|g^k\|)^2. &\quad (4.9)
\end{aligned}$$

Since $g^k \in \partial_y f(z^k, z^k)$ and $f(z^k,\cdot)$ is convex on K, we have

$$\left\langle g^k, x^k - x^*\right\rangle = \left\langle g^k, x^k - z^k + z^k - x^*\right\rangle \geq \left\langle g^k, x^k - z^k\right\rangle + f(z^k, z^k) - f(z^k, x^*).$$

On the other hand, since f is pseudomonotone and $f(x^*, z^k) \geq 0$, it follows that $-f(z^k, x^*) \geq 0$. Then from the last inequality we obtain

$$\left\langle g^k, x^k - x^*\right\rangle \geq \left\langle g^k, x^k - z^k\right\rangle. \tag{4.10}$$

From the formula (4.3), it is easy to see that

$$x^k - z^k = \frac{\theta_k}{1-\theta_k}(z^k - y^k).$$

Therefore

$$\begin{aligned} \left\langle g^k, x^k - z^k \right\rangle &= \frac{\theta_k}{1-\theta_k} \left\langle g^k, z^k - y^k \right\rangle \\ &\geq \frac{\theta_k}{1-\theta_k}[f(z^k, z^k) - f(z^k, y^k)] \\ &= \frac{-\theta_k}{(1-\theta_k)} f(z^k, y^k). \end{aligned} \tag{4.11}$$

Using (ii) in Lemma 4.2 and formula (4.4), we deduce

$$\frac{-\theta_k}{(1-\theta_k)} f(z^k, y^k) = \sigma_k \|g^k\|^2 > 0. \tag{4.12}$$

Then from (4.8), (4.10), (4.11) and (4.12) follows

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma_k(2-\gamma_k)(\sigma_k\|g^k\|)^2 \quad \forall x^* \in K^*,$$

which means that inequality (4.7) was proved.

The statements (ii) and (iii) are immediate from (i) and from fact that $0 < \gamma_k < 2$. To prove (iv) we apply (i) for all k from 0 to n to obtain

$$\sum_{k=0}^{n} \gamma_k(2-\gamma_k)(\sigma_k\|g^k\|)^2 \leq \|x^0 - x^*\|^2 - \|x^{n+1} - x^*\|^2.$$

Since $\left\{\|x^k - x^*\|\right\}_{k\geq 0}$ is convergent, passing $n \to \infty$ we obtain

$$\sum_{k=0}^{\infty} \gamma_k(2-\gamma_k)(\sigma_k\|g^k\|)^2 < \infty.$$

The statement (v) is immediate from the fact that, by (ii), for every solution $x^* \in K^*$, the whole sequence $\left\{x^k - x^*\right\}$ is convergent. The lemma is proved. □

Now we are on a position to prove the following convergence theorem for Algorithm 2.

Theorem 4.1 *In addition to the assumptions of Lemmas 4.2 and 4.3, we assume that f is lower semicontinuous on $K \times K$. Then, Algorithm 2 or terminates at some iteration point x^k yielding a solution to (EP) or else the sequence $\left\{x^k\right\}$ converges to a solution of (EP). Moreover, if $\gamma_k = \gamma \in (0,2), \forall k \geq 0$, then*

$$\lim_{k\to\infty} \inf(\sigma_k\|g^k\|\sqrt{k+1}) = 0.$$

Proof. If Algorithm 2 terminates at some iteration point x^k, then x^k is a solution to (EP) by Lemma 4.1.

Otherwise, by (iv) of Lemma 4.3, we have

$$\gamma_k(2-\gamma_k)(\sigma_k\|g^k\|)^2 \to 0 \quad \text{as } k \to \infty.$$

Since $\liminf \gamma_k(2-\gamma_k) > 0$, it follows that $\sigma_k\|g^k\| \to 0$ as $k \to \infty$. Then

$$\sigma_k\|g^k\| = \frac{-\theta_k}{(1-\theta_k)\|g^k\|} f(z^k, y^k) \to 0 \quad \text{as } k \to \infty.$$

Thus, from Step 2.2 of Algorithm 2 and by Lemma 4.3(iii), we have $\|g^k\| \neq 0$ for all k and bounded, it follows that

$$\frac{-\theta_k}{1-\theta_k} f(z^k, y^k) \to 0 \quad \text{as } k \to \infty. \tag{4.13}$$

On the other hand, according to the rule (4.3), it is easy to see that:

$$\frac{\alpha}{2\rho}\|x^k - y^k\|^2 \leq -f(z^k, y^k). \tag{4.14}$$

We consider two cases:
Case 1: $\limsup \theta_k > 0$. Then there exists $\overline{\theta} > 0$ and a subsequence $\mathcal{K}$ such that $\theta_k \geq \overline{\theta}$ for every $k \in \mathcal{K}$. From (4.13) and inequality (4.14), we obtain:

$$\lim_{k(\in\mathcal{K})\to\infty} \|y^k - x^k\| = 0. \tag{4.15}$$

Again by Lemma 4.3(ii), the subsequence $\{x^k : k \in \mathcal{K}\}$ is bounded. Thus we may assume without loss of generality that $\{x^k : k \in \mathcal{K}\}$ converges to some point $\overline{x}$. Using the limit (4.15) we see that the subsequence $\{y^k : k \in \mathcal{K}\}$ also converges to $\overline{x}$. Hence, from Step 1 of Algorithm 2 and by the upper semicontinuity of $f(\cdot, y)$, we can write

$$\rho f(\overline{x}, y) + \frac{1}{2}\|y - \overline{x}\|^2 \geq 0 \quad \forall y \in K.$$

By Lemma 2.2, $\overline{x}$ is a solution to (EP). Then in virtue of (v) in Lemma 4.3 the whole sequence $\{x^k\}$ converges to a solution to (EP), which proves the theorem in this case.
Case 2: $\lim_{k\to\infty} \theta_k = 0$. Let us set

$$\overline{z}^k = (1-\theta^{m-1})x^k + \theta^{m-1}y^k.$$

As before, since $\{x^k\}$ is bounded, we may be assume without loss of generality that some subsequence $\{x^k : k \in \mathcal{K}\}$ converges to some point $\overline{x}$. From Step 1 of Algorithm 2, by lower semicontinuity of $\rho f(x^k, \cdot) + \frac{1}{2}\|\cdot - x^k\|^2$, the sequence

$\{y^k\}$ is bounded too [1, 2]. Thus, by taking a subsequence, if necessary, we may assume that the subsequence $\{y^k : k \in \mathcal{K}\}$ also converges to some point $\overline{y}$. Taking the limit, by the lower semicontinuity of f and upper semicontinuity of $f(\cdot, y)$, from

$$\rho f(x^k, y^k) + \frac{1}{2}\|y^k - x^k\|^2 \le \rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 \quad \forall y \in K,$$

we can write

$$\overline{y} = \arg\min_{y \in K} \left\{ \rho f(\overline{x}, y) + \frac{1}{2}\|y - \overline{x}\|^2 \right\}. \tag{4.16}$$

On the other hand, by the rules (4.3) of Step 2.2 in Algorithm 2, since m is the smallest natural number satisfying (4.3), by $\{\theta_k\} \to 0$, we have:

$$\rho f(\overline{z}^k, y^k) + \frac{\alpha}{2}\|y^k - x^k\|^2 > 0.$$

By definition of $\overline{z}^k$, it implies $\lim_{k(\in\mathcal{K})\to\infty} \overline{z}^k = \overline{x}$. Passing $k \to \infty$, $k \in \mathcal{K}$ the above inequality, we obtain

$$\rho f(\overline{x}, \overline{y}) + \frac{\alpha}{2}\|\overline{y} - \overline{x}\|^2 \ge 0. \tag{4.17}$$

Note that formula (4.16) is equivalent to

$$\rho f(\overline{x}, y) + \frac{1}{2}\|y - \overline{x}\|^2 \ge \rho f(\overline{x}, \overline{y}) + \frac{1}{2}\|\overline{y} - \overline{x}\|^2 \quad \forall y \in K. \tag{4.18}$$

Substituting $y = \overline{x} \in K$, we then obtain

$$\rho f(\overline{x}, \overline{y}) + \frac{1}{2}\|\overline{y} - \overline{x}\|^2 \le 0. \tag{4.19}$$

Taking into account (4.17) we have

$$\frac{(1-\alpha)}{2}\|\overline{y} - \overline{x}\|^2 \le 0,$$

which together with $\alpha \in (0, 1)$ implies $\overline{x} = \overline{y}$. Then using (4.18) with $\overline{y} = \overline{x}$ we obtain

$$\rho f(\overline{x}, y) + \frac{1}{2}\|y - \overline{x}\|^2 \ge 0 \quad \forall y \in K.$$

Thus, by Lemma 2.2, $\overline{x}$ is a solution to (EP). Then, as before, we can deduce from (v) of Lemma 4.3 that the whole sequence $\{x^k\}$ converges to a solution to (EP).

Finally, we establish the remainder of the theorem. By assumtion $\gamma_k = \gamma \in (0, 2)$ for all $k \ge 0$. We suppose that $\lim_{k\to\infty} \inf(\sigma_k\|g^k\|\sqrt{k+1}) = 0$ does not hold, then there exist a number $\mu > 0$ such that:

$$\sigma_k\|g^k\| \ge \frac{\mu}{\sqrt{k+1}}.$$

And from Lemma 4.3(iv), it is easy to show that:

$$\mu \sum_{k=0}^{\infty} \frac{1}{k+1} < +\infty,$$

which is a contradiction. The theorem is proved. $\square$

Remark 4.1 *In practice to implement the algorithm we take a tolerance $\epsilon > 0$ and we terminate the algorithm when either $\|x^k - y^k\| \leq \epsilon$ or $\|g^k\| \leq \epsilon$.*

5 Numerical Results.

The computational results presented here are obtained by using MALAB Optimization Toolbox for solving the strongly convex quadratic subproblems needed to solve in the proposed algorithms. We tested Algorithm 1 for the following equilibrium problem:

$$\textit{Find } x^* \in K \textit{ such that } f(x^*, y) \geq 0 \quad \textit{for all } y \in K, \tag{5.1}$$

where K is a polyhedral convex set given by

$$K := \{x \in R^n | Ax \leq b, x \geq 0\}, \tag{5.2}$$

and the equilibrium bifunction $f : K \times K \to R \cup \{+\infty\}$ has the form

$$f(x, y) = \langle F(x) + Qy + q, y - x \rangle . \tag{5.3}$$

with $F(\cdot)$ being a continuous mapping on K and $Q \in R^{n\times n}$ being symmetric positive semidefinite and $q \in I\!R^n$. Since Q is symmetric positive semidefinite, $f(x, \cdot)$ is convex for each fixed $x \in K$. For Problem (5.1), we have the following results:

Lemma 5.1 *If $F : K \to I\!R^n$ is τ- strongly monotone on K. Then*

i) f is monotone on K whenever $\tau = \|Q\|$.

ii) f is $\tau - \|Q\|$-strongly monotone on K whenever $\tau > \|Q\|$.

Proof. From the definition of f we have

$$f(x, y) + f(y, x) = \langle Q(y - x), y - x\rangle - \langle F(y) - F(x), y - x\rangle , \tag{5.4}$$

which implies

$$\langle Q(y - x), y - x\rangle \leq \|Q\|\|y - x\|^2. \tag{5.5}$$

Since F is τ-strongly monotone on K, that is

$$\langle F(y) - F(x), y - x\rangle \geq \tau\|y - x\|^2,$$

we have, from (5.4) and (5.4) that $f(x,y)+f(y,x) \le 0$ whenever $\tau = \|Q\|$ and

$$f(x,y) + f(y,x) \le -(\tau - \|Q\|)\|y-x\|^2,$$

whenever $\tau > \|Q\|$. $\square$

Lemma 5.2 *If $F:$ is L-Lipschitz on K,i.e.*

$$\|F(y) - F(x)\| \le L\|y-x\| \quad x, y \in K,$$

Then

$$f(x,y) + f(y,z) \ge f(x,z) - c_1\|y-x\|^2 - c_2\|z-y\| \quad \forall x, y, z \in K,$$

for any $c_1, c_2 > 0$ satisfying

$$2\sqrt{c_1 c_2} \ge L + \|Q\|.$$

Proof. For every $x, y, z \in K$ we have

$$f(x,y) + f(y,z) - f(x,z) = \langle F(y) - F(x), z-y \rangle + \langle Q(y-z), y-x \rangle,$$

which, by Cauchy-Schwartz, inequality implies

$$\langle Q(y-z), y-x \rangle \ge -\|Q\|\|z-y\|\|y-x\|,$$

and

$$\langle F(y) - F(x), z-y \rangle \ge -\|F(y) - F(x)\|\|z-y\|.$$

Since F is L-Lipschitz, we can write

$$\langle F(y) - F(x), z-y \rangle \ge -L\|y-x\|\|z-y\|.$$

Thus from the last three inequalities follows

$$f(x,y) + f(y,z) - f(x,z) \ge -(L + \|Q\|)\|y-x\|\|z-y\|.$$

Then, by the hypothesis we have

$$-(L + \|Q\|)\|y-x\|\|z-y\| \ge -c_1\|y-x\|^2 - c_2\|z-y\|^2.$$

Thus

$$f(x,y) + f(y,z) \ge f(x,z) - c_1\|y-x\|^2 - c_2\|z-y\|^2.$$

$\square$

In a special case when F is a linear mapping of the form $F(x) = Px$ with $P \in R^{n\times n}$, the function f defined above takes the form

$$f(x,y) = \langle Px + Qy + q, y-x \rangle. \tag{5.6}$$

We suppose that the matrices P, Q are chosen such that Q is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite. Then f has the following properties:

(i) f is monotone, $f(\cdot, y)$ is continuous and $f(x, \cdot)$ is differentiable, convex on K.

(ii) For every $x, y, z \in K$ one has

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|z-y\|^2 - c_2\|y-x\|^2. \tag{5.7}$$

where $c_1 = c_2 = \frac{\|P-Q\|}{2}$.

Indeed, for every $x, y \in K$, since $Q - P$ is symmetric negative semidefinite, we have

$$f(x,y) + f(y,x) = \langle (Q-P)(y-x), y-x\rangle \leq 0.$$

On the other hand, since Q is symmetric positive semidefinite, $f(x, \cdot)$ is convex and differentiable on K.

To see (ii) we note that for every $x, y, z \in K$ we have

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) &= \langle (P-Q)(y-x), z-y\rangle \\ &\geq -2\frac{\|P-Q\|}{2}\|y-x\|\|z-y\| \\ &\geq -\frac{\|P-Q\|}{2}\|y-x\|^2 - \frac{\|P-Q\|}{2}\|z-y\|^2. \end{aligned}$$

By setting, for example, $c_1 = c_2 = \frac{\|Q-P\|}{2}$, we obtain (5.7).

Now we consider the optimization problem

$$\min_{y\in K}\left\{\rho f(x,y) + \frac{1}{2}\|y-x\|^2\right\}. \tag{5.8}$$

that we can write in the form

$$\rho f(x,y) + \frac{1}{2}\|y-x\|^2 = \frac{1}{2}\langle Hy, y\rangle + \langle c(x), y\rangle + b(x),$$

where $H = 2\rho Q + I$, $c(x) = (\rho(P-Q)-I)x + \rho q$ and $b(x) = \left\langle (\frac{1}{2}I - \rho P)x - \rho q, x\right\rangle$.

Similarly, for the problem

$$\min_{y\in K}\left\{\rho f(\overline{x},y) + \frac{1}{2}\|y-x\|^2\right\}, \tag{5.9}$$

we have

$$\rho f(\overline{x},y) + \frac{1}{2}\|y-x\|^2 = \frac{1}{2}\left\langle \hat{H}y, y\right\rangle + \langle \hat{c}(x,\overline{x}), y\rangle + \hat{b}(x,\overline{x}),$$

where

$$\hat{H} = 2\rho Q + I,$$

$$\hat{c}(x,\overline{x}) = \rho(P - Q^T)\overline{x} - x + \rho q,$$

and

$$\hat{b}(x, \overline{x}) = \frac{1}{2}\|x\|^2 - \rho \langle P\overline{x} + q, \overline{x} \rangle .$$

Thus both problems (5.8) and (5.9) are convex quadratic programming problems which can be solved efficiently by MALAB Optimization Toolbox

We tested Algorithm 1 with f given as in (5.1) and $n = 5$. The following matrices P and Q are randomly generated

$$Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}; \qquad P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

With $q = (1, -2, -1, 2, -1)^T$,

$$K = \left\{ x \in R^5 | \sum_{i=1}^{5} x_i \geq -1, -5 \leq x_i \leq 5, i = 1, \ldots, 5 \right\},$$

$c_1 = c_2 = \frac{1}{2}\|Q - P\| = 1.4525, \rho = \frac{1}{2}c_1 = 0.7262,\ x_0 = (1, 3, 1, 1, 2)^T$ and $\epsilon = 10^{-3}$ we obtained the following computational results.

Iter(k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k
0	1.00000	3.00000	1.00000	1.00000	2.00000
1	-0.34415	1.59236	0.68742	-0.15427	0.63458
2	-0.67195	1.10393	0.65016	-0.57872	0.30562
3	-0.73775	0.92351	0.66742	-0.74459	0.22567
4	-0.74236	0.85341	0.68785	-0.81261	0.20624
5	-0.73668	0.82486	0.70195	-0.84184	0.20152
6	-0.73168	0.81276	0.71030	-0.85493	0.20037
7	-0.72864	0.80747	0.71491	-0.86100	0.20009
8	-0.72700	0.80511	0.71737	-0.86389	0.20002
9	-0.72617	0.80403	0.71865	-0.86529	0.20001
10	-0.72576	0.80354	0.71931	-0.86598	0.20000

The approximate solution obtained after ten iterations is

$$x^{10} = (-0.72576, 0.80354, 0.71931, -0.86598, 0.20000)^T.$$

If we choose

$$P = \begin{bmatrix} 3.1 & 2.0 & 0.0 & 0.0 & 0.0 \\ 2.0 & 3.6 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 3.5 & 2.0 & 0.0 \\ 0.0 & 0.0 & 2.0 & 3.3 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix},$$

then the eigenvalues of matrix $Q-P$ are $(-0.7192, -2.7808, -2.9050, -0.8950, 0.0000)$. Thus, by observation (i), f is monotone. In this case the obtained computational results are

Iter(k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k
0	1.00000	3.00000	1.00000	1.00000	2.00000
1	-0.34006	1.59892	0.69395	-0.14884	0.69814
2	-0.67118	1.10637	0.65254	-0.57720	0.36476
3	-0.73773	0.92446	0.66833	-0.74422	0.27939
4	-0.74245	0.85380	0.68821	-0.81255	0.25753
5	-0.73676	0.82503	0.70210	-0.84185	0.25193
6	-0.73172	0.81283	0.71037	-0.85495	0.25049
7	-0.72866	0.80751	0.71494	-0.86102	0.25013
8	-0.72701	0.80512	0.71738	-0.86390	0.25003
9	-0.72618	0.80404	0.71866	-0.86530	0.25001
10	-0.72577	0.80354	0.71932	-0.86599	0.25000

and an approximate solution is

$$x^{10} = (-0.72577, 0.80354, 0.71932, -0.86599, 0.25000)^T$$

with the tolerance $\epsilon = 10^{-3}$.

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