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POLYNOMIALLY BOUNDED OPERATORS AND BOUNDED PRODUCTS

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Abstract

Two commuting operators S and T are polynomially bounded if there exists a constant C such that $\|p(S,T)\| \le C \|p\|_{\infty}$ for every polynomial p in two variables, where the sup norm is computed over the product of the spectrums of the two operators. Two operator valued measures E and F defined on the Borel sets of the plane satisfy a boundedness condition if their product has a bounded extension to the algebra generated by the measurable rectangles. In this note we point out a connection between these two properties for scalar operators S and T and their spectral resolutions E and F, respectively.

In this note we point out a connection between two different properties which have been studied in connection with certain pairs of commuting operators between Banach spaces. Namely, we consider the pairs of commuting scalar operators which are polynomially bounded, and the pairs of commuting scalar operators which have spectral resolutions whose products satisfy a boundedness condition.

Let X be a complex Banach space with $L(X)$ the space of continuous linear operators on X. Let $S, T \in L(X)$ be commuting operators with spectrums $\sigma = \sigma(S)$ and $\tau = \tau(T)$ [throughout we adhere to the terminology and notation of Dunford and Schwartz ([4],[5])]. Let $\mathcal{P}$ be the space of all complex valued polynomials in two variables with the sup-norm, $\| p \|_{\infty} = \sup\{| p(s,t) | : s \in \sigma, t \in \tau\}$. The operators S and T are said to be polynomially bounded or satisfy condition (P) if

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(*P*) *there exists a constant* C *such that* $\| p(S,T) \| \leq C \| p \|_{\infty}$ *for all* $p \in \mathcal{P}$.

Polynomially bounded operators have been considered in various settings and used to construct functional calculi ([1],[6]; in condition (P) we have used a different definition for $\| \ \|_{\infty}$). Let S and T be scalar operators with resolutions of the identity E and F, respectively ([5], XV.4.1). Let $\mathcal{B}(\sigma)$ (resp. $\mathcal{B}(\tau)$) be the Borel subsets of σ (resp. τ). If $A \in \mathcal{B}(\sigma)$, $B \in \mathcal{B}(\tau)$, let $A \times B$ be the rectangle generated by A and B. Define the product $E \times F$ of E and F on the rectangles by $(E \times F)(A \times B) = E(A)F(B)$. Let $\mathcal{A}$ be the algebra generated by the rectangles $A \times B$, $A \in \mathcal{B}(\sigma)$, $B \in \mathcal{B}(\tau)$. Then the product $E \times F$ has a unique finitely additive extension $E \times F$ to $\mathcal{A}$. Every element of $\mathcal{A}$ has a representation $\cup_{i=1}^{n} A_i \times B_i$ where the $\{A_i \times B_i\}$ are pairwise disjoint and $A_i \in \mathcal{B}(\sigma)$,$B_i \in \mathcal{B}(\tau)$. We say that S and T (or E and F) satisfy the boundedness condition (B) if

(B): there exists a constant k such that $\| \sum_{i=1}^{n} E(A_i)F(B_i) \| \leq k$ for every pairwise disjoint sequence $\{A_i \times B_i\}$ with $A_i \in \mathcal{B}(\sigma), B_i \in \mathcal{B}(\tau)$ (i.e., if $E \times F$ is bounded on $\mathcal{A}$).

The boundedness property (B) has been considered when the problem of whether the sum and product of two commuting spectral operators is a spectral operator ([3]). In general condition (B) is not satisfied even for reflexive spaces ([7],[10]) although it is always satisfied in Hilbert spaces ([12]).

In this note we consider the relationship between conditions (P) and (B) for commuting scalar operators. We show that for such operators (B) always implies (P) and show that the converse holds for commuting spectral operators with real spectrums.

If $x \in X$, let F_x denote the X-valued measure on $\mathcal{B}(\tau)$ defined by $F_x(B) = F(B)x$; note that F_x is countably additive in the norm topology of X since F is countably additive in the strong operator topology ([5]XV.2.5). If $y \in X'$, let ${}_yE$ be the X'-valued set function defined on $\mathcal{B}(\sigma)$ by $({}_yE)(A) = yE(A)$ note that ${}_yE$ is bounded and countably additive in the weak* topology of X'. We define the product of ${}_yE$ and F_x , ${}_yE \cdot F_x$,on measurable rectangles $A \times B, A \in \mathcal{B}(\sigma)$,$B\in \mathcal{B}(\tau)$, by ${}_yE \cdot F_x(A \times B) = \langle {}_yE(A), F_x(B) \rangle = \langle y, E(A)F(B)x \rangle$. Then ${}_yE \cdot F_x$ has a unique finitely additive extension to the algebra $\mathcal{A}$ generated by the measurable rectangles.

Lemma 1. *Assume that condition (B) is satisfied. Then* ${}_yE \cdot F_x$ *is bounded and countably additive on* $\mathcal{A}$ *and has a unique, countably additive extension to the* σ*-algebra* $\mathcal{B}(\sigma \times \tau)$ *generated by* $\mathcal{A}$ *which is bounded by* $k \| y \| \| x \|$.

Proof Condition (B) implies that ${}_yE \cdot F_x$ is bounded on $\mathcal{A}$ by $k \| y \| \| x \|$. If $A \in \mathcal{B}(\sigma)$ $(B \in \mathcal{B}(\tau))$, then ${}_yE(A) \cdot F_x(\cdot)$ $({}_yE(\cdot) \cdot F_x(B))$ is countably additive since F (E) is countably additive in the strong operator topology. It follows from Theorem2 of [8] that ${}_yE \cdot F_x$ is countably additive on $\mathcal{A}$.

The last statement follows from standard results for the extension of scalar measures.

Let G be an X-valued, bounded, finitely additive set function defined on a σ-algebra Σ of subsets of set S. If $f : S \to \mathbf{C}$ is bounded and Σ-measurable, the integral of f with respect to $G, \int_S f dG$, is then a well-defined element of X satisfying the inequality

$$(*) \quad \| \int_S f dG \| \leq \sup\{| f(t) | : t \in S\} semi - var(G)(S),$$

where $semi - var$ is the semi-variation of G ([4]IV.10.3; we integrate only bounded functions so no elaborate integration theory is required). In particular, if $f : S \to \mathbf{C}$ is a bounded Borel function, then f is both E and E_x integrable with $(\int_S f dE)x = \int_S f dE_x$.

We need the following simple Fubini-type result for the product ${}_yE \cdot F_x$.

Lemma 2. *Assume that condition (B) is satisfied. Let $f : \sigma \to \mathbf{C}$ and $g : \tau \to \mathbf{C}$ be bounded Borel functions and $x \in X, y \in X'$. Then $\int_{\sigma\times\tau} f \otimes g d_yE \cdot F_x = \langle \int_\sigma f d_yE, \int_\tau g dF_x \rangle$, where $f \otimes g : \sigma \times \tau \to \mathbf{C}$ is defined by $f \otimes g(s,t) = f(s)g(t)$.*

Proof If f and g are characteristic functions of Borel sets, the result follows from the definitions of the products so the result holds for simple functions. For the general case pick sequences of simple Borel functions $\{\alpha_k\}, \{\beta_k\}$ such that $\alpha_k \to f$ uniformly on σ and $\beta_k \to g$ uniformly on τ. Then $\alpha_k \otimes \beta_k \to f \otimes g$ uniformly on $\sigma \times \tau$ so the result follows easily from Lemma1 and the inequality (*).

We have our first result relating properties (B) and (P).

Theorem 3. *Assume that condition (B) is satisfied. Then condition (P) is satisfied with constant $C = 4k$.*

Proof Let $x \in X, y \in X'$ with $\| x \| \leq 1, \| y \| \leq 1$. For $i, j \geq 0$, by Lemma2 $\langle y, S^iT^jx \rangle = \langle \int_\sigma s^i d_yE(s), \int_\tau t^j dF_x(t) \rangle = \int_{\sigma\times\tau} s^it^j d_yE \cdot F_x(s,t)$. If p is a polynomial of two variables, we have $\langle y, p(S,T)x \rangle = \int_{\sigma\times\tau} p(s,t) d_yE \cdot F_x(s,t)$. From Lemma1 and [4]III.1.5, we obtain

$$| \langle y, p(S,T)x \rangle | \leq \| p \|_\infty var({}_yE \cdot F_x)(\sigma \times \tau) \leq \| p \|_\infty 4k,$$

where $var(\nu)$ is the variation of a measure ν. Hence, $\| p(S,T) \| \leq \| p \|_\infty 4k$ and condition (P) is satisfied (with constant $4k$ from (B)).

For the converse of Theorem3 we make the additional assumption that the spectrums of S and T are real. In this case it is only necessary to consider real polynomials in condition (P).

Theorem 4. *Assume that condition (P) is satisfied only for real polynomials and that σ and $\tau \subset \mathbf{R}$. Then condition (B) is satisfied with constant $k = C$.*

Proof Let $\mathcal{P}$ be the space of all real polynomials in two variables equipped with the sup-norm, $\| p \| = \sup\{| p(s,t) | : s \in \sigma, t \in \tau\}$. Define $\Psi : \mathcal{P} \to L(X)$ by $\Psi(p) = p(S,T)$. Then Ψ is an algebra homomorphism and by condition (P), Ψ is continuous with $\| \Psi \| \leq k$. Therefore, Ψ has a continuous linear extension, still denoted by Ψ ,from $C(\sigma \times \tau)$, the space of all real valued continuous functions on $\sigma \times \tau$, into $L(X)$ with $\| \Psi \| \leq k$. There exists a finitely additive set function G from the Borel sets of $\sigma \times \tau$, $\mathcal{B}(\sigma \times \tau)$, into $L(X)''$ such that $\langle G(\cdot), z\rangle \in rca(\ \sigma \times \tau)$ for every $z \in L(X)'$, $\langle z, \Psi f\rangle = \int_{\sigma\times\tau} f(s,t) dG(s,t) z$ for every $f \in C(\sigma \times \tau)$, and $\| \Psi \| = semi - var(G)$ ([4]VI.7.2). Since $S^i T^j = \int_\sigma s^i dE(s) \int_\tau t^j dF(t) = \Psi(s^i t^j) = \int_{\sigma\times\tau} s^i t^j dG(s,t)$ for $i, j \geq 0$,

$$(1)\ S^i q(T) = \int_\sigma s^i dE(s) \int_\tau q(t) dF(t) = \Psi(s^i q(t)) = \int_{\sigma\times\tau} s^i q(t) dG(s,t)$$

for $i \geq 0$ and all real valued polynomials q in one variable . If $g \in C(\tau)$, pick a sequence of real valued polynomials $\{q_k\}$ converging to g uniformly on τ. Then replacing q by q_k in (1) and passing to the limit using (*) gives
$S^i g(T) = \int_\sigma s^i dE(s) \int_\tau g(t) dF(t) = \Psi(s^i g(t)) = \int_{\sigma\times\tau} s^i g(t) dG(s,t)$.
Repeating the same argument gives

$$(2)\ f(S)g(T) = \int_\sigma f(s) dE(s) \int_\tau g(t) dF(t) = \Psi(f(s)g(t)) = \int_{\sigma\times\tau} f(s)g(t) dG(s,t)$$

for $f \in C(\sigma), g \in C(\tau)$.

If $x \in X, y \in X'$, define $y \otimes x \in L(X)'$ by $\langle y \otimes x, U\rangle = \langle y, Ux\rangle$ for $U \in L(X)$. Then $\| y \otimes x \| = \| y \| \| x \|$. We claim that $\langle yE(A), F(B)x\rangle = \langle G(A \times B), y \otimes x\rangle = G(A \times B) y \otimes x$ for all Borel sets $A \in \mathcal{B}(\sigma), B \in \mathcal{B}(\tau)$. From (2),

$$(3)\ \langle y \int_\sigma f dE, \int_\tau g dF x\rangle = \int_{\sigma\times\tau} f \otimes g dG y \otimes x$$

for $f \in C(\sigma), g \in C(\tau)$. Let $K \subset \tau$ be compact and pick $g_k \in C(\tau), | g_k | \leq 1$, such that $\{g_k\}$ converges to χ_K pointwise. Replacing g by g_k in (3) and using the fact that Fx and $Gy \otimes x$ are countably additive, we obtain from the Dominated Convergence Theorem ([4]IV.10.10)

$\langle y \int_\sigma f dE, \int_\tau \chi_K dF x\rangle = \langle y \int_\sigma f dE, F(K)x\rangle = \int_{\sigma\times\tau} f \otimes \chi_K dG y \otimes x$. Both $\langle y \int_\sigma f dE, F(\cdot)x\rangle$ and $\int_{\sigma\times\tau} f \otimes \chi_\cdot dG y \otimes x$ are countably additive on $\mathcal{B}(\tau)$, and since they agree on compact sets by the computation above, they are equal on $\mathcal{B}(\tau)$.

Fix $B \in \mathcal{B}(\tau)$ and let $L \subset \sigma$ be compact. Pick $f_k \in C(\sigma), | f_k | \leq 1$, such that $f_k \to \chi_L$ pointwise. Since $\langle y \int_\sigma f_k dE, F(B)x\rangle = \int_{\sigma\times\tau} f_k \otimes \chi_B dG y \otimes x$, passing to the limit as above gives $\langle yE(L), F(B)x\rangle = G(L \times K)y \otimes x$. Both $\langle yE(\cdot), F(B)x\rangle$ and $G(\cdot \times B)y \otimes x$ are countably additive on $\mathcal{B}(\sigma)$, and since they agree on the compact sets, they must be equal on $\mathcal{B}(\sigma)$. This establishes our claim.

Now let $\{A_i \times B_i : i = 1, ..., n\}$ be pairwise disjoint with $A_i \in \mathcal{B}(\sigma), B_i \in \mathcal{B}(\tau)$. Put $h = \sum_{i=1}^n \chi_{A_i} \otimes \chi_{B_i}$. Let $x \in X, y \in X'$ with $\| x \| \leq 1, \| y \| \leq 1$. Since $var(Gy \otimes x) \leq semi - var(G)$, we have from the equality above

$$\begin{aligned} & \left| \int_{\sigma\times\tau} hdGy \otimes x \right| = \left| \sum_{i=1}^n G(A_i \times B_i)y \otimes x \right| = \left| \langle y, \sum_{i=1}^n E(A_i)F(B_i)x\rangle \right| \\ \leq \quad & var(Gy \otimes x) \leq semi - var(G) = \| \Psi \| \leq k \end{aligned}$$

Hence, $\| \sum_{i=1}^n E(A_i)F(B_i) \| \leq k$ and (B) is satisfied with the same constant as in (P).

Remark 5. The proof of Theorem4 shows that if the assumptions of the theorem are satisfied, there is a functional calculus $\Psi : C(\sigma \times \tau) \to L(X)$,i.e., there exists an algebra homomorphism $\Psi : C(\sigma \times \tau) \to L(X)$ such that $\Psi(p) = p(S, T)$ for every real polynomial in two variables, and, moreover, Ψ has a representation as $\langle y, \Psi(f)(x)\rangle = \int_{\sigma\times\tau} fd_yF \cdot E_x$ for $x \in X, y \in X'$. Since the integral $\int_{\sigma\times\tau} fdG$ exists for every bounded Borel function $f : \sigma \times \tau \to \mathbf{R}$ ([2] I.1.12), the functional calculus Ψ can be extended to a functional calculus $\Psi : B(\sigma \times \tau, \mathcal{B}(\sigma \times \tau)) \to L(X)$, where $B(\sigma \times \tau, \mathcal{B}(\sigma \times \tau))$ is the space of bounded Borel functions on $\sigma \times \tau$ with the sup-norm. The calculations above show that

$$\begin{aligned} (3)\ \ \langle y \otimes x, \Psi(f \otimes g)\rangle &= \int_{\sigma\times\tau} f \otimes g(s,t)dG(s,t)y \otimes x \\ &= \left\langle y \int_\sigma fdE, \int_\tau gdFx \right\rangle = \int_{\sigma\times\tau} f \otimes gd_yE \cdot F_x, \end{aligned}$$

when f (g) is a simple Borel function on σ (τ). A simple passage to the limit shows that (3) holds when f (g) is a bounded Borel function. In particular, $G(L \times K)y \otimes x = \langle yE(L), F(K)x\rangle$, when L (K) is a Borel subset of σ (τ), so that G is in some sense an extension of the product $E \times F$ to the Borel sets of $\sigma \times \tau$. If the space X contains no isomorphic copy of c_0, then G actually has a unique extension to $\mathcal{B}(\sigma \times \tau)$ which has values in $L(X)$, not $L(X)''$, and is strongly countably additive ([11]).

It would be of interest to know if the conclusion of Theorem 4 holds without the restrictions on the spectrums. The proof above uses the fact that the real valued polynomials are dense in the space of continuous real valued functions, something not available in the complex case.

If X is a Hilbert space and the spectral resolutions are self-adjoint projections on X, then condition (B) is always satisfied since if $\{A_i \times B_i : i = 1, ..., n\}$

is a pairwise disjoint sequence from $\mathcal{A}$, then $\sum_{i=1}^{n} E(A_i)F(B_i)$ is an orthogonal projection and hence has norm 1.

If X is a Hilbert space and S, T are commuting scalar operators, then by [5]XV.6.2 there is a bounded, invertible self-adjoint U such that $UE(\cdot)U^{-1}$, $UF(\cdot)U^{-1}$ are self-adjoint and thus satisfy condition (B) by the observation above. Thus, we have

Corollary 6. *If X is a Hilbert space and S, T are commuting scalar operators, then conditions (B) and (P) are satisfied.*

Proof Since $UE(\cdot)U^{-1}, UF(\cdot)U^{-1}$ satisfy condition (B), E and F likewise satisfy the condition. The last statement follows from Theorem3.

The first statement in Corollary6 was established by Wermer by different methods ([12]); the proof above uses LemmaXV.6.2 of [5].

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