East-West J. of Mathematics: Vol. 20, No 2 (2018) pp. 158-179

THE HIT PROBLEM AND THE ALGEBRAIC TRANSFER IN SOME DEGREES

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Abstract

Denote by P_k the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$, with the degree of each generator x_i being 1, and let GL_k be the general linear group over the prime field $\mathbb{F}_2$ of two elements which acts naturally on P_k by matrix substitution.

We study the *Peterson hit problem* of determining a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we study the hit problem in terms of the admissible monomials at the degree $(k-1)(2^d-1)$. These results are used to verify Singer's conjecture for the algebraic transfer, which is a homomorphism from the homology of the mod-2 Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2)$, to the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ consisting of all the GL_k-invariant classes of degree n. More precisely, using the results on the hit problem, we prove that Singer's conjecture for the algebraic transfer is true in the case $k = 5$ and the degree $4(2^d - 1)$ with d an arbitrary positive integer.

1 Introduction

Denote by $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1. This algebra arises as the cohomology with coefficients in $\mathbb{F}_2$ of an elementary abelian 2-group of rank k. Hence, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula $Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f)Sq^{n-i}(g)$, for $f, g \in P_k$ (see Steenrod and Epstein [30]).

Key words: Steenrod algebra, Peterson hit problem, algebraic transfer, polynomial algebra.
2010 AMS Mathematics classification: Primary 55S10; Secondary 55S05, 55T15.

A polynomial g in P_k is called *hit* if it belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ is the augmentation ideal of $\mathcal{A}$. That means g can be written as a finite sum $g = \sum_{u\geqslant 0} Sq^{2^u}(g_u)$ for suitable polynomials $g_u \in P_k$.

We are interested in the *hit problem*, set up by F. Peterson, of determining a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to determine a basis of the $\mathbb{F}_2$-vector space $QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. The problem is an interesting and important one. It was first studied by Peterson [21], Wood [41], Singer [28], and Priddy [24], who showed its relation to several classical problems respectively in cobordism theory, modular representation theory, the Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. Then, this problem was investigated by Carlisle and Wood [3], Crabb and Hubbuck [4], Hung and Nam [11], Janfada and Wood [13], Kameko [14, 15], Mothebe [18], Nam [19], Repka and Selick [25], Phuc and Sum [22, 23], Silverman [26], Silverman and Singer [27], Singer [29], Walker and Wood [39, 40], Wood [42] and others.

The vector space QP_k was explicitly calculated by Peterson [21] for $k = 1, 2$, by Kameko [14] for $k = 3$, and recently by the present author [31, 33] for $k = 4$.

Let GL_k be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on P_k by matrix substitution. Since the two actions of $\mathcal{A}$ and GL_k upon P_k commute with each other, there is an inherited action of GL_k on QP_k.

Denote by $(P_k)_n$ the subspace of P_k consisting of all the homogeneous polynomials of degree n in P_k and by $(QP_k)_n$ the subspace of QP_k consisting of all the classes represented by the elements in $(P_k)_n$. In [28], Singer defined the algebraic transfer, which is a homomorphism

$$\varphi_k : \mathrm{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_k)_n^{GL_k}$$

from the homology of the Steenrod algebra to the subspace of $(QP_k)_n$ consisting of all the GL_k-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2)$. This transfer was studied by Boardman [1], Bruner, Hà and Hung [2], Hà [9], Hung [10], Chon-Hà [6, 7, 8], Minami [17], Nam [20], Hung and Quỳnh [12], Tin and Sum [38], the present author [32, 34, 35] and others.

It was shown that the transfer is an isomorphism for $k = 1, 2$ by Singer in [28] and for $k = 3$ by Boardman in [1] . However, for any $k \geqslant 4$, φ_k is not a monomorphism in infinitely many degrees (see Singer [28], Hung [10].) Singer made the following conjecture.

Conjecture 1.1 (Singer [28])**.** *The algebraic transfer φ_k is an epimorphism for any $k \geqslant 0$.*

The conjecture is true for $k \leqslant 3$. We are studying this conjecture for $k = 4$ by using the results in [31, 33]. We hope that it is also true in this case.

From the results of Wood [41] and Kameko [14], the hit problem is reduced to the case of degree n of the form

$$n = s(2^d - 1) + 2^d m, \tag{1.1}$$

where s, d, m are non-negative integers and $1 \leqslant s < k$, (see [33]). For $s = k-1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [4], Nam [19], Repka and Selick [25] and the present author [31, 33, 35].

In the present paper, we study the hit problem in degree n of the form (1.1) with $s = k-1$ and $m = 0$. This result is used to verify Singer's conjecture for algebraic transfer. More precisely, using this result we prove that Singer's conjecture is true in the case $k = 5$ and the degree $4(2^d - 1)$ with d an arbitrary positive integer.

From the result of Carlisle and Wood [3] on the boundedness conjecture, we can see that for d big enough, the dimension of $(QP_k)_n$ does not depend on d; it depends only on k. One of the main results of the paper is the following.

Theorem 1.2. *Let $n = (k-1)(2^d - 1)$ with d a positive integer. If $d \geqslant k \geqslant 3$, then*

$$\dim(QP_k)_n \geqslant \left((k-3)\binom{k}{2} + 1\right)(2^k - 1). \tag{1.2}$$

For either $k = 3$ or $k = 4$, the results of Kameko [14] and the present author [31, 33] imply that the inequality (1.2) is an equality. Note that this theorem has been proved in [22] for $d > k$. However, for the case $d = k$, the theorem is new and the proof of it is more complicated.

Using Theorem 1.2 for $k = 5$ and the results in [23], we obtain the following.

Theorem 1.3. *If $n = 4(2^d - 1)$ with d a positive integer, then $(QP_5)_n^{GL_5} = 0$.*

By a simple computatuion using the results in Tangora [37], Lin [16] and Chen [5], we see that $\mathrm{Ext}_{\mathcal{A}}^{5,2^{d+2}+1}(\mathbb{F}_2, \mathbb{F}_2) = \langle h_0 h_d^4 \rangle$, where h_d denote the Adams element in $\mathrm{Ext}_{\mathcal{A}}^{1,2^d}(\mathbb{F}_2, \mathbb{F}_2)$. Since $h_d^4 = 0$ for $d > 0$, we get $\mathrm{Ext}_{\mathcal{A}}^{5,2^{d+2}+1}(\mathbb{F}_2, \mathbb{F}_2) = 0$. Hence, $\mathrm{Tor}^{\mathcal{A}}_{5,2^{d+2}+1}(\mathbb{F}_2, \mathbb{F}_2) = 0$. By Theorem 1.3, the homomorphism

$$\varphi_5 : \mathrm{Tor}^{\mathcal{A}}_{5,2^{d+2}+1}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_5)^{GL_5}_{4(2^d-1)}$$

is a trivial isomorphism. So, we get the following.

Corollary 1.4. *Singer's conjecture is true for $k = 5$ and the degree $4(2^d - 1)$ with d an arbitrary positive integer.*

This paper is organized as follows. In Section 2, we recall some needed information on the weight vectors of monomials, the admissible monomials in P_k and Singer's criterion on the hit monomials. Theorems 1.2 and 1.3 are respectively proved in Sections 3 and 4.

2 Preliminaries

In this section, we recall some needed information from Kameko [14] and Singer [29], which will be used in the next section.

2.1 The weight vectors of monomials

Notation 2.1.1. We denote $\mathbb{N}_k = \{1, 2, \ldots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j \in \mathbb{N}_k \setminus \mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k,$$

In particular, $X_{\mathbb{N}_k} = 1,\ X_\emptyset = x_1 x_2 \ldots x_k,\ X_j = x_1 \ldots \hat{x}_j \ldots x_k,\ 1 \leqslant j \leqslant k$, and $X := X_k \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 with $i \geqslant 0$.

Let $x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant k$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geqslant 0$. Then, we have $x = \prod_{t \geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.1.2. For a monomial x in P_k, define two sequences associated with x by

$$\begin{aligned}
\omega(x) &= (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots),\\
\sigma(x) &= (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),
\end{aligned}$$

where $\omega_i(x) = \sum_{1 \leqslant j \leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)},\ i \geqslant 1$. The sequences $\omega(x)$ and $\sigma(x)$ is respectively called the weight vector and the exponent vector of x.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence ω are called the weight vector if $\omega_i = 0$ for $i \gg 0$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

For a weight vector ω, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$. If there are $i_0 = 0, i_1, i_2, \ldots, i_r > 0$ such that $i_1 + i_2 + \ldots + i_r = m$, $\omega_{i_1 + \ldots + i_{s-1} + t} = b_s, 1 \leqslant t \leqslant i_s, 1 \leqslant s \leqslant r$, and $\omega_i = 0$ for all $i > m$, then we write $\omega = (b_1^{(i_1)}, b_2^{(i_2)}, \ldots, b_r^{(i_r)})$. Denote $b_u^{(1)} = b_u$. For example, $\omega = (3, 3, 2, 2, 2, 1, 0, \ldots) = (3^{(2)}, 2^{(3)}, 1)$.

Denote by $P_k(\omega)$ the subspace of P_k spanned by all monomials y such that $\deg y = \deg \omega,\ \omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by all monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

Definition 2.1.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$ then f is called *hit*.

ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

For a polynomial $f \in P_k$, we denote by $[f]$ the class in QP_k represented by f. If ω is a weight vector and $f \in P_k(\omega)$, then denote by $[f]_\omega$ the class in $QP_k(\omega)$ represented by f. Denote by $|S|$ the cardinal of a set S.

It is easy to see that

$$QP_k(\omega) \cong QP_k^\omega := \langle\{[x] \in QP_k : x \text{ is admissible and } \omega(x) = \omega\}\rangle.$$

So, we get

$$(QP_k)_n = \bigoplus_{\deg \omega = n} QP_k^\omega \cong \bigoplus_{\deg \omega = n} QP_k(\omega).$$

Hence, we can identify the vector space $QP_k(\omega)$ with $QP_k^\omega \subset QP_k$.

For $1 \leqslant i \leqslant k$, define the $\mathcal{A}$-homomorphism $\rho_i : P_k \to P_k$, which is determined by $\rho_i(x_i) = x_{i+1}, \rho_i(x_{i+1}) = x_i$, $\rho_i(x_j) = x_j$ for $j \neq i, i+1,\ 1 \leqslant i < k$, and $\rho_k(x_1) = x_1 + x_2$, $\rho_k(x_j) = x_j$ for $j > 1$.

It is easy to see that the general linear group GL_k is generated by the matrices associated with $\rho_i,\ 1 \leqslant i \leqslant k$, and the symmetric group Σ_k is generated by the ones associated with $\rho_i,\ 1 \leqslant i < k$. So, a class $[f]_\omega$ represented by a homogeneous polynomial $f \in P_k(\omega)$ is an GL_k-invariant if and only if $\rho_i(f) \equiv_\omega f$ for $1 \leqslant i \leqslant k$. $[f]_\omega$ is an Σ_k-invariant if and only if $\rho_i(f) \equiv f$ for $1 \leqslant i < k$.

We note that the weight vector of a monomial is invariant under the permutation of the generators x_i, hence $QP_k(\omega)$ has an action of the symmetric group Σ_k. Furthermore, we have the following.

Lemma 2.1.4. *Let ω be a weight vector. Then, $QP_k(\omega)$ is an GL_k-module.*

Proof. We prove the lemma by showing that if x is a monomial in P_k, then $g_k(x) \in P_k(\omega(x))$.

If $\nu_1(x) = 0$, then $x = g_k(x)$ and $\omega(g_k(x)) = \omega(x)$. Suppose $\nu_1(x) > 0$ and $\nu_1(x) = 2^{t_1} + \ldots + 2^{t_b}$, where $0 \leqslant t_1 < \ldots < t_b,\ b \geqslant 1$.

Since $x = \prod_{t \geqslant 0} X_{\mathbb{J}_t(x)}^{2^t} \in P_k$ and g_k is a homomorphism of algebras,

$$g_k(x) = \prod_{t \geqslant 0}(g_k(X_{\mathbb{J}_t(x)}))^{2^t} = \left(\prod_{u=1}^{b}\big((x_1{+}x_2)X_{\mathbb{J}_{t_u}(x)\cup 1}\big)^{2^{t_u}}\right)\left(\prod_{t \neq t_1, t_2, \ldots, t_b} X_{\mathbb{J}_t(x)}^{2^t}\right).$$

Then, $g_k(x)$ is a sum of monomials of the form

$$\bar{y} = \left(\prod_{j=1}^{c}\left(x_2 X_{\mathbb{J}_{t_{u_j}}(x)\cup 1}\right)^{2^{t_u}}\right)\left(\prod_{t\neq t_{u_1},\ldots,t_{u_c}} X_{\mathbb{J}_t(x)}^{2^t}\right),$$

where $0 \leqslant c \leqslant b$. If $c = 0$, then $\bar{y} = x$ and $\omega(\bar{y}) = \omega(x)$. Suppose $c > 0$.

If $2 \in \mathbb{J}_{t_{u_j}}(x)$ for all j, $1 \leqslant j \leqslant c$, then $\omega(\bar{y}) = \omega(x)$ and $\bar{y} \in P_k(\omega(x))$. Suppose there is an index j such that $2 \notin \mathbb{J}_{t_{u_j}}(x)$. Let j_0 be the smallest index such that $2 \notin \mathbb{J}_{t_{u_{j_0}}}(x)$. Then, we have

$$\omega_i(\bar{y}) = \begin{cases}\omega_i(x), & \text{if } i \leqslant t_{u_{j_0}},\\ \omega_i(x) - 2, & \text{if } i = t_{u_{j_0}} + 1.\end{cases}$$

Hence $\omega(\bar{y}) < \omega(x)$ and $\bar{y} \in P_k(\omega(x))$. The lemma is proved. $\square$

2.2 The admissible monomials

Definition 2.2.1. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds:
i) $\omega(x) < \omega(y)$;
ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.2.2. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_m$ such that $y_t < x$ for $t = 1, 2, \ldots, m$ and $x - \sum_{t=1}^{m} y_t \in \mathcal{A}^+P_k$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

Theorem 2.2.3 (See Kameko [14]). *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*
i) *If w is inadmissible, then xw^{2^r} is also inadmissible.*
ii) *If w is strictly inadmissible, then wy^{2^s} is also strictly inadmissible.*

Now, we recall a result of Singer [29] on the hit monomials in P_k.

Definition 2.2.4. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{d_j} - 1$ for d_j a non-negative integer and $j = 1, 2, \ldots, k$. If z is a spike with $d_1 > d_2 > \ldots > d_{r-1} \geqslant d_r > 0$ and $d_j = 0$ for $j > r$, then it is called the minimal spike.

For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1\leqslant i\leqslant r}(2^{d_i} - 1)$, where $d_i > 0$. Singer showed in [29] that if $\mu(n) \leqslant k$, then there exists uniquely a minimal spike of degree n in P_k.

Lemma 2.2.5 (See [22]). *All the spikes in P_k are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector ω is weakly decreasing and $\omega_1 \leqslant k$, then there is a spike z in P_k such that $\omega(z) = \omega$.*

The following is a criterion for the hit monomials in P_k.

Theorem 2.2.6 (See Singer [29]). *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

This result implies a result of Wood, which originally is a conjecture of Peterson [21].

Theorem 2.2.7 (See Wood [41]). *If $\mu(n) > k$, then $(QP_k)_n = 0$.*

We end this section by recalling some notations which will be used in the next sections. We set

$$P_k^0 = \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \ : \ a_1 a_2 \dots a_k = 0\} \rangle,$$
$$P_k^+ = \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \ : \ a_1 a_2 \dots a_k > 0\} \rangle.$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have the following.

Proposition 2.2.8. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $QP_k = QP_k^0 \oplus QP_k^+$. Here $QP_k^0 = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0$ and $QP_k^+ = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+$.*

3 On the generators of P_k in degree $(k-1)(2^d-1)$

First of all, we recall some notations and definitions in [33], which will be used in the next sections.

3.1 Construction for the generators

Denote $\mathcal{N}_k = \big\{(i; I); I = (i_1, i_2, \dots, i_r), 1 \leqslant i < i_1 < \dots < i_r \leqslant k,\ 0 \leqslant r < k\big\}$.

Definition 3.1.1. Let $(i; I) \in \mathcal{N}_k$, let $r = \ell(I)$ be the length of I, and let u be an integer with $1 \leqslant u \leqslant r$. A monomial $x \in P_{k-1}$ is said to be u-compatible with $(i; I)$ if all of the following hold:

i) $\nu_{i_1-1}(x) = \nu_{i_2-1}(x) = \dots = \nu_{i_{(u-1)}-1}(x) = 2^r - 1$,
ii) $\nu_{i_u-1}(x) > 2^r - 1$,
iii) $\alpha_{r-t}(\nu_{i_u-1}(x)) = 1,\ \forall t,\ 1 \leqslant t \leqslant u$,
iv) $\alpha_{r-t}(\nu_{i_t-1}(x)) = 1,\ \forall t,\ u < t \leqslant r$.

Clearly, a monomial x can be u-compatible with a given $(i;I) \in \mathcal{N}_k$ for at most one value of u. By convention, x is 1-compatible with $(i;\emptyset)$.

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i,\\ x_{j+1}, & \text{if } i \leqslant j < k.\end{cases}$$

Definition 3.1.2. Let $(i;I) \in \mathcal{N}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1}+\ldots+2^{r-u}} \prod_{u<t\leqslant r} x_{i_t}^{2^{r-t}}$ for $r = \ell(I) > 0$, $x_{(\emptyset,1)} = 1$. For a monomial x in P_{k-1}, we define the monomial $\phi_{(i;I)}(x)$ in P_k by setting

$$\phi_{(i;I)}(x) = \begin{cases} (x_i^{2^r-1} f_i(x))/x_{(I,u)}, & \text{if there exists } u \text{ such that}\\ & x \text{ is } u\text{-compatible with } (i,I),\\ 0, & \text{otherwise.}\end{cases}$$

Then we have an $\mathbb{F}_2$-linear map $\phi_{(i;I)} : P_{k-1} \to P_k$. In particular, $\phi_{(i;\emptyset)} = f_i$.

For a subset $B \subset P_k$, we denote $[B] = \{[f] : f \in B\}$. If $B \subset P_k(\omega)$, then we set $[B]_\omega = \{[f]_\omega : f \in B\}$.

We denote

$$\Phi^0(B) = \bigcup_{1\leqslant i\leqslant k} \phi_{(i;\emptyset)}(B) = \bigcup_{1\leqslant i\leqslant k} f_i(B).$$

$$\Phi^+(B) = \bigcup_{(i;I)\in\mathcal{N}_k, 0<\ell(I)\leqslant k-1} \phi_{(i;I)}(B) \setminus P_k^0.$$

$$\Phi(B) = \Phi^0(B) \bigcup \Phi^+(B).$$

It is easy to see that if B is a minimal set of generators for $\mathcal{A}$-module P_{k-1} in degree n, then $\Phi^0(B)$ is a minimal set of generators for $\mathcal{A}$-module P_k^0 in degree n and $\Phi^+(B) \subset P_k^+$.

For a positive integer b, denote

$$\omega_{(k,b)} = ((k-1)^{(b)}),\ \bar{\omega}_{(k,b)} = ((k-1)^{(b-1)}, k-3, 1).$$

Lemma 3.1.3 (See [33])**.** *Let b be a positive integer and let $j_0, j_1, \ldots, j_{b-1} \in \mathbb{N}_k$. We set $i = \min\{j_0, \ldots, j_{b-1}\}$ and $I = (i_1, \ldots, i_r)$ with $\{i_1, \ldots, i_r\} = \{j_0, \ldots, j_{b-1}\} \setminus \{i\}$. Then, we have*

$$\prod_{0\leqslant t<b} X_{j_t}^{2^t} \equiv_{\omega_{(k,b)}} \phi_{(i;I)}(X^{2^b-1}).$$

Definition 3.1.4. For any $(i;I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i,\\ \sum_{s\in I} x_{s-1}, & \text{if } j = i,\\ x_{j-1}, & \text{if } i < j \leqslant k.\end{cases}$$

Then, $p_{(i;I)}$ is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = \emptyset$, $p_{(i;\emptyset)}(x_i) = 0$ and $p_{(i;I)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

Lemma 3.1.5 (See [22]). *If x is a monomial in P_k, then $p_{(i;I)}(x) \in P_{k-1}(\omega(x))$.*

Lemma 3.1.5 implies that if ω is a weight vector and $x \in P_k(\omega)$, then $p_{(i;I)}(x) \in P_{k-1}(\omega)$. Moreover, $p_{(i;I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$. In particular, we have

Lemma 3.1.6 (See [33]). *Let b be a positive integer and let $(j;J),(i;I) \in \mathcal{N}_k$ with $\ell(I) < b$.*

i) *If $(i;I) \subset (j;J)$, then $p_{(j;J)}\phi_{(i;I)}(X^{2^b-1}) = X^{2^b-1} \mod(P_{k-1}^-(\omega_{(k,b)}))$.*
ii) *If $(i;I) \not\subset (j;J)$, then $p_{(j;J)}\phi_{(i;I)}(X^{2^b-1}) \in P_{k-1}^-(\omega_{(k,b)})$.*

3.2 Proof of Theorem 1.2

For $0 < h \leqslant k$, set $\mathcal{N}_{k,h} = \{(i;I) \in \mathcal{N}_k : \ell(I) < h\}$. Then, $|\mathcal{N}_{k,h}| = \sum_{t=1}^{h}\binom{k}{t}$.

Proposition 3.2.1 (See [22]). *Let d be a positive integer and let $p = \min\{k,d\}$. Then, the set*

$$B(d) := \left\{\left[\phi_{(i;I)}(X^{2^d-1})\right]_{\omega_{(k,d)}} : (i;I) \in \mathcal{N}_{k,p}\right\}$$

is a basis of the $\mathbb{F}_2$-vector space $QP_k(\omega_{(k,d)})$. Consequently

$$\dim QP_k(\omega_{(k,d)}) = \sum_{t=1}^{p}\binom{k}{t}.$$

Set $C_k = \{x_{j_1}x_{j_2}\ldots x_{j_{k-3}}x_j^2 : 1 \leqslant j_1 < j_2 < \ldots < j_{k-3} < k,\ j_1 \leqslant j < k\} \subset P_{k-1}$. It is easy to see that $|C_k| = (k-3)\binom{k}{2}$.

Lemma 3.2.2 (See [22]). *C_k is the set of the admissible monomials in P_{k-1} such that their weight vectors are $\bar{\omega}_{(k,1)} = (k-3,1)$. Consequently,*

$$\dim QP_{k-1}(\bar{\omega}_{(k,1)}) = (k-3)\binom{k}{2}.$$

Corollary 3.2.3. *Let d be a positive integer. Then,*

$$D_{(k,d)} = \{X^{2^{d-1}-1}z^{2^{d-1}} : z \in C_k\}$$

is the set of the admissible monomials in P_{k-1} such that their weight vectors are $\bar{\omega}_{(k,d)}$. Consequently, $\dim QP_{k-1}(\bar{\omega}_{(k,d)}) = (k-3)\binom{k}{2}$.

Proposition 3.2.4. *Let d be an integer such that $d \geqslant k \geqslant 4$. Then, the set*

$$\bar{B}(d) := \bigcup_{z \in C_k} \big\{ \big[\phi_{(i;I)}(X^{2^{d-1}-1} z^{2^{d-1}})\big]_{\bar{\omega}_{(k,d)}} : \ (i;I) \in \mathcal{N}_k \big\}$$

is a basis of $QP_k(\bar{\omega}_{(k,d)})$. Consequently $\dim QP_k(\bar{\omega}_{(k,d)}) = (k-3)\binom{k}{2}(2^k-1)$.

This proposition has been proved in [22] for $d > k$. We prove the proposition for $d = k$ in the end of this section.

Proof of Theorem 1.2. For $k = 3$, the theorem follows from the results of Kameko [14]. For $k = 4$, it follows from the results in [31, 33].

Suppose $k \geqslant 5$ and $d \geqslant k$. Since $\deg(\omega_{(k,d)}) = \deg(\bar{\omega}_{(k,d)}) = (k-1)(2^d-1) = n$ and $(QP_k)_n \cong \bigoplus_{\deg \omega = n} QP_k(\omega)$. Hence, using Propositions 3.2.1 and 3.2.4, we get

$$\begin{aligned}
\dim(QP_k)_n &= \sum_{\deg \omega = n} \dim QP_k(\omega) \\
&\geqslant \dim QP_k(\omega_{(k,d)}) + \dim QP_k(\bar{\omega}_{(k,d)}) \\
&= \left((k-3)\binom{k}{2} + 1\right)(2^k - 1).
\end{aligned}$$

The theorem is proved. □

Note that if $k > 5$, then the sequence $\widetilde{\omega}_{(k,d)} = ((k-1)^{(d-2)}, k-3, k-4, 2)$ is weakly decreasing and $\deg \widetilde{\omega}_{(k,d)} = (k-1)(2^d-1)$. By Lemma 2.2.5, $QP_k(\widetilde{\omega}_{(k,d)}) \neq 0$. Hence, the inequality (1.2) is not an equality.

The following is a modification of a results in [33].

Lemma 3.2.5. *Let d be a positive integer and let y_0 be a monomial in $(P_k)_{k-2}$, $y_j = y_0 x_j$ for $1 \leqslant j \leqslant k$, and $(i;I) \in \mathcal{N}_k$.*

i) *If $r = \ell(I) < d-1$, then*

$$\phi_{(i;I)}(X^{2^{d-1}-1}) y_i^{2^{d-1}} \equiv_{\bar{\omega}_{(k,d)}} \sum_{1 \leqslant j < i} \phi_{(j;I)}(X^{2^{d-1}-1}) y_j^{2^{d-1}} + \sum_{i < j \leqslant k} \phi_{(t_j;I^{(j)})}(X^{2^{d-1}-1}) y_j^{2^{d-1}},$$

where $t_j = \min(j, I)$, and $I^{(j)} = (I \cup j) \setminus \{t_j\}$ for $j > i$.

ii) *If $r < d-2$, then*

$$\phi_{(i;I)}(X^{2^{d-1}-1}) y_i^{2^{d-1}} \equiv_{\bar{\omega}_{(k,d)}} \sum_{1 \leqslant j < i} \phi_{(j;I \cup i)}(X^{2^{d-1}-1}) y_j^{2^{d-1}} + \sum_{i < j \leqslant k} \phi_{(i;I \cup j)}(X^{2^{d-1}-1}) y_j^{2^{d-1}}.$$

Proof. Applying the Cartan formula, we have

$$Sq^1(X_\emptyset^{2^c-1}y_0^{2^c}) = \sum_{1\leqslant j\leqslant k} X_j^{2^c-1}y_j^{2^c}, \tag{3.1}$$

where c is a positive integer. If $r=0$, then $t_j=j$ and $I^{(j)}=\emptyset$ for $j>i$. Then, the the first part of the lemma follows from the relation (3.1) with $c=d$.

If $0<r<d-1$, then $\phi_{(i;I)}(X^{2^{d-1}-1})y_i^{2^{d-1}} = \phi_{(i_1;I\setminus i_1)}(X^{2^r-1})(X_i^{2^c-1}y_i^{2^c})^{2^r}$, with $c=d-r-1>0$ and $i_1=\min I$. It is easy to see that

$$\phi_{(i_1;I\setminus i_1)}(X^{2^r-1})(Sq^1(X_\emptyset^{2^c-1}y_0^{2^c}))^{2^r} \equiv_{\bar\omega_{(k,d)}} 0.$$

Hence, using the relation (3.1), we get

$$\begin{aligned}\phi_{(i;I)}(X^{2^d-1})y_i^{2^d} \equiv_{\bar\omega_{(k,d)}} &\sum_{1\leqslant j<i}\phi_{(i_1;I\setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r}\\ &+\sum_{i<j\leqslant k}\phi_{(i_1;I\setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r}.\end{aligned}$$

A simple computation shows $\phi_{(i_1;I\setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} = \phi_{(j;I)}(X^{2^d-1})y_j^{2^d}$, for $j<i$. By computing from Lemma 3.2.2 in [33], we have

$$\phi_{(i_1;I\setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} \equiv_{\bar\omega_{(k,d)}} \phi_{(t_j;I^{(j)})}(X^{2^d-1})y_j^{2^d}, \text{ for } j>i.$$

Hence, the first part of the lemma follows.

If $0<r<d-2$, then $\phi_{(i;I)}(X^{2^{d-1}-1})y_i^{2^{d-1}} = \phi_{(i;I)}(X^{2^{r+1}-1})(X_i^{2^c-1}y_i^{2^c})^{2^{r+1}}$, with $c=d-r-2>0$. Hence, by a direct computation from the relation (3.1), we get

$$\begin{aligned}\phi_{(i;I)}(X^{2^{d-1}-1})y_i^{2^{d-1}} \equiv_{\bar\omega_{(k,d)}} &\sum_{1\leqslant j<i}\phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}}\\ &+\sum_{i<j\leqslant k}\phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}}.\end{aligned}$$

We have $\phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} = \phi_{(j;I\cup i)}(X^{2^{d-1}-1})y_j^{2^{d-1}}$, for $j<i$. Applying Lemma 3.2.2 in [33], we obtain

$$\phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} \equiv_{\bar\omega_{(k,d)}} \phi_{(i;I\cup j)}(X^{2^{d-1}-1})y_j^{2^{d-1}} \quad \text{for } j>i.$$

So, the second part of the lemma is proved. $\square$

Denote by $I_t=(t+1,t+2,\ldots,k)$ for $1\leqslant t\leqslant k$. Set

$$Y_{(t,k,d)} = \sum_{u=t}^{k}\phi_{(t;I_t)}(X^{2^{d-1}-1})x_u^{2^{d-1}},\ d\geqslant k-t+2.$$

Lemma 3.2.6. *Let $y_0 \in (P_k)_{k-2}$, $1 \leqslant t \leqslant k$, and $y_j = y_0x_j$, $1 \leqslant j \leqslant k$. Then,*

$$Y_{(t,k,d)}y_0^{2^{d-1}} \equiv_{\bar\omega_{(k,d)}} \sum_{(j;J)} \phi_{(j;J)}(X^{2^{d-1}-1})y_j^{2^{d-1}},$$

where the sum runs over all $(j;J) \in \mathcal{N}_k$ with $1 \leqslant j < t$, $J \subset I_{t-1}$, $J \neq I_{t-1}$.

Proof. By Lemma 3.6 in [33], we have

$$Y_{(t,k,d)} = \sum_{(j;J)} \phi_{(j;J)}(X^{2^{d-1}-1})x_j^{2^{d-1}} + \sum_{u=0}^{k-t} Sq^{2^u}(g_u) \mod(P_k^-(\omega)), \tag{3.2}$$

where the sum runs over all $(j;J) \in \mathcal{N}_k$ with $1 \leqslant j < t$, $J \subset I_{t-1}$, $J \neq I_{t-1}$, $\omega = ((k-1)^{(d-1)},1)$ and g_u are suitable polynomials in P_k.

Observe that if y is a monomial in $P_k^-(\omega)$, then there is an index i, $1 \leqslant i \leqslant d-1$ such that $\omega_i(y) < k-1$ (see the proof of Proposition 2.5 in [33]), hence $yy_0^{2^{d-1}} \in P_k^-(\bar\omega_{(k,d)})$. Since $k-t < d-1$, $Sq^{2^u}(g_u)y_0^{2^{d-1}} = Sq^{2^u}(g_uy_0^{2^{d-1}})$ for all $0 \leqslant u \leqslant k-t$.

The lemma now follows from the above equalities and the relation (3.2). $\square$

We set

$$D_{(k,d)}^{(j)} = \{X^{2^{d-1}-1}z^{2^{d-1}} : z \in C_k, \nu_i(z) = 0,\ i < j,\ \nu_j(z) > 0\},\ 1 \leqslant j \leqslant 3.$$

It is easy to see that $D_{(k,d)} = D_{(k,d)}^{(1)} \cup D_{(k,d)}^{(2)} \cup D_{(k,d)}^{(3)}$.

The following is a modification of Lemma 3.7 in [33].

Lemma 3.2.7. *Let d be a positive integer, let $y \in D_{(k,d)}$ and let $(i;I),(j;J) \in \mathcal{N}_k$ with $\ell(J) \leqslant \ell(I)$.*

i) *If either $d > k$ or $d = k$ and $I \neq I_1$, then*

$$p_{(j;J)}(\phi_{(i;I)}(y)) \equiv_{\bar\omega_{(k,d)}} \begin{cases} y, & \text{if } (j;J) = (i;I),\\ 0, & \text{if } (j;J) \neq (i;I).\end{cases}$$

ii) *If $y \in D_{(k,d)}^{(1)}$ and $d = k$, then*

$$p_{(i;I)}(\phi_{(1;I_1)}(y)) \equiv_{\bar\omega_{(k,d)}} \begin{cases} y, & \text{if } (i;I) = (1;I_1),\\ 0 \bmod\langle D_{(k,d)}^{(2)} \cup D_{(k,d)}^{(3)}\rangle, & \text{if } (i;I) = (2;I_2),\\ 0, & \text{otherwise}\ .\end{cases}$$

iii) *If $y \in D_{(k,d)}^{(2)}$ and $d = k$, then*

$$p_{(i;I)}(\phi_{(1;I_1)}(y)) \equiv_{\bar\omega_{(k,d)}} \begin{cases} y, & \text{if } (i;I) = (1;I_1),(1;I_2),(2;I_2),\\ 0 \bmod\langle D_{(k,d)}^{(3)}\rangle, & \text{if } (i;I) = (3;I_3),\\ 0, & \text{otherwise}\ .\end{cases}$$

iv) *If* $y \in D^{(3)}_{(k,d)}$ *and* $d = k$, *then*

$$p_{(i;I)}(\phi_{(1;I_1)}(y)) \equiv_{\bar\omega_{(k,d)}} \begin{cases} y & \text{if } I_3 \subset I,\\ 0, & \text{otherwise.}\end{cases}$$

Proof of Proposition 3.2.4. Set $D^*_{(k,d)} = D_{(k,d)} \cup \{X^{2^d-1}\}$. Denote by $\mathcal{P}_{(k,d)}$ the subspace of P_k spanned by the set

$$\big\{\phi_{(i;I)}(y) : y \in D^*_{(k,d)},\ (i;I) \in \mathcal{N}_k\big\}.$$

Let x be a monomial of degree $n = (k-1)(2^d-1)$ in P_k and $[x]_{\bar\omega_{(k,d)}} \neq 0$. Then, we have $\omega_i(x) = k-1$ for $1 \leqslant i \leqslant d-1$. Hence, we obtain $x = \Big(\prod_{0\leqslant t<d-1} X_{j_t}^{2^t}\Big)z^{2^{d-1}}$, for suitable monomial $z \in (P_k)_{k-1}$.

By a simple computation using Lemmas 3.1.3, we see that there is $(i;I) \in \mathcal{N}_k$ such that

$$x = \Big(\prod_{0\leqslant t<d-1} X_{j_t}^{2^t}\Big)z^{2^{d-1}} \equiv_{\bar\omega_{(k,d)}} \phi_{(i;I)}(X^{2^{d-1}-1})z^{2^{d-1}},$$

where $r = \ell(I) < d-1$.

We need to prove $[x]_{\bar\omega_{(k,d)}} \in [\mathcal{P}_{(k,d)}]_{\bar\omega_{(k,d)}}$. The proof of this fact is based on Lemmas 3.2.5 and 3.2.6. It is divided into many cases, which are similar to the ones of Proposition 3.3 in [33]. However, the relation $\equiv_{\bar\omega_{(k,d)}}$ is used in the proof instead of $\equiv$.

Suppose $x \in P_k(\bar\omega_{(k,d)})$. Since $[x]_{\bar\omega_{(k,d)}} \in [\mathcal{P}_{(k,d)}]_{\bar\omega_{(k,d)}}$, we have

$$x \equiv_{\bar\omega_{(k,d)}} \sum_{(y,(i;I))\in D_{(k,d)}\times\mathcal{N}_k} \gamma_{(y,(i;I))}\phi_{(i;I)}(y) + \sum_{(i;I)\in\mathcal{N}_k} \delta_{(i;I)}\phi_{(i;I)}(X^{2^d-1}),$$

where $\gamma_{(y,(i;I))}, \delta_{(i;I)} \in \mathbb{F}_2$. Since $[x]_{\omega_{(k,d)}} = [\phi_{(i;I)}(y)]_{\omega_{(k,d)}} = 0$ for all $y \in D_{(k,d)}$, from the last equality, one gets

$$\sum_{(i;I)\in\mathcal{N}_k} \delta_{(i;I)}\phi_{(i;I)}(X^{2^d-1}) \equiv_{\omega_{(k,d)}} 0.$$

Now using Proposition 3.2.1, we obtain $\delta_{(i;I)} = 0$ for all $(i;I) \in \mathcal{N}_k$. So, the space $QP_k(\bar\omega_{(k,d)})$ is spanned by the set $\bar B(d)$.

To prove the set $\bar B(d)$ is linearly independent in $QP_k(\bar\omega_{(k,d)})$, we assume that there is a linear relation

$$\mathcal{S} = \sum_{y\in D_{(k,d)}} \gamma_y\phi_{(i;I)}(y) \equiv_{\bar\omega_{(k,d)}} 0,$$

where $\gamma_y \in \mathbb{F}_2$ for all $y \in D_{(k,d)}$. By Lemma 3.1.5, $p_{(j;J)}(\mathcal{S}) \equiv_{\bar\omega_{(k,d)}} 0$ for all $(j;J) \in \mathcal{N}_k$. Based on Lemma 3.2.7, we explicitly compute $p_{(j;J)}(\mathcal{S})$ in terms of the monomials in $D_{k,d}$. Using Corollary 3.2.3 and the relations $p_{(j;J)}(\mathcal{S}) \equiv_{\bar{\omega}_{(k,d)}} 0$ for all $(j;J) \in \mathcal{N}_k$, we will obtain $\gamma_y = 0$ for all $y \in D_{(k,d)}$.

Note that the argument in this proof is similar to the one as given in the proof of Proposition 3.3 in [33]. We refer the reader to [33] for the similar details of the proof. $\square$

4 An application to the fifth Singer algebraic transfer

In this section, we prove Theorem 1.3 by a direct computation. Note that the computations are very complicated, so present here some main results. We refer the readers to the online version [36] for intermediate calculations.

From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k,

$$B_k^0(n) = B_k(n) \cap P_k^0,\ B_k^+(n) = B_k(n) \cap P_k^+.$$

For a weight vector ω of degree n, we set

$$B_k(\omega) = B_k(n) \cap P_k(\omega),\ B_k^0(\omega) = B_k^0(n) \cap P_k(\omega),\ B_k^+(\omega) = B_k^+(n) \cap P_k(\omega).$$

Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the basses of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

For any monomials $z, z_1, z_2, \ldots, z_m$ in $P_k(\omega)$ with $m \geqslant 1$, we denote

$$\begin{aligned}
\Sigma_k(z_1, z_2, \ldots, z_m) &= \{\sigma z_t : \sigma \in \Sigma_k, 1 \leqslant t \leqslant m\} \subset P_k(\omega),\\
\langle[\Sigma_k(z_1, z_2, \ldots, z_m)]_\omega\rangle &= \mathrm{Span}([\Sigma_k(z_1, z_2, \ldots, z_m)]_\omega),\\
[B(z_1, z_2, \ldots, z_m)]_\omega &= [B_k(\omega)]_\omega \cap \langle[\Sigma_k(z_1, z_2, \ldots, z_m)]_\omega\rangle,\\
p(z) &= \sum_{y \in B_k(n) \cap \Sigma_k(z)} y.
\end{aligned}$$

Obviously, $\langle[\Sigma_k(z_1, z_2, \ldots, z_m)]_\omega\rangle$ is an Σ_k-submodule of $QP_k(\omega)$. By Theorem 2.2.6, if ω is the weight vector of the minimal spike of degree n, then $[B]_\omega = [B]$. So, we write $[B(z_1, z_2, \ldots, z_m)]_\omega = [B(z_1, z_2, \ldots, z_m)]$.

4.1 Computation of $QP_k(\omega_{(k,d)})^{GL_k}$

Denote $u_{d,j} = \phi_{(1;I_{k+1-j})}(X^{2^d-1}),\ j \leqslant \min\{k, d\}$. In this subsection we prove the following.

Proposition 4.1.1. *Let d be a positive integer. Then*

$$QP_k(\omega_{(k,d)})^{GL_k} = \begin{cases} 0, & \text{if } d < k,\\ \langle[q_d]_{\omega_{(k,d)}}\rangle, & \text{if } d \geqslant k.\end{cases}$$

Here $q_d = \sum_{(i;I)\in\mathcal{N}_k}\phi_{(i,I)}(X^{2^d-1}) = \sum_{j=1}^k p(u_{d,j})$.

From the results in Section 3, we have the direct summand decomposition of the Σ_k-modules:

$$QP_k(\omega_{(k,d)}) = \bigoplus_{j=1}^{\min\{k,d\}} \langle[\Sigma_k(u_{d,j})]_{\omega_{(k,d)}}\rangle.$$

By a direct computation, we easily obtain the following.

Lemma 4.1.2. *For any* $j \leqslant \min\{k,d\}$, $\langle[\Sigma_k(u_{d,j})]_{\omega_{(5,d)}}\rangle^{\Sigma_k} = \langle[p(u_{d,j})]_{\omega_{(k,d)}}\rangle$.

Proof of Proposition 4.1.1. Let a polynomial $f \in P_k(\omega_{(k,d)})$ such that $[f]_{\omega_{(k,d)}} \in QP_5(\omega_{(k,d)})^{GL_k}$. Then, $[f]_{\omega_{(h,d)}} \in QP_k(\omega_{(k,d)})^{\Sigma_k}$. Using Lemma 4.1.2, we have

$$f \equiv_{\omega_{(k,d)}} \sum_{j=1}^{\min\{k,d\}} \gamma_j p(u_{d,j}),$$

with $\gamma_j \in \mathbb{F}_2$. By computing $g_k(f) + f$ in terms of the admissible monomials and using Lemma 2.1.4 we see that if $d < k$, then

$$\begin{aligned} g_k(f) + f \equiv_{\omega_{(k,d)}} &\gamma_d\phi_{(1;I_{k+1-d})}(X^{2^d-1}) \\ &+ \sum_{j<d}(\gamma_j + \gamma_{j+1})\phi_{(1;I_{k+1-j})}(X^{2^d-1}) + \text{ other terms } \equiv_{\omega_{(k,d)}} 0. \end{aligned}$$

The last equality implies $\gamma_j = 0$ for $1 \leqslant j \leqslant d$. If $d \geqslant k$, then

$$g_k(f) + f \equiv_{\omega_{(k,d)}} \sum_{1\leqslant t<k}(\gamma_t + \gamma_{t+1})\left(\sum_{I\subset I_2,\ell(I)=t}\phi_{(1;I)}(X^{2^d-1})\right) \equiv_{\omega_{(k,d)}} 0.$$

From this we obtain $\gamma_j = \gamma_1$ for $2 \leqslant j \leqslant k$. The proposition follows. $\square$

4.2 Computation of $(QP_5(\bar\omega_{(5,d)})^{GL_5}$

Note that for any $d > 0$, we have the direct summand decomposition of the Σ_5-modules:

$$QP_5(\bar\omega_{(5,d)}) = QP_5^0(\bar\omega_{(5,d)}) \bigoplus QP_5^+(\bar\omega_{(5,d)}).$$

From the results in [23], $QP_5^0(\bar\omega_{(5,d)})$ has a basis $[B_5(\bar u_{d,1})] \cup [B_5(\bar u_{d,2})]$, where

$$\bar u_{d,1} = x_1^{2^{d-1}-1}x_2^{2^{d-1}-1}x_3^{2^d-1}x_4^{2^{d+1}-1},\ \bar u_{d,2} = x_1^{2^{d-1}-1}x_2^{2^d-1}x_3^{2^d-1}x_4^{2^d+2^{d-1}-1}.$$

Using the results in [23], we see that $\dim\langle[\Sigma_5(\bar u_{d,1})]\rangle = 60, \langle[\Sigma_5(\bar u_{d,2})]\rangle = 40$ and there is a direct summand decompositions of the Σ_5-modules:

$$QP_5^0(\bar\omega_{(5,d)}) = \langle[\Sigma_5(\bar u_{d,1})]\rangle \bigoplus \langle[\Sigma_5(\bar u_{d,2})]\rangle.$$

By a direct computation, we get

Lemma 4.2.1. $\langle[\Sigma_5(\bar u_{d,1})]\rangle^{\Sigma_5} = \langle[p(\bar u_{d,1})]\rangle$, $\langle[\Sigma_5(\bar u_{d,2})]\rangle^{\Sigma_5} = 0$.

For $d = 1$, $QP_5(\bar\omega_{(5,1)}) = QP_5^0(\bar\omega_{(5,1)})$. So, one gets the following.

Corollary 4.2.2. $QP_5(\bar\omega_{(5,1)})^{\Sigma_5} = \langle[p(u_{1,1}]\rangle$.

For $d \geqslant 3$, we set

$$\bar u_{d,3} = x_1x_2^{2^{d-1}-2}x_3^{2^{d-1}-1}x_4^{2^d-1}x_5^{2^{d+1}-1},\ \bar u_{d,4} = x_1x_2^{2^{d-1}-2}x_3^{2^d-1}x_4^{2^d-1}x_5^{2^d+2^{d-1}-1}.$$

B computing from the results in [23], we have

$$\dim\langle[\Sigma_5(\bar u_{d,3})]\rangle = 60,\ \dim\langle[\Sigma_5(\bar u_{d,4})]\rangle = 20.$$

Lemma 4.2.3. *For any $d \geqslant 3$, $\langle[\Sigma_5(\bar u_{d,3})]\rangle^{\Sigma_5} = \langle[p(\bar u_{d,3})]\rangle$, $\langle[\Sigma_5(\bar u_{d,4})]\rangle^{\Sigma_5} = 0$.*

For $d \geqslant 4$, we set

$$\bar u_{d,5} = x_1^3x_2^{2^{d-1}-3}x_3^{2^{d-1}-2}x_4^{2^d-1}x_5^{2^{d+1}-1},\ \bar u_{d,6} = x_1x_2^{2^{d-1}-1}x_3^{2^{d-1}-1}x_4^{2^d-2}x_5^{2^{d+1}-1},$$
$$\bar u_{d,7} = x_1^3x_2^{2^{d-1}-3}x_3^{2^{d-1}-1}x_4^{2^d-2}x_5^{2^{d+1}-1},\ \bar u_{d,8} = x_1^7x_2^{2^{d-1}-5}x_3^{2^{d-1}-3}x_4^{2^d-2}x_5^{2^{d+1}-1}.$$

Lemma 4.2.4. *For any integer $d \geqslant 4$,*

$$\langle[\Sigma_5(\bar u_{d,5})]\rangle^{\Sigma_5} = \langle[p(\bar u_{d,5})]\rangle,\ \langle[\Sigma_5(\bar u_{d,6}, \bar u_{d,7}, \bar u_{d,8})]\rangle^{\Sigma_5} = \langle[p_{d,1}]\rangle.$$

Here the polynomial $p_{d,1}$ is determined as in Section 5 of the online version [36].

4.2.1 The case $d = 2$

For $d = 2$, by a direct computation using the results in [23], we have the direct summand decomposition of the Σ_5-modules:

$$QP_5^+(\bar\omega_{(5,2)}) = \langle[\Sigma_5(\bar u_{2,3})]\rangle \bigoplus \langle[\Sigma_5(u_{2,4})],$$

where $\bar u_{2,3} = x_1x_2x_3x_4^2x_5^7$, $\bar u_{2,4} = x_1x_2x_3^2x_4^3x_5^5$. By a direct computation, we have

Lemma 4.2.5. $\langle[\Sigma_5(\bar u_{2,3})]\rangle^{\Sigma_5} = 0$ *and* $\langle[\Sigma_5(\bar u_{2,4})]\rangle^{\Sigma_5} = \langle[p_{2,1}]\rangle$, *where the polynomial $p_{2,1}$ is determined as in Section 5 of the online version [36].*

From Lemmas 4.2.1 and 4.2.5, we obtain the following.

Proposition 4.2.6. $QP_5(\bar\omega_{(5,2)})^{\Sigma_5} = \langle[p(u_{2,1}], [p_{2,1}]\rangle$.

4.2.2 The case $d = 3$

For $d = 3$, by a direct computation using the results of [23], we get the direct summand decompositions of the Σ_5-modules:

$$QP_5^+(\bar{\omega}_{(5,3)}) = \bigoplus_{j=3}^{5} \langle [\Sigma_5(\bar{u}_{3,j})] \rangle \bigoplus \langle [\Sigma_5(\bar{u}_{3,6}, \bar{u}_{3,7}, \bar{u}_{3,8}, \bar{u}_{3,9})] \rangle,$$

where

$$\bar{u}_{3,5} = x_1x_2^3x_3^3x_4^6x_5^{15},\ \bar{u}_{3,6} = x_1x_2^3x_3^3x_4^7x_5^{14},\ \bar{u}_{3,7} = x_1^3x_2^3x_3^5x_4^6x_5^{11},$$
$$\bar{u}_{3,8} = x_1x_2^3x_3^6x_4^7x_5^{11},\ \bar{u}_{3,9} = x_1^3x_2^3x_3^5x_4^7x_5^{10}.$$

We have

$$\dim\langle [\Sigma_5(\bar{u}_{3,5})] \rangle = 55, \dim\langle [\Sigma_5(\bar{u}_{3,6}, \bar{u}_{3,7}, \bar{u}_{3,8}, \bar{u}_{3,9})] = 220.$$

By a direct computation, we obtain the following.

Lemma 4.2.7. $\langle [\Sigma_5(\bar{u}_{3,5})] \rangle^{\Sigma_5} = 0, \langle [\Sigma_5(\bar{u}_{3,6}, \bar{u}_{3,7}, \bar{u}_{3,8}, \bar{u}_{3,9})] \rangle^{\Sigma_5} = \langle [p_{3,1}], [p_{3,2}] \rangle$, *where the polynomials $p_{3,1}$ and $p_{3,2}$ are determines as in Section 5 of the online version [36].*

Combining this lemma and Lemmas 4.2.1, 4.2.3, we get the following result.

Proposition 4.2.8. $QP_5(\bar{\omega}_{(5,3)})^{\Sigma_5} = \langle [p(\bar{u}_{3,1})], [p(\bar{u}_{3,3})], [p_{3,1}], [p_{3,2}] \rangle$.

4.2.3 The case $d = 4$

By a direct computation using the results in [23], we have the direct summand decomposition of the Σ_5-modules:

$$QP_5^+(\bar{\omega}_{(5,4)}) = \bigoplus_{j=3}^{5} \langle [\Sigma_5(\bar{u}_{4,j})] \rangle \bigoplus \langle [\Sigma_5(\bar{u}_{4,6}, \bar{u}_{4,7}, \bar{u}_{4,8})] \rangle \bigoplus \langle [\Sigma_5(\bar{u}_{4,9}, \ldots, \bar{u}_{4,15})] \rangle,$$

where

$$\bar{u}_{4,9} = x_1x_2^7x_3^7x_4^{15}x_5^{30},\ \bar{u}_{4,10} = x_1x_2^7x_3^{14}x_4^{15}x_5^{23},\ \bar{u}_{4,11} = x_1^3x_2^5x_3^7x_4^{15}x_5^{30},$$
$$\bar{u}_{4,12} = x_1^3x_2^5x_3^{14}x_4^{15}x_5^{23},\ \bar{u}_{4,13} = x_1^3x_2^7x_3^7x_4^{13}x_5^{30},\ \bar{u}_{4,14} = x_1^3x_2^7x_3^{13}x_4^{15}x_5^{22},$$
$$\bar{u}_{4,15} = x_1^7x_2^7x_3^{11}x_4^{13}x_5^{22}.$$

Furthermore, we have $\dim\langle [\Sigma_5(\bar{u}_{4,9}, \ldots, \bar{u}_{4,15})] \rangle = 335$.

Lemma 4.2.9. $\langle[\Sigma_5(\bar{u}_{4,9},\ldots,\bar{u}_{4,15})]\rangle^{\Sigma_5} = \langle[p_{4,2}],[p_{4,3}],[p_{4,4}]\rangle$, *where the polynomials* $p_{4,t}, t = 2,3,4$, *are defined as in Section 5 of the online version [36].*

The proof of this lemma is straightforward.

Combining this lemma and Lemmas 4.2.1-4.2.4, we obtain the following.

Proposition 4.2.10. *We have*

$$QP_5(\bar{\omega}_{(5,4)})^{\Sigma_5} = \langle[p(\bar{u}_{4,1})],[p(u_{4,3})],[p(u_{4,5})],[p_{4,1}],[p_{4,2}],[p_{4,3}],[p_{4,4}]\rangle.$$

4.2.4 The case $d \geqslant 5$

For $d \geqslant 5$, by a direct computation using the results in [23], we have the direct summand decomposition of the Σ_5-modules:

$$QP_5^+(\bar{\omega}_{(5,d)}) = \bigoplus_{j=3}^{5}\langle[\Sigma_5(\bar{u}_{d,j})]\rangle \bigoplus\langle[\Sigma_5(\bar{u}_{d,6},\bar{u}_{d,7},\bar{u}_{d,8})]\rangle \bigoplus\langle[\Sigma_5(\bar{u}_{d,9},\ldots,\bar{u}_{4,20})]\rangle,$$

where the monomials $\bar{u}_{d,t}, 9 \leqslant t \leqslant 20$ are determined as in Section 4 of the online version [36].

We also have

$$\dim\langle[\Sigma_5(\bar{u}_{d,9},\ldots,\bar{u}_{d,20})]\rangle = 335.$$

Lemma 4.2.11. $\langle[\Sigma_5(\bar{u}_{d,9},\ldots,\bar{u}_{d,20})]\rangle^{\Sigma_5} = \langle[p_{d,2}],[p_{d,3}],[p_{d,4}],[p_{d,5}]\rangle$, *where the polynomials* $p_{d,j}, j = 2,3,4,5$, *are determined as in Section 5 of the online version [36].*

The proof of this lemma is very complicated. It is proved by a direct computation.

Combining this result and Lemmas 4.2.1-4.2.4, we have the following.

Proposition 4.2.12. *For any* $d \geqslant 5$, *we have*

$$QP_5(\bar{\omega}_{(5,d)})^{\Sigma_5} = \langle[p(\bar{u}_{d,1})],[p(u_{d,3})],[p(u_{d,5})],[p_{d,1}],[p_{d,2}],[p_{d,3}],[p_{d,4}],[p_{d,5}]\rangle.$$

4.3 Proof of Theorem 1.3

Let $f \in (P_5)_{4(2^d-1)}$ such that $[f] \in (QP_5)^{GL_5}_{4(2^d-1)}$. Then, we have $[f] \in QP_5(\omega_{(5,d)})^{GL_5}$.

The case $d < 5$. By Proposition 4.1.1, $[f]_{\omega_{(5,d)}} = 0$, hence $[f] \in QP_5(\bar{\omega}_{(5,d)})^{GL_5}$.

For $d = 1$, using Corollary 4.2.2, we have $f \equiv \gamma p(\bar{u}_{1,1})$ with $\gamma \in \mathbb{F}_2$. By computing $g_5(f) + f$ in terms of the admissible monomials, we obtain

$$g_5(f) + f \equiv \gamma x_1x_2^3 + \text{ other terms} \equiv 0.$$

This relation implies $\gamma = 0$. The theorem is true for $d = 1$.

For $d = 2$, from Proposition 4.2.6, we have $f \equiv \gamma p(\bar u_{2,1}) + \delta p_{2,1}$ with $\gamma,\ \delta \in \mathbb{F}_2$. By computing $g_5(f) + f$ in terms of the admissible monomials, we get

$$g_5(f) + f \equiv \gamma x_2x_3x_4^3x_5^7 + (\gamma+\delta)x_1x_2^3x_3x_4^2x_5^5 + \text{other terms} \equiv 0.$$

This relation implies $\gamma = \delta = 0$. The theorem is proved for $d = 2$.

For $d = 3$, using Proposition 4.2.8, we have $f \equiv \gamma_1 p(\bar u_{3,1}) + \gamma_2 p(\bar u_{3,3}) + \gamma_3 p_{3,1} + \gamma_4 p_{3,2}$ with $\gamma_j \in \mathbb{F}_2, j = 1,2,3,4$. A direct computation shows

$$\begin{aligned} g_5(f) + f \equiv\ &\gamma_1 x_2^7x_3^3x_4^3x_5^{15} + \gamma_2 x_1x_2^3x_3^2x_4^7x_5^{15} + \gamma_3 x_1x_2^3x_3^3x_4^7x_5^{14}\\ &+ (\gamma_1+\gamma_2+\gamma_3+\gamma_4)x_1x_2^7x_3^3x_4^3x_5^{14} + \text{ other terms} \equiv 0. \end{aligned}$$

This relation implies $\gamma_j = 0,\ j = 1,2,3,4$. The theorem holds for $d = 3$.

For $d = 4$, Proposition 4.2.10 implies that $f \equiv \gamma_1 p(\bar u_{4,1}) + \gamma_2 p(\bar u_{4,3}) + \gamma_3 p(\bar u_{4,5}) + \sum_{j=1}^4 \gamma_{3+j} p_{4,j}$ with $\gamma_j \in \mathbb{F}_2, j = 1,2,\ldots,7$. By computing $g_5(f)+f$ in terms of the admissible monomials, we get

$$\begin{aligned} g_5(f) + f \equiv\ &\gamma_1 x_1^{15}x_2^{31}x_3^7x_4^7 + \gamma_2 x_1x_2^7x_3^7x_4^{15}x_5^{30} + \gamma_3 x_1^3x_2^{15}x_3^{13}x_4^7x_5^{22}\\ &+ \gamma_4 x_1^3x_2^7x_3^5x_4^{14}x_5^{31} + (\gamma_5+\gamma_6)x_2^{15}x_3^7x_4^{15}x_5^{23} + \gamma_6 x_1^7x_2^{15}x_3x_4^{23}x_5^{14}\\ &+ \gamma_7 x_1x_2^{15}x_3^7x_4^{15}x_5^{22} + \text{ other terms } \equiv 0. \end{aligned}$$

From this relation we obtain $\gamma_j = 0,\ j = 1,2,\ldots,7$. The theorem is proved for $d = 4$.

The case $d \geqslant 5$. According to Proposition 4.1.1, $[f]_{\omega_{(5,d)}} = \gamma[q_d]_{\omega_{(5,d)}}$, with $\gamma \in \mathbb{F}_2$. Hence, $f = \gamma q_d + h$, with $h \in QP_5(\bar\omega_{(5,d)})$. By a direct computation from the relations $g_i(f) + f \equiv 0,\ i = 1,2,3$, we get $\gamma = 0$. So, $[f] = [h] \in QP_5(\bar\omega_{(5,d)})^{GL_5}$. Using Proposition 4.2.12, we obtain

$$f \equiv \gamma_1 p(\bar u_{d,1}) + \gamma_2 p(\bar u_{d,3}) + \gamma_3 p(\bar u_{d,5}) + \sum_{j=1}^5 \gamma_{3+j} p_{d,j},$$

where $\gamma_j \in \mathbb{F}_2, j = 1,2,\ldots,8$. By a direct computation, we have

$$\begin{aligned} &g_5(f) + f \equiv \gamma_1 x_1^{2^{d-1}-1}x_2^{2^d-1}x_3^{2^{d-1}-1}x_4^{2^{d+1}-1}\\ &+ \gamma_2 x_1^3x_2^{2^d-1}x_3^{2^{d-1}-3}x_4^{2^{d-1}-1}x_5^{2^{d+1}-2} + \gamma_3 x_1^7x_2^{2^d-1}x_3^{2^{d-1}-5}x_4^{2^{d+1}-3}x_5^{2^{d-1}-2}\\ &+ \gamma_4 x_1^3x_2^{2^d-1}x_3^{2^{d-1}-1}x_4^{2^d-3}x_5^{2^{d+1}-2^{d-1}-2} + \gamma_5 x_1^3x_2^{2^{d+1}-1}x_3^{2^{d-1}-3}x_4^{2^{d-1}-1}x_5^{2^d-2}\\ &+ \gamma_6 x_1x_2^{2^{d-1}-1}x_3^{2^d-2}x_4^{2^d-1}x_5^{2^{d+1}-2^{d-1}-1} + \gamma_7 x_1^7x_2^{2^d-1}x_3^{2^{d-1}-5}x_4^{2^{d+1}-2^{d-1}-3}x_5^{2^d-2}\\ &+ (\gamma_1+\gamma_3+\gamma_7+\gamma_8)x_1^3x_2^{2^{d-1}-1}x_3^{2^d-3}x_4^{2^d-2}x_5^{2^{d+1}-2^{d-1}-1} + \text{ other terms } \equiv 0. \end{aligned}$$

The last equality implies that $\gamma_j = 0,\ j = 1,2,\ldots,8$. The theorem is completely proved.

Acknowledgment

The author was supported in part by the National Foundation for Science and Technology Development (NAFOSTED) of Viet Nam under the grant number 101.04-2017.05.

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