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CENTER OF NOETHERIAN RINGS

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Abstract

For a right Noetherian ring A with the center $R = Z(A)$, and a finitely generated right A-module M, we show:

(1) $P \in Ass(M)$ implies that $P \cap R \in Supp(M)$.

(2) $P \in Min.Supp(M)$ implies that there exists $Q \in Ass(M)$ such that $Q \cap R = P$.

This result has several applications in determining the nilradical of the center of a Noetherian ring. We also give a conceptually simple proof of the fact that the center of an Artinian ring is semiprimary. Some other related results are obtained for irreducible rings.

1 INTRODUCTION

A ring R means always a ring with identity. For a Noetherian ring R, $N(R)$ denotes the nil radical or the prime radical of R. For an ideal I of R, $C(I)$ denotes the set of elements of R that are regular modulo I. Let M be a right R-module and S be a subset of M, then the right annihilator of S in R is denoted by $Ann(S)$. Let B be a subset of a set A, then B^c denotes the complement of B in A. For any ring A we denote by $Z(A)$ and $J(A)$ the center and the Jacobson radical of A, respectively.

It is well known that the center of a right Artinian ring R is usually not Artinian, however considering the center $Z(R)$ as the endomorphism ring of $(R \otimes R^{op})$-module, R of finite length; it follows that $Z(R)$ is necessarily semiprimary ;i.e. the Jacobson radical $J(R)$ is nilpotent and $R/J(R)$ is Artinian. For these results see [1]. In this paper we determine the center of a right Noetherian

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ring. We first define $Ass(M)$ for a right R-module. We use central localisation to prove Theorem 2.12 which is crucial for determining the minimal prime ideals of the center of a right Noetherian ring. Let A be a right Noetherian ring with R a subring of the center of A with same identity as in A. We then show that for any minimal prime ideal P of R there exists a minimal prime ideal Q of A such that $P = Q \cap R$. We also show that the number of minimal prime ideals of R is finite. This yields a result that $N(A) \cap R = N(R)$. Such results are general results for a right Noetherian ring A. In fact we easily show that if A is an Artinian ring; then the center of A is a semiprimary ring. This proof is conceptually clear and simpler than that given by Jensen and Jondrup in [1]. We finally prove that if P is a minimal prime ideal of a right Noetherian and irreducible ring A, then $P \cap R$ is a minimal prime ideal of R where R is a subring of the center of A.

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We begin with the following Proposition.

Proposition 2.1 *Let A be a right Noetherian ring and M be a finitely generated right A-module, and let R be a subring of $Z(A)$ that contains the identity of A. Then the set $S = \{$Ideals I of A such that $I = Ann(U)$, $0 \neq U \subseteq M$ a submodule$\}$ contains a maximal element P which is a prime ideal of A. Moreover, if $P = Ann(V)$, $0 \neq V \subseteq M$, then $P = Ann(K)$, for all $0 \neq K \subseteq M$.*

Proof Choose $P \in S$ maximal by Noetherian condition on A. Now $P = Ann(V)$ for some $0 \neq V \subseteq M$. Let $IJ \subseteq P$ where I and J are ideals of A. If $I \not\subseteq P$ then $VI \neq 0$. Now $IJ \subseteq P = Ann(V)$ implies that $VIJ = 0$, so $J \subseteq Ann(VI)$. Let $T = Ann(VI)$ so that $J \subseteq T$. Now $T \in S$ and since $P = Ann(V)$, $P \subseteq T$. But P is maximal in S, so $T = P$ and so $J \subseteq P$. So P is a prime ideal of A. Other things are obvious. $\square$

Definition 2.2 *Let V be an A-module. $V \neq 0$ such that $Ann(V) = P$ and $Ann(T) = P$ for all $0 \neq T \subseteq V$, then we call V a prime module.*

Definition 2.3 *Let M be an A-module and $0 \neq V \subseteq M$ be a prime module with $Ann(V) = P$, we say P is the assasinator of M. The set of all assasinators of M is denoted by $Ass(M)$.*

Remark 2.4 Proposition 2.1 says that for a Noetherian module M over a right Noetherian ring A, $Ass(M) \neq \phi$. Also if V is a prime module, we can choose V to be cyclic such that $V \approx (A/I)$, I a right ideal of A and $P = Ann(A/I)$.

Definition 2.5 *A right ideal I of a ring A is said to be a prime right ideal of A if $xAy \subseteq I$, $x \notin I$ implies that $y \in I$.*

Definition 2.6 *Let I be a right ideal of a ring A. An ideal J is called the bound of I if $J \subseteq I$ and J is the largest ideal contained in I.*

Proposition 2.7 *Let A be a right Noetherian ring. Let M be a cyclic prime module with $Ann(M) = P$. If $M \approx (A/I)$, then P is a bound of I and I is a prime right ideal of A.*

Proof $P = Ann(M)$ implies that $P = Ann(A/I)$, so clearly P is a bound of I. Now let $xAy \subseteq I$, $x \notin I$. Now xA is a right ideal of A and $xA \not\subset I$. Now $(xA+I)/I$ is a right A- submodule of A/I. Also $((xA+I)/I)y = xAy+I = I$, the zero of A/I. So $y \in Ann((xA+I)/I)$. So $y \in P \subseteq I$ and so I is a prime right ideal. □

Let A be a ring and S be a multiplicative closed subset of R, where R is a subring of $Z(A)$ such that $1 \in S$ where 1 is the identity of R with R and S having no zero divisors. We construct M_S for any right A-module M as the equivalence class of pairs (m,s) where $m \in M$ and $s \in S$ and (m, s) (k, q) if there exists $t \in S$ such that $(mq - ks)t = 0$. Denote equivalence class of (m,s) by (m/s). With suitable addition and multiplication M_S becomes a right A-module. If $M = A$, then A_S is a ring and M_S is a right A_S-module with multiplication (m/t)(r/s)=(mr/ts). Let $f : M \rightarrow M_S$ be the map given by f(m)=(m/1). It is easily seen that f is an A-map. If $M = A$, then f is a ring homomorphism. With this we state the following two Propositions which can be proved easily:

Proposition 2.8 *The functor $M \rightarrow M_S$ is exact ;i.e. A_S is a flat A-module.*

Proposition 2.9 *The map $I \rightarrow I_S$ gives a one-one correspondence between the prime right ideals of A with $I \cap S = \phi$ and all the proper prime right ideals of A_S. The inverse map is given by $I \rightarrow f^{-1}(I)$, where f is the natural map $A \rightarrow A_S$.*

Proposition 2.10 *Let A be a right Noetherian ring. Let $R \subseteq Z(A)$ be a subring of $Z(A)$ and $S \subseteq R$ be a multiplicative closed set. If M is a finitely generated right A-module and if $P_S \in Ass(M*_{A*})$, where $M* = M_S$ and $A* = A_S$, then for $P \cap S = \phi$, $f^{-1}(P_S) = P \in Ass(M_A)$, where f is the usual map $A \rightarrow A_S$.*

Proof $P_S \in Ass(M*_{A*})$, so there exists a prime submodule $0 \neq U* \subseteq M*$ such that $Ann(U*) = P_S \in A_S$ and $Ann(V*) = Ann(U*)$ for all $0 \neq V* \subseteq U*$. We can choose $U*$ cyclic and such that $U* = C_S$ for some non zero submodule $0 \neq C \subseteq M$. So by 2.7 and 2.8 $C_S \approx (A_S/I_S)$ and I_S a prime right ideal with bound P_S. Now $Ann(C_S) = P_S$ implies that $Ann(A_S/I_S) = P_S$. Note that $((1/1)+I_S).P_S = 0$ as $P_S \subseteq I_S$. So $(1+I)P.t = 0$, $t \in S$. So $(1+I)t \neq 0$, so $t \notin I$ as $I \cap S = \phi$. Now generate a non zero R-module by $t + I$ and call it J/K. Note that $J/K = (t + I)R \neq 0$ and $(t + I)R = (tR + I)/I$.

Now $P \subseteq Ann((tR+I)/I)$ and since $I \cap S = \phi$ with I a prime right ideal with bound P, we have $P = Ann((tR+I)/I)$ because if $P \subseteq L$ such that $((tR+I)/I)L = 0$,then $tL \subseteq I$. But $t \notin I$, so $L \subseteq I$ which is a contradiction. Similarly we can show that $P \in Ass(M)$ and $P \cap S = \phi$. □

Definition 2.11 *Let $R \subseteq Z(A)$ as usual. Let M be an A-module and $P \subseteq R$ be a prime ideal of R. Let $S = (P \cap R)^c$, the complement of $P \cap R$ in R. Denote localisation of M at S by M_S. Define support$(M) = \{$Prime ideals P of R such that $M_S \neq 0\}$.We denote support(M) by Supp(M). Min.Supp$(M) = \{$Prime ideals P of R such that P is minimal with the property that $M_S \neq 0\}$.*

Theorem 2.12 *Let A be a right Noetherian ring and M a finitely generated A-module, then:*

(1) $P \in Ass(M)$ implies that $P \cap R \in Supp(M)$.

(2) If $P \in Min.Supp(M)$, then there exists $Q \in Ass(M)$ such that $Q \cap R = P$, where R is a subring of $Z(A)$.

Proof **(1)**. Let $P \in Ass(M)$, then $P = Ann(C)$, $0 \neq C \subseteq M$ and C prime cyclic right A-module, so $C \approx (A/I)$, I a prime right ideal with P as bound of I. Now $0 \to (A/I) \to M$ is exact and $0 \to S^{-1}(A/I) \to S^{-1}(M)$ is injection where $S^{-1}(M)$ etc denotes the usual localisation at $S = (P \cap R)^c$. Now $S^{-1}(A/I) \neq 0$ implies that $S^{-1}(M) \neq 0$. So $(P \cap R) \in Supp(M)$.

(2). Let $P \in Min.Supp(M)$. For any prime ideal P of A, $S^{-1}(P) \cap S^{-1}(R) = S^{-1}(P \cap R)$, where $S^{-1}(M)$ denotes the usual localisation at $S = P^c$. Denote $S^{-1}(M)$ and $S^{-1}(P)$ by T and L respectively. Now localize T at L which is unique maximal ideal of $S^{-1}(R)$. Now $T \neq 0$ implies that $TL \neq 0$. Now for any prime ideal B of R with $B \subset P$, we have $T_U = 0$, because $U = S^{-1}(B) \in Min.Supp(T)$. Now by 2.4 $T \neq \phi$. Let D be a prime ideal such that $V = S^{-1}(D) \in Ass(T)$, then $T_F \neq 0$, where $F = S^{-1}(D \cap R)$. Also $V \cap S^{-1}(R) \subseteq L$ because L is the unique maximal ideal of $S^{-1}(R)$. So $F = L$ as $F \in Supp(T)$ by above. Thus $V \cap S^{-1}(R) = L$. It can be easily seen from this that $D \cap R = P$ and $V = S^{-1}(D) \in Ass(T)$ easily yields that $D \in Ass(M)$. □

We now give several applications of our main Theorem. We determine the Nilradical of the center of a Noetherian ring. We also give a conceptually simple proof of the fact that the center of an Artinian ring is semiprimary. Finally we apply the main Theorem to give a precise form of the minimal prime ideals of the center of an irreducible right Noetherian ring.

Proposition 2.13 *Let A be a Noetherian ring and R be a subring of $Z(A)$. Then for any minimal prime ideal Q of R, $Q = P \cap R$ where P is a minimal prime ideal of A.*

Proof Note that $AQ \neq 0$ and $Q \in Min.Supp(A)$. So there exists $P \in Ass(A)$ such that $P \cap R = Q$ by 2.12. Let $U \subseteq P$ be a minimal prime ideal of A, then $U \cap R = Q = P \cap R$, because Q is a minimal prime ideal. □

Proposition 2.14 *Let A be a Noetherian ring and R be a subring of $Z(A)$, then:*

(1) $N(A) \cap R = N(R)$.

(2) The number of minimal prime ideals of R is finite.

Proof (1). We have $N(R) = \cap P_i$, P_i all minimal prime ideals of R. Now each $P_i = Q_i \cap R$ for some minimal prime ideal Q_i of A and since Q_i are finite, so $N(R) = (Q_1 \cap R) \cap (Q_2 \cap R) \cap ... \cap (Q_n \cap R)$. Now $N(A) \cap R \subseteq N(R)$ and since A is right Noetherian, so for A nil radical= prime radical. Let $x \in N(R)$, so x is nilpotent. So that $xA \subseteq N(A)$ implies that $\sum x_i A \subseteq N(A)$ for all $x_i \in N(R)$. So $N(R)A \subseteq N(A)$ and $N(R) \subseteq N(R)A \cap R \subseteq N(A) \cap R$. So $N(R) = N(A) \cap R$.

(2). Obvious by 2.12. □

We now use 2.12 to prove that the centre of an Artinian ring is semiprimary. We also recall that in an Artinian ring all prime ideals are minimal (and indeed maximal). One may see Theorem (2.3.9) and Corollary (3.2.26) of [2].

Proposition 2.15 *Let A be a right Artinian ring and $R = Z(A)$. If Q is a maximal ideal of R, then $Q = P \cap R$ for some minimal prime ideal P of A.*

Proof Let $P_j, 1 \leq j \leq n$ be the minimal prime ideals of A. If $Q \subseteq (\cup(P_j \cap R))$, $1 \leq j \leq n$, then $Q \subseteq P_j \cap R$, for some P_j and so $Q = P_j \cap R$ as Q is maximal. Now suppose $Q \not\subset (\cup(P_j \cap R))$, then there exists $c \in Q$ such that $c \notin (\cup(P_j \cap R))$ and so $c \notin (P_j \cap R)$ for all j. Now let $cx \in N(A)$ so that $cx \in P_j$ for all j, so $cAx \subseteq P_j$. Now $c \in R$, so $c \notin P_j$ implies that $x \in P_j$ for all j, $1 \leq j \leq n$. Hence $x \in N(A)$ and so $c \in C(N(A))$. But A is right Artinian, so $c \in C(0)$. So c is a unit of A. So ct=1, $t \in A$ implies that $t \in Z(A) = R$. But $c \in Q$ which contradicts the fact that Q is a maximal ideal of $Z(A) = R$. So $Q \subseteq (\cup(P_j \cap R))$, $1 \leq j \leq n$. Hence as above $Q = P_j \cap R$ for some j, $1 \leq j \leq n$. □

Theorem 2.16 *Let A be a right Artinian ring and $R = Z(A)$, then $N(R) = J(R)$. Also $J(R)$ is nilpotent and R is semiprimary.*

Proof We have $N(R) = N(A) \cap R$ by 2.14. Now for a minimal prime ideal Q of R there exists a minimal prime ideal P of A such that $Q = P \cap R$ by 2.12. Let M be a maximal ideal of R, then $M = T \cap R$ for some minimal prime ideal T of A. So number of maximal ideals of R is finite, say M_j, $1 \leq j \leq m$ are the maximal ideals of R. So $J(R) = \cap M_j$, $1 \leq j \leq m$. Now given any minimal prime ideal P of A, $P \cap R$ is a prime ideal of R. In fact if S is a maximal ideal of R such that $S \supseteq P \cap R$, then $S = P \cap R$ by 2.15. So $N(R) = J(R) = N(A) \cap R$ and so $J(R)$ is nilpotent. Now by Chinese Remainder Theorem, namely Proposition (2.2.1) of [2] $(R/J(R)) \approx (\prod(R/M_j))$, where M_j, $1 \leq j \leq m$ are distinct maximal ideals of R. So $R/J(R)$ is Artinian and hence R is semiprimary. □

Remark 2.17 Such results are not true for subrings of Artinian rings as it is well known by considering the examples of the ring of integers Z and the field of rational numbers Q. We are unable to prove in 2.13 that if P is a minimal prime ideal of A , then $P \cap Z(A)$ is a minimal prime ideal of $Z(A)$. However one can do this for irreducible rings. Notice that a ring A is called an irreducible ring if the intersection of any two non-zero ideals of A is non-zero. Irreducible rings are important in the sense that if A is a right Noetherian ring, then there exist ideals I_j, $1 \leq j \leq n$ such that $\cap I_j = 0$ and each R/I_j is an irreducible ring. We leave the proof of these obvious results to the reader.

Proposition 2.18 *Let A be a right Noetherian ring which is irreducible and $R \subseteq Z(A)$. Let $c \in R$ be non-nilpotent, then c is regular in A.*

Proof Since c is central, the right and left annihilators of c in A coincide. Call each $Ann_A(c)$. Suppose $Ann_A(c) \neq 0$. Now by ascending chain condition there exists an integer $t \geq 1$ such that $Ann_A(c^t) = Ann_A(c^{t+s})$ for all integers $s \geq 1$. Since c is not nilpotent, $c^t \neq 0$ and by our hypothesis on A, we have $(c^t A) \cap Ann_A(c) \neq 0$. So there is an $a \in A$ such that $0 \neq c^t a \in Ann_A(c)$ which implies that $a \in Ann_A(c^{t+1}) = Ann_A(c^t)$ which is a contradiction as $c^t a \neq 0$. So c is regular in A. □

Theorem 2.19 *Let A be a right Noetherian ring which is irreducible and $R \subseteq Z(A)$. Let P be a minimal prime ideal of A, then $P \cap R$ is a minimal prime ideal of R.*

Proof Note that number of minimal prime ideals of R is finite as any minimal prime ideal Q_j of R is such that $Q_j = P_j \cap R$ for some minimal prime ideal P_j of A. So let Q_j, $1 \leq j \leq m$ be the minimal prime ideals of R. Now let P be a minimal prime ideal of A and suppose $P \cap R \subseteq (\cup Q_j)$, then $P \cap R \subseteq Q_j$ for some j, $1 \leq j \leq m$ and so $P \cap R = Q_j$. So $P \cap R$ is a minimal prime ideal of R. Now suppose $(P \cap R) \not\subseteq (\cup Q_j)$, then there exists $c \in P \cap R$ such that $c \notin (\cup Q_j)$, so $c \notin Q_j$ for all j, $1 \leq j \leq m$. So $c \in C(Q_j)$ for all j, $1 \leq j \leq m$. So $c \in C(N(R)) = C(N(A) \cap R)$. So c is not nilpotent, so by 2.18 c is a regular element of A. So $c \notin P$ which is a contradiction. So $P \cap R \subseteq (\cup Q_j)$ and as above $P \cap R$ is a minimal prime ideal of R. □

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