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DEFORMATIONS OF r-RETICULAR MAP GERMS

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Abstract

The purpose of this paper is to study the theory of germ of $r-$reticular map under the action of the group $\mathrm{Diff}^r(n)$. The main result is to find the relations between versal deformations and infinitesimal deformations of a germ $r-$reticular map.

1. INTRODUCTION AND BASIC CONCEPTS

One of these questions of Catastrophe theory is an investigation of the singularities of map-germ after perturbation. The aim of this paper is to generalize Mather's deformation theory of reticular map-germ. This is a preparation to classify reticular map-germs under an algebraic criterion.

The paper contains two sections. The first section deals with the basis notations, and the second gives some results on the relations between versal deformation and an infinitesimal deformation of a germ of $r-$reticular map in real case.

Let $f, g\colon \mathbb{R}^n \to \mathbb{R}^p$ be C^∞-differentiable maps. We say that f and g define a *germ* at $x_0 \in U \subset \mathbb{R}^n$, if there exists a neighbourhood V of x_0 in U such that $f \equiv g$ on V. We denote by $\varepsilon(n,p)$ the set of germs of differentiable maps at 0 from $\mathbb{R}^n$ to $\mathbb{R}^p$. We write $\varepsilon(n)$ instead of $\varepsilon(n,1)$.

Fix an $r \in \mathbb{N}, 0 \leq r \leq n$, and consider X_i to be a germ of the set $\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i = 0\}$. Denote $P(I_r)$, the family of all subsets of

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the set $I_r = \{1, 2, ..., r\}$.

The collection $\underline{X} = (X_\sigma)_{\sigma \in P(I_r)}$ where $X_\sigma = \cap_{i \in \sigma} X_i$, is called a germ of $r-$reticular manifold. Denote by $\mathrm{Diff}^r(n)$, the set $\{\Phi \colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \mid \Phi$ is a diffeomorphism germ at *zero*, such that $\Phi(X_\sigma) = X_\sigma, \sigma \in P(I_r)\}$.

Let $f \colon X \to Y$ be a map from X into Y, where X, Y are smooth manifold-germs. Denote by $\underline{f}$ the collection $(f_\sigma)_{\sigma \in P(I_r)}$, where $f_\sigma = f|_{X_\sigma}$. We call it a *r-reticular map-germ,* and we write $\underline{f} \colon \underline{X} \to Y$.

We say that two germs $\underline{f}, \underline{g}$ are $r-$reticular equivalent if there exists $\Phi \in \mathrm{Diff}^r(n)$ such that $\underline{f} = \underline{g} \circ \Phi$.

Definition 1.1 Let $\underline{f} \colon \underline{X} \to Y$ be a $r-$reticular map-germ and let U be an open subset of $\mathbb{R}^p$, we say that a map-germ $F \colon \underline{X} \times U \ni (x, u) \mapsto F(x, u) \in Y$ is a $q-$*parameter deformation* of $\underline{f}$ if $F(x, 0) = \underline{f}(x)$.

Definition 1.2 A deformation F of $\underline{f}$ is said to be *a constant deformation* if F is of the following form: $F(x, u) = \underline{f}(x)$

Definition 1.3 We say two $q-$parameter deformations F_1, F_2 of the $r-$reticular germ $\underline{f}$ are $r-$*reticularly equivalent* if there exists $g \colon (\underline{X} \times \mathbb{R}^q, 0) \to (\underline{X}, 0)$, $g(X_i, 0) = X_i, \forall i = 1, ..., r$ and $g(x, 0) = x$ such that $F_1(x, u) = F_2(g(x, u), u)$.

Definition 1.4 Let $F \colon \underline{X} \times U \to Y$ be a $q-$parameter deformation of $r-$reticular germ $\underline{f} \colon \underline{X} \mapsto Y$ and U, U' be open subsets of $\mathbb{R}^q$, and $h \colon (U', 0) \to (U, 0), u' \mapsto h(u')$ be a smooth map-germ. Then, the map $h * F \colon \underline{X} \times U' \ni (x, u') \mapsto F(x, h(u')) \in Y$ is called a *deformation induced from F by the map h.*

Definition 1.5 An r-reticular germ $\underline{f}$ is *stable* if any deformation of $\underline{f}$ is reticularly equivalent to the constant deformation of $\underline{f}$.

Definition 1.6 A deformation F of $\underline{f}$ is said to be *versal* if every deformation of $\underline{f}$ is r-reticularly equivalent to one induced from F.

Definition 1.7 We denote by $\mathbb{R}\{\dot{F}_1, ..., \dot{F}_q\}$ the vector subspace of $\varepsilon(n)$, which is generated by $\dot{F}_1, ..., \dot{F}_q$, where $\dot{F}_i = \frac{\partial F}{\partial u_i}(x, 0)$ and $(e_1, \ldots, e_p)$ is the canonical basis of $\varepsilon(n, p)$. We write

$$T_r f = \varepsilon(n)\{x_1 \frac{\partial f}{\partial x_1}, ..., x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial x_{r+1}}, ..., \frac{\partial f}{\partial x_n}\} + f^* \varepsilon(p)(e_1, \ldots, e_p),$$

where $f^*\varepsilon(p)$ is the subring of $\varepsilon(n)$ defined by $f^*\varepsilon(p) = \{k \circ f \ : \ k \in \varepsilon(p)\}\varepsilon(n)$.

We say that a q-parameter deformation F of $\underline{f}$ is an *infinitesimally versal deformation* if:

$$T_r f + \mathbb{R}\{\dot{F}_1, ..., \dot{F}_q\} = \varepsilon(n)$$

2. RESULTS

Theorem 2.1 *Let $\underline{f}\colon \underline{X} \to Y$ be a $r-$reticular germ and let U be open set in $\mathbb{R}^q$, and $F\colon \underline{X} \times U \ni (x,u) \mapsto F(x,u) \in Y$ be a $q-$parameter deformation of $\underline{f}$. Then the deformation F is infinitesimally versal if it is a versal deformation.*

Proof For any $\alpha \in \varepsilon(n,p)$, we construct an 1-parameter deformation $F'(x,t) = \underline{f}(x) + t\alpha(x)$ of $\underline{f}$. Since F is versal, there exist g, φ such that $\underline{f}(x) + t\alpha(x) = F(g(t,x), \varphi(t))$, where $g\colon \underline{X} \times \mathbb{R} \to \underline{X}$, $g(x,0) = x$ and g is of the form

$$g(x,t) = (x_1 g_1(x,t), \ldots, x_r g_r(x,t), g_{r+1}(x,t), \ldots, g_n(x,t)),$$

and

$$\varphi\colon \mathbb{R} \to \mathbb{R}^q,\ \varphi(0) = 0,$$

such that g and φ are smooth germs. Differentiating by t at 0, we get:

$$\begin{aligned} \alpha(x) &= F'(g(x,t), \varphi(t))|_{t=0} \\ &= \sum_{i=1}^{r} x_i \frac{\partial F}{\partial x_i}.\frac{\partial g_i(x,t)}{\partial t}|_{t=0} + \sum_{i=r+1}^{n} \frac{\partial F}{\partial x_i}.\frac{\partial g_i(x,t)}{\partial t}|_{t=0} + \sum_{i=1}^{q} \frac{\partial F}{\partial u_i}.\frac{\partial \varphi_i}{\partial t}|_{t=0} \\ &= \sum_{i=1}^{r} x_i \frac{\partial \underline{f}}{\partial x_i} h_i(x) + \sum_{i=r+1}^{n} \frac{\partial \underline{f}}{\partial x_i} h_i(x) + \sum_{i=1}^{q} C_i \dot{F}_i(x) \end{aligned}$$

where $h_i(x) = \frac{\partial g_i(x,t)}{\partial t}|_{t=0}, \dot{F}_i(x) = \frac{\partial F}{\partial u_i}, C_i = \frac{\partial \varphi_i(t)}{\partial t}|_{t=0}$. The proof of the theorem is now completed. $\square$

Before going to the next result, we need the following lemmas.

Lemma 2.2 *Let $\underline{f} \in \varepsilon(n,p), \Phi \in \varepsilon(n+q,p)$ and let Φ be $q-$parameter deformation of $\underline{f}$, and $F \in \varepsilon(n+q-1,p)$ is defined by $F(x, t_2, ..., t_q) = \Phi(x, 0, t_2, ..., t_q)$. If there exists a vector field germ X at $0 \in \mathbb{R}^n \times \mathbb{R}^q$ such that*

1) $X = \frac{\partial \Phi}{\partial t_1} + \sum_{i=2}^{q} \varepsilon_i(t) \frac{\partial \Phi}{\partial t_i} + \sum_{i=1}^{n} X_i(x,t) \frac{\partial \Phi}{\partial x_i}$

2) $X \circ \phi = 0,$

then there exists a submersion map-germ $h\colon (\mathbb{R}^q, 0) \to (\mathbb{R}^{q-1}, 0)$ such that Φ is $r-$reticularly equivalent to $h * F$.

Proof If follows readily from the condition 2 that ϕ is a constant deformation of F. Hence X is constant of vector field

$$v = \frac{\partial \Phi}{\partial t_1} + \sum_{i=2}^{q} \varepsilon_i(t) \frac{\partial \Phi}{\partial t_i}$$

in a neighbourhood of $0 \in \mathbb{R}^q$. If $\Psi(t_1, ..., t_q) = (t_1, \Psi_{t_1}(t_2, ..., t_q))$ is a locally diffeomorphism, defined by the integral of vector field v, then $h\colon (t_1, ..., t_q) \to \Psi_{t_1}^{-1}(t_2, ..., t_q)$ is the required map and we have

$$\phi = (h * F)(\Pi_{\underline{X}}(x,t), t)$$

where $\Pi_{\underline{X}} : \underline{X} \times \mathbb{R}^q \ni (x,t) \to x \in \underline{X}$ is canonically projective. Therefore ϕ is r-reticularly equivalent to $h * F$. $\square$

Lemma 2.3 *Let F be an infinitesimally versal $q-$parameter deformation of $\underline{f}$ and Φ be an 1-parameter deformation of F. Then the deformation Φ is $r-$reticularly equivalent to the deformation induced from F.*

Proof Since F is an infinitesimally versal deformation of f, we have

$$\varepsilon(n) = \sum_{i=1}^{r} x_i \frac{\partial f}{\partial x_i}\varepsilon(n) + \sum_{i=r+1}^{n} \frac{\partial f}{\partial x_i}\varepsilon(n) + \mathbb{R}\{\dot{F}_1, ..., \dot{F}_q\}.$$

It is equivalent to

$$\varepsilon(n+1,p) = \sum_{i=1}^{r} x_i \frac{\partial \Phi}{\partial x_i}\varepsilon(n+1) + \sum_{i=r+1}^{n+1} \frac{\partial \Phi}{\partial x_i}\varepsilon(n+1) + \varepsilon(1)\{\dot{\Phi}_1, ..., \dot{\Phi}_q\}. \quad (*)$$

In particular, if $\frac{\partial \Phi}{\partial t} \in \varepsilon(n+1,p)$, by (*) we get

$$\frac{\partial \Phi}{\partial t} = \sum_{i=1}^{r} x_i X_i(x,t) \frac{\partial \Phi}{\partial x_i} + \sum_{i=r+1}^{n+1} X_i(x,t) \frac{\partial \Phi}{\partial x_i} + \sum_{i=1}^{q} \varepsilon(t) \frac{\partial \Phi}{\partial \lambda_i}$$

Taking $X = \frac{\partial \Phi}{\partial t} - \sum_{i=1}^{q} \varepsilon(t) \frac{\partial \Phi}{\partial \lambda_i} - \sum_{i=1}^{r} x_i X_i(x,t) \frac{\partial \Phi}{\partial x_i} - \sum_{i=r+1}^{n} X_i(x,t) \frac{\partial \Phi}{\partial x_i}$, then by Lemma 2.2, we have $X \circ \phi = 0$. Hence the deformation Φ is equivalent to $h * F$ for some h, proving our Lemma. $\square$

Theorem 2.4 *Any infinitesimally versal deformation is versal.*

Proof Let F be an infinitesimally versal $q-$parameter deformation of $\underline{f}$ and suppose that F' is any deformation of $\underline{f}$ with parameter $\lambda' \in \mathbb{R}^{q'}$. Consider the deformation $\hat{F}(x,\lambda,\lambda') = F(x,\lambda) + F'(x,\lambda') - \underline{f}(x)$, and the sequence of subspaces $\mathbb{R}^q \subset \mathbb{R}^{q+1} \subset ... \subset \mathbb{R}^{q+q'}$. Since F is an infinitesimally versal, we see that $\hat{F}_1 = \hat{F}|_{\lambda'_2 = ... = \lambda'_{q'} = 0}$ is equivalent to the induced deformation of F.

We will prove that if F is infinitesimally versal, then $\hat{F}_1$ is infinitesimally versal too. Indeed, since F is infinitesimally versal, we have

$$\varepsilon(n,p) = T_r f + \mathbb{R}[\dot{F}_1, ..., \dot{F}_q] \quad (*')$$

and $\dot{F}_i = [\dot{\hat{F}}_1]_i, i = 1,2,...,q$. Therefore, $\{\mathbb{R}[\dot{F}_1, ..., \dot{F}_q]\} \subset \mathbb{R}\{[\dot{\hat{F}}_1]_1, ..., [\dot{\hat{F}}_1]_{q+1}\}$. Hence the condition $(*')$ is satisfied.

By the same argument above, we see that $\hat{F}_2 = \hat{F}|_{\lambda'_3 = ... = \lambda'_{q'} = 0}$ is equivalent to the induced deformation from $\hat{F}_1$. Similarly, we get $\hat{F}_i$, is equivalent to the induced deformation from $\hat{F_{i-1}}, i = 2, ..., q'$, and therefore $\hat{F}$ is equivalent to the induced deformation of F, proving our Theorem. $\square$

From now on, we denote by I_f^r, the ideal generated by $(x_1\frac{\partial f}{\partial x_1}, \dots, x_r\frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial x_{r+1}}, \frac{\partial f}{\partial x_n})$ and O_f^r to be the ring defined by $\varepsilon(n,p)/I_f^r$. The symbol $\cong$ means an isomorphism between two rings.

Theorem 2.5 *Let $f\colon \underline{X} \to Y, g\colon \underline{X} \to Y$ be two stable r-reticular map-germs. If f is reticularly equivalent to g, then O_f^r and O_g^r are isomorphic.*

Before giving the proof of theorem 2.5, we have the following remark. Given a germ $\Phi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, we obtain a mapping $\Phi^*\colon \varepsilon(p) \to \varepsilon(n)$ by the formula $\lambda \to \lambda \circ \Phi$. One can easily check that Φ^* is an algebra homomorphism. It is said to be induced by Φ.

Since f, g are equivalent, there exists $\Phi \in \mathrm{Diff}^r(n)$ such that $f = g \circ \Phi$. Defferentiating by x_i, we get $\frac{\partial f}{\partial x_i} = \frac{\partial (g\circ\Phi)}{\partial x_i} = \sum_{j=1}^n (\frac{\partial g}{\partial x_j} \circ \Phi)\frac{\partial \Phi_j}{\partial x_i} = \sum_{j=1}^n \Phi^*(\frac{\partial g}{\partial x_j})\frac{\partial \Phi_j}{\partial x_i}$ It implies that

$$I_f^r \subseteq \Phi^*(I_g^r) \tag{1}$$

By the same argument above, we see that if $f = g \circ \Phi$, then $g = f \circ \Phi^{-1}$. Hence $\frac{\partial g}{\partial x_i} = \frac{\partial (f\circ\Phi^{-1})}{\partial x_i} = \sum_{j=1}^n (\frac{\partial f}{\partial x_j} \circ \Phi^{-1})\frac{\partial \Phi_j^{-1}}{\partial x_i} = \sum_{j=1}^n (\Phi^{-1})^*(\frac{\partial f}{\partial x_j})\frac{\partial \Phi_j^{-1}}{\partial x_i}$. This shows that $I_g^r \subseteq (\Phi^{-1})^*(I_f^r)$. It implies that

$$\Phi^*(I_g^r) \subseteq \Phi^*[(\Phi^{-1})^*(I_f^r)] = (\Phi^{-1} \circ \Phi)^*(I_f^r) = I_f^r$$

This shows that

$$\Phi^*(I_g^r) \subseteq I_f^r \tag{2}$$

From (1) and (2), it follows that $I_f^r = \Phi^*(I_g^r)$ and therefore $I_f^r \cong I_g^r$. Hence $O_f^r \cong O_g^r$, proving our theorem. $\square$

Problem Suppose that $O_f^r \cong O_g^r$. Are f and g reticularly equivalent?

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References

[1] V.I.Arnold, "Singularity Theory", Cambridge Uni. Press, 1981.

[2] V.I.Arnold, S.M.Gusein-Zade and A.N.Varchenco, "Singularity of Differentiable Maps", volume I, Birkhäuser, Boston-Basel-Stuttgart, 1985.

[3] P. L. Domennico, S. Castigiano and A. Hayes, "Catastrophe Theory", Addison-Wesley, 1993.

[4] Nguyen Tien Dai, Nguyen Huu Duc et Frédéric Pham, *Appendice de Nguyen Tu Cuong, Singularités non Dégégégées Dés systèmes de gauss-mains réticules,* Memoire de la Société mathématique de France, Nouvelle série $n^o 6$, Supplément au Bulletin de la Société Mathématique de France, Tome 109/Fascicule 3.

[5] Nguyen Huu Duc, *Singularity theory and applications,* Extended Abstracts, mini seminar in Applied Mathematics, Ho Chi Minh City, 26 April 1997, Vol I.

[6] M. Kashiware, T. Kawai, "On the holonomic systems with singularities and its relevance to physical problems in copmlex analysis", Lecture Notes in Physics 126, Springer-Verlag, 1980.

[7] N.H.Duc and F. Pham, *Configurations Legendriennesstables et fonctions d'Air y-Weber generalisées*, Ann. Inst. Fourier, Grenoble **41**(4), (1991) 905-936.