

A COMPACT EMBEDDING OF SEMISIMPLE SYMMETRIC SPACES

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Abstract

Let G be a connected real semisimple Lie group with finite center and σ be an involutive automorphism of G . Suppose that H is a closed subgroup of G with $G_e^\sigma \subset H \subset G^\sigma$, where G^σ is the fixed points group of σ and G_e^σ denotes its identity component. The coset space $\mathbf{X} = G/H$ is then a semisimple symmetric space. Let θ be a Cartan involution which commutes with σ and K be the set of all fixed points of θ . Then K is a σ -stable maximal compact subgroup of G and the coset space G/K becomes a Riemannian symmetric space of noncompact type. By using the action of the Weyl group, we have constructed a compact real analytic manifold in which the Riemannian symmetric space G/K is realized as an open subset and that G acts analytically on it. The purpose of this note is to apply the above construction to the case of semisimple symmetric spaces $\mathbf{X} = G/H$. Our construction is similar to those of Schlichtkrull, Lizhen Ji, Oshima for Riemannian symmetric spaces and similar to those of Kosters, Sekiguchi, Oshima for semisimple symmetric spaces.

1 Introduction

Let G be a connected real semisimple Lie group with finite center, σ be an involutive automorphism of G and $\mathbf{X} = G/H$ be the corresponding semisimple symmetric space. Here H is a closed subgroup of G with $G_e^\sigma \subset H \subset G^\sigma$, where G^σ is the fixed points group of σ and G_e^σ denotes its identity component.

Denote by θ the Cartan involution which commutes with σ and K the set of all fixed points of θ . Then K is a σ -stable maximal compact subgroup of G .

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Let \mathcal{G} be the Lie algebra of G . The involutions of \mathcal{G} induced by σ and θ are denoted by the same letters, respectively.

Suppose that $\mathcal{G} = \mathcal{H} \oplus \mathcal{Q} = \mathcal{K} \oplus \mathcal{P}$ are the decompositions of \mathcal{G} into +1 and -1 eigenspaces for σ and θ , respectively, where \mathcal{H} (resp. \mathcal{K}) is the Lie algebra of H (resp. K). Fix a maximal Abelian subspace \mathcal{A} in $\mathcal{P} \cap \mathcal{Q}$ and let \mathcal{A}^* denote the dual space of \mathcal{A} . The corresponding analytic subgroup A of \mathcal{A} in G is then called the vectorial part of X . For a $\alpha \in \mathcal{A}^*$, put

$$\mathcal{G}_\alpha = \{Y \in \mathcal{G} \mid [H, Y] = \alpha(H)Y, \forall H \in \mathcal{A}\}.$$

Then the set $\Sigma = \{\alpha \in \mathcal{A}^* \mid \mathcal{G}_\alpha \neq \{0\}, \alpha \neq 0\}$ defines a root system with the inner product induced by the Killing form \langle, \rangle of \mathcal{G} . Moreover, the Weyl group W of Σ is defined with the normalizer $N_K(\mathcal{A})$ of \mathcal{A} in K modulo the centralizer $M = Z_K(\mathcal{A})$ of \mathcal{A} in K . It acts naturally on \mathcal{A} and coincides via this action with the reflection group of the root system Σ .

Choose a fundamental system $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of Σ , where the number l which equals $\dim \mathcal{A}$ is called the split rank of the symmetric space X and denote Σ^+ the corresponding set of all positive roots in Σ . Denote by $W_{K \cap H}$ the normalizer $N_{K \cap H}(\mathcal{A})$ of \mathcal{A} in K modulo the centralizer $Z_{K \cap H}(\mathcal{A})$ of \mathcal{A} in K . We see that $W_{K \cap H}$ is a subgroup of W . For each element ω of W we fix a representative $\underline{\omega}$ in $N_K(\mathcal{A})$ so that $\underline{\omega} \in N_{K \cap H}(\mathcal{A})$ if $\omega \in W_{K \cap H}$.

The purpose of this note is to construct a compact real analytic manifold $\widehat{\mathbf{X}}$ in which the semisimple symmetric space $\mathbf{X} = G/H$ is realized as an open subset and that G acts analytically on it. By the Cartan decomposition $G = KAH$, we must compactify the vectorial part A . Our construction is a motivation of Oshima's construction. In [5] we proposed a construction for the Riemannian symmetric spaces G/K . Here we apply the construction for the semisimple symmetric spaces $\mathbf{X} = G/H$.

Denote by $\mathcal{G}_{\mathbf{C}}$ the complexification of \mathcal{G} and $G_{\mathbf{C}}$ the corresponding analytic group. For simplicity, we assume that G is the real form of the complex Lie group $G_{\mathbf{C}}$. Let $\mathcal{A}_{\mathbf{C}}$ be the complexification of \mathcal{A} and $A_{\mathbf{C}}$ be the analytic subgroup of $\mathcal{A}_{\mathbf{C}}$ in $G_{\mathbf{C}}$. For each $a \in A_{\mathbf{C}}$ and $\alpha \in \Sigma$ we define $a^\alpha = e^{\alpha \cdot \log a} \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and consider the subset

$$A_{\mathbb{R}} = \{a \in A_{\mathbf{C}} \mid a^\alpha \in \mathbb{R}, \forall \alpha \in \Sigma\}.$$

This note is organized as follows. In Section 1, we consider the compactification of the vectorial part based on the construction in [5]. By this way, we firstly construct an embedding of $A_{\mathbb{R}}$ into a compact real analytic manifold $\widehat{A}_{\mathbb{R}}$ which is called a compactification of $A_{\mathbb{R}}$. Then we illustrate the construction via an example. In Section 2, we construct the compact manifold $\widehat{\mathbf{X}}$ based on the action of Weyl group on $\widehat{A}_{\mathbb{R}}$ and consider the real analytic structure of $\widehat{\mathbf{X}}$ induced from the real analytic structure of $\widehat{A}_{\mathbb{R}}$.

2 Compactification of the vectorial part

Let $(\mathbf{C}^*)^\Sigma$ be the set of complexes $z = (z_\beta)_{\beta \in \Sigma}$, where $z_\beta \in \mathbf{C}^*$ and $\mathbf{C}\mathbb{P}^1$ be the 1-dimensional complex projective space. Consider the map $\varphi : A_{\mathbf{C}} \longrightarrow (\mathbf{C}^*)^\Sigma$ defined by $\varphi(a) = (a^\alpha)_{\alpha \in \Sigma}$, $\forall a \in A_{\mathbf{C}}$. Then, for every $z = (z_\alpha)_{\alpha \in \Sigma} \in \varphi(A_{\mathbf{C}})$ we have

$$z_{-\alpha} = (z_\alpha)^{-1}, \forall \alpha \in \Sigma \quad (2.1)$$

$$z_\alpha = \prod_{\gamma \in \Delta} (z_\gamma)^{k(\alpha, \gamma)}, \forall \alpha \in \Sigma^+, \alpha = \sum_{\gamma \in \Delta} k(\alpha, \gamma) \cdot \gamma. \quad (2.2)$$

By using the natural embedding of $(\mathbf{C}^*)^\Sigma$ into $(\mathbf{C}\mathbb{P}^1)^\Sigma$, we get an embedding map of $A_{\mathbf{C}}$ into $(\mathbf{C}\mathbb{P}^1)^\Sigma$ denoted also by φ .

Let $\mathbf{M} = \{z \in (\mathbb{R}\mathbb{P}^1)^\Sigma \mid z_{-\alpha} = z_\alpha^{-1}, \forall \alpha \in \Sigma\}$, where $\mathbb{R}\mathbb{P}^1$ is the 1-dimensional real projective space. By definition, \mathbf{M} is compact. Moreover, the subset

$$\mathcal{U}_{\Sigma^+} = \left\{ m = (m_\alpha, m_{-\alpha}) \in \mathbf{M} \mid m_\alpha \in \mathbb{R}, m_{-\alpha} \in \mathbb{R}^* \cup \{\infty\}, \forall \alpha \in \Sigma^+ \right\}$$

is an open subset in $(\mathbb{R}\mathbb{P}^1)^\Sigma$ and we get a homeomorphism $\chi_{\Sigma^+} : \mathcal{U}_{\Sigma^+} \longrightarrow \mathbb{R}^{\Sigma^+}$ defined by $\chi_{\Sigma^+}(m) = (m_\alpha)_{\alpha \in \Sigma^+}$, $\forall m \in \mathcal{U}_{\Sigma^+}$.

Recall that W acts on \mathbf{M} by $(w.z)_\alpha = z_{w^{-1}\alpha}$, $\forall \alpha \in \Sigma$, $w \in W$, $z \in \mathbf{M}$. So we have $\mathcal{U}_{w(\Sigma^+)} = w(\mathcal{U}_{\Sigma^+})$, $\forall w \in W$. By a similar way as in [5, Lemma 1.2], we see that the pair $\{\mathcal{U}_{\Sigma^+}, \chi_{\Sigma^+}\}$ is a chart on \mathbf{M} and $\{\mathcal{U}_{w(\Sigma^+)}, \chi_{w(\Sigma^+)}\}_{w \in W}$ defines an atlas of charts on \mathbf{M} such that \mathbf{M} becomes a real analytic submanifold.

By definition, $\varphi(A_{\mathbb{R}})$ is a subset of $(\mathbb{R}\mathbb{P}^1)^\Sigma$. Denote by $\widehat{A}_{\mathbb{R}}$ the closure of $\varphi(A_{\mathbb{R}})$ in $(\mathbb{R}\mathbb{P}^1)^\Sigma$. It follows from (2.1) and (2.2) that $\widehat{A}_{\mathbb{R}}$ is a compact subset of \mathbf{M} . We now define an atlas of charts on $\widehat{A}_{\mathbb{R}}$ induced from the atlas on \mathbf{M} .

Let \mathcal{U}_Δ be the subset of \mathcal{U}_{Σ^+} consists of elements $m = (m_\alpha, m_{-\alpha})$ such that $m_\alpha = \prod_{\gamma \in \Delta} (m_\gamma)^{k(\alpha, \gamma)}$, $\forall \alpha \in \Sigma^+$, $\alpha = \sum_{\gamma \in \Delta} k(\alpha, \gamma) \cdot \gamma$. Then \mathcal{U}_Δ is an open subset in $\widehat{A}_{\mathbb{R}}$. It follows that $\chi_{\Sigma^+}(\mathcal{U}_\Delta) = \{x \in \mathbb{R}^{\Sigma^+} \mid x_\alpha = \prod_{\gamma \in \Delta} (x_\gamma)^{k(\alpha, \gamma)}\}$ and we get

a homeomorphism $\chi_\Delta : \mathcal{U}_\Delta \longrightarrow \mathbb{R}^\Delta$ defined by $\chi_\Delta(m) = (m_\gamma)_{\gamma \in \Delta}$, $\forall m \in \mathcal{U}_\Delta$. Moreover, by a similar argument as that given in [5, Theorem 1.4], we have

Theorem 2.1. *$\widehat{A}_{\mathbb{R}}$ is a compact real analytic manifold that is called a compactification of $A_{\mathbb{R}}$. The set of charts $\{\mathcal{U}_{w(\Delta)}, \chi_{w(\Delta)}\}_{w \in W}$ defines an atlas of charts on $\widehat{A}_{\mathbb{R}}$ so that the manifold $\widehat{A}_{\mathbb{R}}$ is covered by $|W|$ -many charts.*

Remark 2.2. Denote by $\mathcal{G}_+ = \mathcal{G}^{\sigma\theta} = \mathcal{K} \cap \mathcal{H} \oplus \mathcal{P} \cap \mathcal{Q}$ and let

$$\Sigma_+ = \Sigma(\mathcal{A}, \mathcal{G}_+) = \{\alpha \in \Sigma \mid \mathcal{G}_\alpha \cap \mathcal{G}_+ \neq \{0\}, \alpha \neq 0\}$$

be the set of restricted roots of \mathcal{A} in \mathcal{G}_+ .

Suppose that H is connected. Since $H = (H \cap K).exp(\mathcal{H} \cap \mathcal{P})$ we have that $H \cap K$ is connected. Then $G_+ = (H \cap K).exp(\mathcal{P} \cap \mathcal{Q})$ is connected and the corresponding Weyl group $W_{K \cap H}$ of \mathcal{A} in \mathcal{G}_+ defined by

$$W_{K \cap H} = W_{(K \cap H)_e} = W(\mathcal{A}, \mathcal{G}_+) = \langle s_\alpha \mid \alpha \in \Sigma_+ \rangle.$$

Since $\mathcal{G}_+ = \mathcal{K} \cap \mathcal{H} \oplus \mathcal{P} \cap \mathcal{Q}$ is the Cartan decomposition and $K \cap H$ is a maximal compact subgroup of G_+ , the coset $\mathbf{X}_+ := G_+ / (K \cap H)$ becomes a Riemannian symmetric space of non-compact type. We then can apply the construction for the vectorial part of \mathbf{X}_+ .

Denote $A_{\mathbb{R},+} = \left\{ a \in A_{\mathbb{C}} \mid a^\gamma \in \mathbb{R}, \forall \gamma \in \Sigma_+ \right\}$. By construction, the corresponding compactification $\widehat{A}_{\mathbb{R},+}$ is contained in the compact manifold

$$\mathbf{M}_+ = \left\{ z \in (\mathbb{P}^1(\mathbb{R}))^{\Sigma_+} \mid z_{-\gamma} = (z_\gamma)^{-1}, \forall \gamma \in \Sigma_+ \right\}.$$

Moreover, $\widehat{A}_{\mathbb{R},+}$ is a compact real analytic manifold which is covered by $|W_{K \cap H}|$ -many charts. Here, each Weyl chamber of \mathcal{A} for Σ_+ contains $|W_{K \cap H} \setminus W|$ -corresponding Weyl chambers of \mathcal{A} for Σ and these subchambers are parameterized by $W_{K \cap H} \setminus W$.

Example Consider the real semi-simple Lie group $G = SL(3, \mathbb{R})$ and denote by $\mathcal{G} = sl(3, \mathbb{R})$ the corresponding Lie algebra of G . Suppose that θ is the Cartan involution defined by $\theta(X) = ({}^t X)^{-1}$, $\forall X \in \mathcal{G}$ and $K = SO(3, \mathbb{R})$ is the maximal compact subgroup in G with respect to θ . Then $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is the Cartan decomposition of \mathcal{G} with respect to θ , where $\mathcal{K} = so(3, \mathbb{R})$ is the Lie algebra of K .

Let σ be the involution of \mathcal{G} defined by

$$\sigma(X) = J\theta(X)J, \forall X \in \mathcal{G}, \text{ where } J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\mathcal{G} = \mathcal{H} \oplus \mathcal{Q}$ be the decomposition of \mathcal{G} with respect to σ , where $\mathcal{H} = so(1, 2)$ is the Lie algebra of the corresponding fixed points group $H = SO_0(1, 2)$.

It follows that $\mathbf{X} = G/H = SL(3, \mathbb{R})/SO_0(1, 2)$ is a semisimple symmetric space of rank two and we get a maximal Abelian subspace of $\mathcal{P} \cap \mathcal{Q}$ defined by

$$\mathcal{A} = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mid t_1 + t_2 + t_3 = 0 \right\}.$$

By definition, the root system Σ of \mathcal{A} in \mathcal{G} is $\Sigma = \{ e_i - e_j \mid 1 \leq i \neq j \leq 3 \}$ and the Weyl group W is isomorphic to S_3 , the symmetric group of order three.

Moreover, the corresponding analytic subgroup in G of \mathcal{A} defined by

$$A = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \mid a_1 a_2 a_3 = 1, a_i > 0 \right\} \cong (0, \infty)^2$$

and we get

$$A_{\mathbb{R}} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \mid a_1 a_2 a_3 = 1 \right\} \cong (\mathbb{R}^*)^2.$$

Denote $\alpha = e_1 - e_2$, $\beta = e_2 - e_3$. We see that $\Delta = \{\alpha, \beta\}$ and $\Sigma = \Sigma^+ \cup (-\Sigma^+)$, where $\Sigma^+ = \{\alpha, \beta, \alpha + \beta\}$. Then we have

$$\mathbf{M} = \left\{ (z_\gamma, z_{-\gamma}) \mid z_\gamma \in \mathbb{P}^1(\mathbb{R}), \gamma \in \Sigma^+ \right\} \cong \mathbb{P}^1(\mathbb{R})^3.$$

It follows that the pair $\{\mathcal{U}_{\Sigma^+}, \chi_{\Sigma^+}\}$ is a chart on \mathbf{M} , where

$$\mathcal{U}_{\Sigma^+} = \left\{ (m_\alpha, m_\beta, m_{\alpha+\beta}, m_{-\alpha}, m_{-\beta}, m_{-\alpha-\beta}) \mid m_\alpha, m_\beta, m_{\alpha+\beta} \in \mathbb{R} \right\} \cong \mathbb{R}^3$$

is an open subset in $(\mathbb{R}\mathbb{P}^1)^{\Sigma^+}$ and $\chi_{\Sigma^+} : \mathcal{U}_{\Sigma^+} \rightarrow \mathbb{R}^{\Sigma^+}$ is the homeomorphism defined by $\chi_{\Sigma^+}(m) = (m_\alpha, m_\beta, m_{\alpha+\beta})$, $\forall m \in \mathcal{U}_{\Sigma^+}$. Then we get

$$\mathcal{U}_\Delta = \left\{ (m_\alpha, m_\beta, m_{\alpha+\beta}, m_{-\alpha}, m_{-\beta}, m_{-\alpha-\beta}) \mid m_\alpha, m_\beta, m_{\alpha+\beta} \in \mathbb{R}, m_{\alpha+\beta} = m_\alpha \cdot m_\beta \right\}$$

and $\chi_{\Sigma^+}(\mathcal{U}_\Delta) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = x_1 \cdot x_2\} \cong \mathbb{R}^2$. Hence

$\widehat{A}_{\mathbb{R}} \cong \mathbb{R}^2 \cup \{\infty\} \cong S^2$ is a compact smooth manifold that is covered by 6-many charts.

By definition, we see that

$$G_+ = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} \mid \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2, \mathbb{R}), a^{-1} = \det(A) > 0 \right\}.$$

Then $X_+ = G_+/K \cap H$ is a Riemannian symmetric space of non-compact type.

Moreover, we have $\Sigma_+ = \{e_i - e_j \mid 2 \leq i \neq j \leq 3\} = \{\beta, -\beta\}$ and the Weyl group $W_{K \cap H}$ is isomorphic to S_2 , where S_2 is the symmetric group of order two. It follows that the coset $W_{K \cap H} \setminus W$ has three elements. Note that

$$A_{\mathbb{R},+} = \left\{ a \in A_{\mathbb{C}} \mid a^\gamma \in \mathbb{R}, \forall \gamma \in \Sigma_+ \right\} \cong \mathbb{R}^*.$$

Then the corresponding compactification $\widehat{A}_{\mathbb{R},+}$ is contained in the compact manifold

$$\mathbf{M}_+ = \left\{ (z_\gamma, z_\gamma^{-1}) \mid z_\gamma \in \mathbb{P}^1(\mathbb{R}), \forall \gamma \in \Sigma_+ \right\} \cong \mathbb{P}^1(\mathbb{R}).$$

It follows that $\widehat{A}_{\mathbb{R},+} \cong \mathbb{R} \cup \{\infty\} \cong S^1$ is a compact smooth manifold that is covered by 2-many charts.

Here each Weyl chamber of \mathcal{A} for Σ_+ contains three corresponding Weyl chambers of \mathcal{A} for Σ and these subchambers are parameterized by $W_{K \cap H} \setminus W$.

3 Construction of a compact embedding

In this Section, we will construct an embedding of the semisimple symmetric space \mathbf{X} into a compact real analytic manifold $\widehat{\mathbf{X}}$ such that the action of G on $\widehat{\mathbf{X}}$ is analytic. Our construction is based on the indicated construction in [5] for Riemannian symmetric spaces and similar to those of Kosters, Sekiguchi, Oshima for semisimple symmetric spaces.

First we recall some notations concerning the compactification $\widehat{A}_{\mathbb{R}}$ as in [5]. Consider the subset $\widehat{A}_{\mathbb{R}}^- = \{ \tilde{a} \in \widehat{A}_{\mathbb{R}} \mid (\tilde{a})^\alpha \in [-1, 1] \}$ and recall that the Weyl group W acts on $\widehat{A}_{\mathbb{R}}$ as follows $(w.\tilde{a})_\alpha = (\tilde{a})_{w^{-1}(\alpha)}$, $\forall w \in W, \forall \tilde{a} \in \widehat{A}_{\mathbb{R}}$. Then we have $W.\widehat{A}_{\mathbb{R}}^- = \widehat{A}_{\mathbb{R}}$ (see [5, Lemma 2.1]). Moreover, for each element $\tilde{a} \in \widehat{A}_{\mathbb{R}}$ we have a unique decomposition $\tilde{a} = (\tilde{a})_{fin} \cdot \epsilon(\tilde{a})$, where $a_{fin} \in A_{\mathbb{R}}$ and $\epsilon(\tilde{a}) \in \widehat{A}_{\mathbb{R}}$ such that $\epsilon(\tilde{a})^\gamma \in \{-1, 0, +1, \infty\}$, $\forall \gamma \in \Delta$.

Note that $\epsilon(\tilde{a}) \in \{-1, 0, +1, \infty\}^\Delta$ and for all $\alpha = \sum_{\gamma \in \Delta} k(\alpha, \gamma) \cdot \gamma \in \Sigma$ we have

$$\epsilon(\tilde{a})^\alpha = \prod_{\gamma \in \Delta} (\epsilon(\tilde{a})^\gamma)^{|k(\alpha, \gamma)|}.$$

Motivating the Oshima's definition, $\epsilon(\tilde{a})$ is called an extended signature of the element \tilde{a} . Now we define parabolic subalgebras with respect to extended signatures $\epsilon(\tilde{a})$, for all $\tilde{a} \in \widehat{A}_{\mathbb{R}}$.

First we consider $\tilde{a}_t \in \widehat{A}_{\mathbb{R}}^-$ and let $F_\epsilon = \{ \gamma \mid \epsilon_\gamma = \epsilon(\tilde{a}_t)^\gamma \neq 0 \}$ be a subset of the simple root system Δ with respect to the extended signature $\epsilon = \epsilon(\tilde{a}_t)$. Denote $\Sigma_\epsilon = (\sum_{\gamma \in F_\epsilon} \mathbb{R}\gamma) \cap \Sigma$ and suppose that W_ϵ is the subgroup of W generated by reflections with respect to γ in F_ϵ . Let P_σ be the parabolic subgroup of G with the corresponding Langlands decomposition $P_\sigma = M_\sigma A_\sigma N_\sigma$ so that $M_\sigma A_\sigma$ is the centralizer of \mathcal{A} in G and the Lie algebra \mathcal{N}_σ of N_σ equals $\sum_{\alpha \in \Sigma^+} \mathcal{G}_\alpha$. Then we can define a parabolic subalgebra

$$\mathcal{P}_\epsilon = \mathcal{M}_\sigma + \mathcal{A}_\sigma + \sum_{\alpha \in \Sigma_\epsilon} \mathcal{G}_\alpha + \sum_{\alpha \in \Sigma^+ - \Sigma_\epsilon} \mathcal{G}_\alpha$$

of \mathcal{G} and its Langlands decomposition $\mathcal{P}_\epsilon = \mathcal{M}_\epsilon + \mathcal{A}_\epsilon + \mathcal{N}_\epsilon$ so that $\mathcal{A}_\epsilon \subset \mathcal{A}_\sigma$ (see [11]).

Let P_ϵ denote the corresponding parabolic subgroup of \mathcal{P}_ϵ in G . It follows that $P_\epsilon = M_\epsilon A_\epsilon N_\epsilon$ is the corresponding Langlands decomposition of P_ϵ and we

define a closed subgroup $P(\epsilon)$ of G by $P(\epsilon) = (M_\epsilon \cap \underline{\omega}^{-1}H\underline{\omega})A_\epsilon N_\epsilon$, where $\underline{\omega}$ is a representative of $\omega \in W$ in $N_K(\mathcal{A})$ so that $\underline{\omega} \in N_{K \cap H}(\mathcal{A})$ if $\omega \in W_{K \cap H}$.

In general, for each $\eta = \eta(\tilde{a})$ with $\tilde{a} = w.\tilde{a}_t \in \widehat{A}_\mathbb{R}$, we firstly consider the parabolic subalgebra P_ϵ with respect to the element $\epsilon = \epsilon(\tilde{a}_t)$. Then we define a parabolic subalgebra $P_\eta = w.P_\epsilon.w^{-1}$ based on the action of the Weyl group W on the parabolic subalgebra P_ϵ (see [3]).

We now define an equivalent relation on the product manifold $G \times \widehat{A}_\mathbb{R}$. Let $x = (g, \tilde{a})$ be an element of $G \times \widehat{A}_\mathbb{R}$, where $\tilde{a} = w.\tilde{a}_t$ and $t = (t_\gamma)_{\gamma \in \Delta}$, $t_\gamma \in [-1, 1]$. Then we put $\text{sgn } x = \epsilon(\tilde{a}_t) = \text{sgn } t$, which is an element of $\{-1, 0, 1\}^\Delta$. Here $\text{sgn } t = (\text{sgn } t_\gamma)_{\gamma \in \Delta}$ and for an s in \mathbb{R} we define $\text{sgn } s = 1$ (resp. $0, -1$) if $s > 0$ (resp. $s = 0, s < 0$).

Denote F_x, Σ_x and W_x instead of $F_\epsilon, \Sigma_\epsilon$ and W_ϵ , respectively, we define a parabolic subalgebra

$$\mathcal{P}_x = \mathcal{M}_\sigma + \mathcal{A}_\sigma + \sum_{\alpha \in \Sigma_x} \mathcal{G}_\alpha + \sum_{\alpha \in \Sigma^+ - \Sigma_x} \mathcal{G}_\alpha$$

of \mathcal{G} and its Langlands decomposition $\mathcal{P}_x = \mathcal{M}_x + \mathcal{A}_x + \mathcal{N}_x$ so that $\mathcal{A}_\epsilon \subset \mathcal{A}_\sigma$.

Moreover, denoting by P_x the corresponding subgroup of \mathcal{P}_x in G , we get the Langlands decomposition $P_x = M_x A_x N_x$ and $P(x) = (M_x \cap \underline{\omega}^{-1}H\underline{\omega})A_x N_x$ is a closed subgroup of G . Let $\{H_1, H_2, \dots, H_l\}$ denote the dual basis of $\Delta = \{\alpha_1, \dots, \alpha_l\}$, that is, $H_j \in \mathcal{A}$ and $\alpha_i(H_j) = \delta_{ij}$, $\forall i, j = 1, 2, \dots, l$ and put $a(x) = \exp(-\sum_{\gamma \in F_x} \log|t_\gamma| H_\gamma)$, where $H_\gamma \in \{H_1, H_2, \dots, H_l\}$ with respect to γ .

Note that for all elements $x = (g, \omega.\tilde{a}_t)$ and $x' = (g', \omega'.\tilde{a}_{t'})$ of $G \times \widehat{A}_\mathbb{R}$ such that $\text{sgn } x = \text{sgn } x'$, we have $W_x = W_{x'}$.

Definition 3.1. We say that two elements $x = (g, \omega.\tilde{a}_t)$ and $x' = (g', \omega'.\tilde{a}_{t'})$ of $G \times \widehat{A}_\mathbb{R}$ are equivalent if and only if the following conditions hold:

- (i) $\text{sgn } x = \text{sgn } x'$
- (ii) $W_{K \cap H} \omega W_x = W_{K \cap H} \omega' W_x$
- (iii) $ga(x)P(x) = g'a'(x')(M_x \cap \underline{\omega}'^{-1}H\underline{\omega}')A_x N_x$.

Remark 3.2. Let $x = (g, \omega.\tilde{a}_t)$ and $x' = (g', \omega'.\tilde{a}_{t'})$ be elements of $G \times \widehat{A}_\mathbb{R}$ and suppose that x and x' satisfy the above condition (iii). Then there exist $u, u' \in W_{K \cap H}$ and $v, v' \in W_x$ and $m, m' \in M$ such that $\underline{u} \underline{\omega} \underline{v} m = \underline{u}' \underline{\omega}' \underline{v}' m'$. Moreover, (see [11, Lemma 1.2]) the condition (iii) in Definition 2.1 is equivalent to

$$ga(x)P(x)\underline{v}m = g'a'(x')P(x')\underline{v}'m'. \quad (3.1)$$

It follows from Remark 3.2 that the Definition 3.1 really gives an equivalent relation, and we write $x \sim x'$. The quotient space of $G \times \widehat{A}_\mathbb{R}$ by this equivalent relation becomes a topological space with the quotient topology and is denoted by $\widehat{\mathbf{X}}$.

Let $\pi : G \times \widehat{A}_{\mathbb{R}} \longrightarrow \widehat{\mathbf{X}}$ be the natural projection. Because the action of G on $G \times \widehat{A}_{\mathbb{R}}$ are compatible with the equivalent relation, we can define an action of G on $\widehat{\mathbf{X}}$ by

$$g_1 \pi(g, \tilde{a}) = \pi(g_1 g, \tilde{a}), \quad \forall g, g_1 \in G, \tilde{a} \in A_{\mathbb{R}}. \quad (3.2)$$

Put $\widehat{A}_{\mathbb{R}, \epsilon} = \{ \tilde{a} \in \widehat{A}_{\mathbb{R}} \mid \epsilon(\tilde{a}) = \epsilon \}$ for each $\epsilon \in \{ -1, 0, 1 \}^{\Delta}$ and denote $\mathbf{X}_{\epsilon} = \pi(G \times \widehat{A}_{\mathbb{R}, \epsilon})$. Then we have

Proposition 3.3. (i) $\widehat{\mathbf{X}}$ is a compact connected G -space and $\widehat{\mathbf{X}} = \bigcup_{\epsilon \in \{ -1, 0, 1 \}^{\Delta}} \mathbf{X}_{\epsilon}$

gives the orbital decomposition of $\widehat{\mathbf{X}}$ for the action of G on it.

(ii) Each \mathbf{X}_{ϵ} is homeomorphic to $G/P(\epsilon)$. There are just $2^{|\Delta|}$ open orbits that are isomorphic to G/H and the number of compact orbits in $\widehat{\mathbf{X}}$ equal that of the elements of the coset $W_{K \cap H} \setminus W$.

Proof (i) Since $\pi(G \times \mathcal{U}_{\Delta})$ is connected and contains any open orbit, the connectedness of $\widehat{\mathbf{X}}$ is clear. Denote by \mathcal{A}^+ the positive chamber corresponding to Σ^+ and put $A^+ = \exp \mathcal{A}^+$. Let $\overline{A^+} = \{ \exp X \mid X \in \mathcal{A} \text{ with } \alpha(X) \geq 0 \text{ for all } \alpha \in \Sigma^+ \}$ be the closure of A^+ , we see that

$$\overline{A^+} = \left\{ \exp \left(- \sum_{\gamma} (\log t_{\gamma}) H_{\gamma} \right) \mid t_{\gamma} \in (0, 1] \right\}.$$

Consider the compact subset $K \times \widehat{A}_{\mathbb{R}} = K \times W \cdot \widehat{A}_{\mathbb{R}} \cong K \times [-1, 1]^{\Delta} \times W$ of $G \times \widehat{A}_{\mathbb{R}}$. Then the subset $\pi(K \times \widehat{A}_{\mathbb{R}})$ is also compact because it is the image of a compact set under the continuous map. Moreover, it follows from the Cartan decomposition $G = KAH$ that $G = \bigcup_{w \in W} K \overline{A^+} \underline{w} H$ (see [11]). Hence, Definition

3.1 and Remark 3.2 prove that the compact set $\pi(K \times \widehat{A}_{\mathbb{R}})$ contains all open G -orbits in $\widehat{\mathbf{X}}$. In other words, this compact set is dense in $\widehat{\mathbf{X}}$ and therefore it must be coincided with $\widehat{\mathbf{X}}$.

(ii) Put $\tilde{a} \in \widehat{A}_{\mathbb{R}, \epsilon}$ for each $\epsilon \in \{ -1, 0, 1 \}^{\Delta}$ and define a map $\Psi : G/P(\epsilon) \longrightarrow \mathbf{X}_{\epsilon}$ by $\Psi(gP(\epsilon)) = \pi(g, \tilde{a})$, $\forall g \in G$. Then by a similar way as in [5, Proposition 2.4], we get that the map is well defined and becomes an homeomorphism which is equivariant for the action of G .

We now define an analytic structure on $\widehat{\mathbf{X}}$ based on the analytic structure on $\widehat{A}_{\mathbb{R}}$.

Let $\mathcal{A}_{\mathcal{P}}$ be a maximal Abelian subspace of \mathcal{P} containing \mathcal{A} and let $\Sigma(\mathcal{A}_{\mathcal{P}}, \mathcal{G})$ be the set of corresponding restricted roots. By [11, Lemma 1.4], we can assume that the representatives \underline{w} of elements $w \in W$ satisfy $Ad(\underline{w})(\mathcal{A}_{\mathcal{P}}) = \mathcal{A}_{\mathcal{P}}$. Denote by $\mathcal{G}(\sigma)$ the reductive Lie algebra generated by $\{ \mathcal{G}(\mathcal{A}_{\mathcal{P}}; \lambda) \mid \lambda \in \Sigma(\mathcal{A}_{\mathcal{P}}, \mathcal{G}), \lambda|_{\mathcal{A}} = 0 \}$, where $\mathcal{G}(\mathcal{A}_{\mathcal{P}}; \lambda) = \{ X \in \mathcal{G} \mid [X, Y] = \lambda(Y)X \text{ for all } Y \in \mathcal{A}_{\mathcal{P}} \}$ and put

$$\mathcal{M}(\sigma) = \{ X \in \mathcal{M}_{\sigma} \mid [X, Y] = \lambda(Y)X \text{ for all } Y \in \mathcal{G}(\sigma) \}.$$

Let $G(\sigma)$ and $M(\sigma)_0$ be the analytic subgroups of G corresponding to $\mathcal{G}(\sigma)$ and $\mathcal{M}(\sigma)$, respectively, and denote

$$M(\sigma) = M(\sigma)_0 \cdot Ad_G^{-1}(Ad(K) \cap \exp(\text{ad}(\sqrt{-1}\mathcal{A}_p))).$$

Then $G(\sigma) \subset H$, $M(\sigma) \subset N_K(\mathcal{A}_p)$ and the representative $\underline{\omega}$ normalizes $G(\sigma)$ and $M(\sigma)$ for any $\omega \in W$. Moreover, it follows from [11, Lemma 1.5] that $M_\sigma = M(\sigma)G(\sigma)$ and

$$M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \simeq M_\sigma/(M_\sigma \cap \underline{\omega}^{-1}H\underline{\omega}) \simeq M/(M \cap \underline{\omega}^{-1}H\underline{\omega}).$$

For every $g \in G$ and $w \in W$, we put $\Omega_g^w = \pi(g\overline{N}_\sigma M(\sigma) \times \mathcal{U}_{w(\Delta)})$, where \overline{N}_σ is the analytic subgroup of G corresponding to $\overline{N}_\sigma = \theta(\mathcal{N}_\sigma)$ and define a map

$$\Phi_g^w : \overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta \longrightarrow \Omega_g^w$$

by $\Phi_g^w(n, m, t) = \pi(gnm, w.\tilde{a}_t)$, $\forall (n, m, t) \in \overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta$. \square By the same argument as that given in [5, Lemma 2.6], we get

Lemma 3.4. *For every $g \in G$ and $w \in W$, the map Φ_g^w is a homeomorphism of $\overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta$ onto an open subset $\Omega_g^w = \pi(g\overline{N}_\sigma M(\sigma) \times \mathcal{U}_{w(\Delta)})$ of \widehat{X} .*

For brevity, we denote $\Omega^w = \overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta$. Then we have

Lemma 3.5. *Let $g, g' \in G$ and $w, w' \in W$. Then the map*

$$(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w) : (\Phi_g^w)^{-1}(\Omega_g^w \cap \Omega_{g'}^{w'}) \longrightarrow (\Phi_{g'}^{w'})^{-1}(\Omega_g^w \cap \Omega_{g'}^{w'})$$

define an analytic diffeomorphism between the open subsets of the set Ω^w .

Proof By definition, Φ_g^w is bijective and continuous. Then the map $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w)$ is bijective and its inverse is of the same form. So we need only to show that the map $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w)$ is analytic. Since $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w) = (\Phi_e^{w'})^{-1} \circ (\Phi_{g'^{-1}g}^w)$, we can assume that $g' = e$.

Fix an arbitrary element $u = (n_o, m_o, t_o)$ of the domain of $(\Phi_e^{w'})^{-1} \circ (\Phi_g^w)$ and put $x = (gn_o m_o, w\tilde{a}_{t_o}) \in \widehat{X}$ and $u' = (n'_o, m'_o, t'_o) = ((\Phi_e^{w'})^{-1} \circ \Phi_g^w)(u)$ in $\Omega^{w'}$, we will show that the map is analytic in a neighborhood of u .

We first consider the case where $w' = w$ and $g \in \overline{N}_\sigma M(\sigma)A^x$. Here A^x is the analytic subgroup of G corresponding to $A^x = \sum_{\gamma \in F_x} \mathbb{R}H_\gamma$. Suppose that $g = n_1 m_1 a_1$, where $n_1 \in \overline{N}_\sigma, m_1 \in M(\sigma)$ and $a_1 \in A^x$. Then we have

$$(\Phi_e^w)^{-1} \circ (\Phi_g^w)(n, m, t) = (n_1 m_1 a_1 n(m_1 a_1))^{-1}, m_1 m, a_1 t,$$

where $a_1 t = (\exp < -a_1, \log a_1 > t_1, \dots, \exp < -a_l, \log a_1 > t_l)$. It follows that the map is analytic.

Next we assume that $w' = w$ and $u' = u = (e, 1, \epsilon)$ with an $\epsilon \in \{-1, 0, 1\}^\Delta$. Then we have $g \in P(x)$ and [11, Lemma 1.9 (i)] there exist neighborhoods V of the origin in $\mathcal{P}(x)$ and U_0 of u in Ω^w such that for any $Y \in V$ and $s \in [0, 1]$, the map $(\Phi_e^w)^{-1} \circ \exp(sY) \circ \Phi_e^w$ defines an analytic diffeomorphism of U_0 onto a neighborhood of u . Hence we have the claim if $g \in \exp V$. Moreover, any $g \in P(x)$ can be written in the form $g = g_0 g_1 \dots g_k$ with $g_0 \in M(\sigma) \cap \underline{\omega}^{-1} H \underline{\omega}$ and $g_i \in \exp V (i = 1, \dots, k)$, where k is a suitable positive integer. Then we have the relation

$$(\Phi_e^w)^{-1} \circ \Phi_g^w = ((\Phi_e^w)^{-1} \circ \Phi_{g_0}^w) \circ ((\Phi_e^w)^{-1} \circ \Phi_{g_1}^w) \circ \dots \circ ((\Phi_e^w)^{-1} \circ \Phi_{g_k}^w)$$

and $(\Phi_e^w)^{-1} \circ \Phi_{g_i}^w$ map the point u to the same point for $i = 0, 1, \dots, k$. It follows that $(\Phi_e^w)^{-1} \circ \Phi_{g_i}^w$ are analytic in some neighborhoods of u in Ω^w and we have the claim.

We consider the case where $w' \neq w, g = e$ and $u = (e, 1, \epsilon)$. Then it follows that $u' = (e, 1, \epsilon)$ when $g' = \underline{v}' m' m^{-1} \underline{v}^{-1}$, with $v, v' \in W_x$. By a similar way as in [11, Lemma 1.9 (ii)], we can prove that the map $\Phi_{g'}^{w'} \circ (\Phi_e^w)$ is analytic in the set

$$\Omega^w(\epsilon) = \overline{N}_\sigma \times M(\sigma) / (M(\sigma) \cap \underline{\omega}^{-1} H \underline{\omega}) \times \mathbb{R}_\epsilon^\Delta,$$

where $\mathbb{R}_\epsilon^\Delta = \{t \in \mathbb{R}^\Delta \mid \text{sgn } t_\gamma = \epsilon_\gamma \text{ if } \epsilon_\gamma \neq 0\}$.

Now we consider the general case. Put $g_1 = (n_o \underline{m}_o a(t_o))^{-1}$, $g_2 = g_3^{-1} g_4^{-1} g g_1^{-1}$, where $g_3 = \underline{v}' m' m^{-1} \underline{v}^{-1}$, $g_4 = n'_o \underline{m}_o a(t'_o)$ and consider the maps

$$\Phi_1 = (\Phi_e^w)^{-1} \circ \Phi_{g_1}^w, \Phi_2 = (\Phi_e^w)^{-1} \circ \Phi_{g_2}^w, \Phi_3 = (\Phi_{g_3}^{w'})^{-1} \circ (\Phi_e^w), \Phi_4 = (\Phi_e^{w'})^{-1} \circ \Phi_{g_4}^{w'}.$$

Then we have

$$(\Phi_e^{w'})^{-1} \circ \Phi_g^w = \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1.$$

Since $\Phi_1(u) = (e, 1, \text{sgn } x)$ and $g_2 \in P(x)$, it follows from what we have proved that the maps Φ_1, Φ_2, Φ_3 and Φ_4 are analytic in a neighborhood of $u = (n_o, m_o, t_o)$. This implies that the map $(\Phi_e^{w'})^{-1} \circ \Phi_g^w$ is analytic in a neighborhood of u and we have the Lemma. \square Lemma 3.4 and Lemma 3.5

assures that we can define an analytic structure on $\widehat{\mathbf{X}}$ through the maps Φ_g^w so that they define analytic diffeomorphisms onto open subsets Ω_g^w of $\widehat{\mathbf{X}}$ and the action of G on $\widehat{\mathbf{X}}$ is analytic. On the other hand $\widehat{\mathbf{X}}$ is Hausdorff because Φ_g^w is homeomorphic and $\overline{N}_\sigma \times M(\sigma) / (M(\sigma) \cap \underline{\omega}^{-1} H \underline{\omega}) \times \mathbb{R}^\Delta$ is Hausdorff. Combining this with Proposition 3.3 we get

Theorem 3.6. (i) $\widehat{\mathbf{X}}$ is a compact connected real analytic manifold and $\bigcup_{w \in W, g \in G} \Omega_g^w$ is an open covering of $\widehat{\mathbf{X}}$ such that the maps Φ_g^w are real analytic diffeomorphisms.

(ii) The action of G on $\widehat{\mathbf{X}}$ is analytic and the orbit $G\pi(x)$ for a point x in $\widehat{\mathbf{X}}$ is isomorphic to $G/P(x)$.

(iii) There are just $2^{|\Delta|}$ open orbits that are isomorphic to G/H and the number of compact orbits in $\widehat{\mathbf{X}}$ equal that of the elements of the coset $W_{K \cap H} \backslash W$.

Remark 3.7. (i) By a similar way, we can construct the compactification of the corresponding Riemannian symmetric spaces $\mathbf{X}_+ = G^+/K \cap H$.

(ii) For the special case $H = K$, i.e., for the case of Riemannian symmetric spaces of non-compact type, we obtain the corresponding compactification indicated in [5].

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