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ESSENTIAL OPERATION SYMBOLS IN TERMS

N. Pabhapote* and **K. Denecke**†

**Department of Mathematics,*
The University of the Thai Chamber of Commerce
126/1 Vibhavadee Rangsit Road
Bangkok, 10400 Thailand
e-mail: nittiya_pab@utcc.ac.th

†Institute of Mathematics
University of Potsdam
Am Neuen Palais, 14415 Potsdam, Germany
e-mail: kdenecke@rz.uni-potsdam.de

Abstract

Generalizing the concept of an essential variable in a term with respect to an algebra or a variety, we define essential operation symbols in a term with respect to an algebra or with respect to a variety of algebras. Using the concept of a unitary Menger algebra of rank n we define so-called operator terms and prove that essential operation symbols in terms with respect to an algebra correspond to essential variables in operator terms with respect to the Menger algebra of all n-ary term operations of this algebra. After proving some elementary propositions, we extend our definition to essential operation symbols in hypersubstitutions and determine some monoids consisting of hypersubstitutions which contain the same essential operation symbols.

Using the isomorphism between the monoid of all endomorphisms of the unitary Menger algebra of all n-ary terms of type τ and the monoid of all hypersubstitutions, we obtain an equivalence between essential variables in operator terms with respect to the Menger algebra $n - cloneV$ and essential operation symbols in terms with respect to the variety V.

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1 Introduction

The study of *essential* and *strongly essential* variables in functions defined on finite sets is a part of k-valued Logic (see e.g. [1], [2], [3], [8]) and plays an important role in Computer Science. Let A be an arbitrary non-empty set. The unary function $f : A \to A$ depends essentially on its input x if it takes on at least two values, i.e. if f is not constant. The n-ary function $f : A^n \to A$ depends essentially on its i-th input x_i if there are elements $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ such that the unary function defined by

$$x_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$$

is not constant on A. In [11] the concept of an essential variable in a function was extended to terms. This allows to use concepts and results from Universal Algebra. At first we recall some basic facts on terms and term operations.

Let $\tau = (n_i)_{i \in I}$ be an arbitrary type and let $W_\tau(X_n)$ be the set of all $n{-}ary$ terms of type τ built up by the n_i - ary operation symbols $f_i, i \in I$, and by variables from an alphabet $X_n = \{x_1, \ldots, x_n\}$. Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ where $X = \{x_1, \ldots, x_n, \ldots\}$ is an arbitrary countably infinite alphabet. These two sets are the universes of two absolutely free algebras,

$$\mathcal{F}_\tau(X) := (W_\tau(X); (\overline{f_i})_{i \in I}) \quad \text{and} \quad \mathcal{F}_\tau(X_n) := (W_\tau(X_n); (\overline{f_i})_{i \in I}),$$

respectively. Here the operations $\overline{f_i}$ are defined by setting

$$\overline{f_i}(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i}).$$

Another operation on sets of terms is the *composition* or *superposition* of terms which plays an important role in Universal Algebra, Clone Theory and Theoretical Computer Science. For each pair of natural numbers m and n greater than zero, the superposition operation S^n_m maps one n-ary term and n m-ary terms to an m-ary term, so that

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m).$$

The operation S^n_m is defined inductively, by setting
$S^n_m(x_j, t_1, \ldots, t_n) := t_j$ for any variable $x_j \in X_n$, and
$S^n_m(f_r(s_1, \ldots, s_{n_r}), t_1, \ldots, t_n) := f_r(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_r}, t_1, \ldots, t_n))$.
Using these operations, we form the heterogeneous or multi-based algebra

$$clone\tau \ := \ (\,(W_\tau(X_n))_{n>0};\ (S^n_m)_{n,m>0}, (x_i)_{0<i\leq n}\,).$$

It is well-known and easy to check that this algebra satisfies the clone axioms

(C1) $\overline{S^p_m}(\tilde{Z}, \overline{S^n_m}(\tilde{Y_1}, \tilde{X_1}, \ldots, \tilde{X_n}), \ldots, \overline{S^n_m}(\tilde{Y_p}, \tilde{X_1}, \ldots, \tilde{X_n}))$
$\approx \overline{S^n_m}(\overline{S^p_n}(\tilde{Z}, \tilde{Y_1}, \ldots, \tilde{Y_p}), \tilde{X_1}, \ldots, \tilde{X_n})$, for $m, n, p = 1, 2, 3, \ldots$,

(C2) $\overline{S^n_m}(\lambda_j, \tilde{X_1}, \ldots, \tilde{X_n}) \approx \tilde{X_j}$, for $1 \leq j \leq n$ and $m, n = 1, 2, 3, \ldots$,

(C3) $\overline{S^m_m}(\tilde{X}_j, \lambda_1, \ldots, \lambda_m) \approx \lambda_j$, for $1 \leq j \leq m$ and $m = 1, 2, 3, \ldots$,
where $\overline{S^p_m}$, $\overline{S^n_m}$ and $\overline{S^p_n}$ are operation symbols corresponding to the operations S^p_m, S^n_m and S^p_n of *clone* τ, where $\lambda_1, \ldots, \lambda_m$ are nullary operation symbols and where $\tilde{Z}$, $\tilde{Y}_1, \ldots, \tilde{Y}_p$, $\tilde{X}_1, \ldots, \tilde{X}_m$ are variables. The algebra *clone*τ is also called clone of terms of type τ.

Since the set $W_\tau(X_n)$ of all n-ary terms of type τ is closed under the superposition operation $S^n := S^n_n$, there is a homogeneous analogue of this structure. The algebra $(W_\tau(X_n); S^n, x_1, \ldots, x_n)$ is an algebra of type $(n+1, 0, \ldots, 0)$, which still satisfies the clone axioms above for the case that $p = m = n$. Such an algebra is called a *unitary Menger algebra of rank* n (see [12] or [6]). Another example of a unitary Menger algebra of rank n can be obtained as follows:
We define a superposition operation $S^{n,A}$ on the set $O^{(n)}(A)$ of all $n-ary$ operations $f^A : A^n \to A$ defined on A. The operation $S^{n,A} : O^{(n)}(A)^{n+1} \to O^{(n)}(A)$ is defined by

$$S^{n,A}(f_0^A, f_1^A, \ldots, f_n^A)(a_1, \ldots, a_n) := f_0^A(f_1^A(a_1, \ldots, a_n), \ldots, f_n^A(a_1, \ldots, a_n))$$

for all $a_1, \ldots, a_n \in A$. Then it is not difficult to check that (C1), (C2), (C3) are satisfied and thus $\mathcal{O}^{(n)}(A) := (O^{(n)}(A); S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$ is a unitary Menger algebra of rank n.

To every term $t \in W_\tau(X_n)$ and every algebra $\mathcal{A}$ of type τ there belongs an induced $n-$ *ary term operation* which is inductively defined by
$x_i^{\mathcal{A}} := e_i^{n,A}$ where $e_i^{n,A} : A^n \to A$ and $e_i^{n,A}(a_1, \ldots, a_n) = a_i$ is the $n - ary$ projection on the i-th component, for $1 \leq i \leq n$.
For compound terms $f_i(s_1, \ldots, s_{n_i})$ we define
$(f_i(s_1, \ldots, s_{n_i}))^{\mathcal{A}} := S^{n_i,A}(f_i^{\mathcal{A}}, s_1^{\mathcal{A}}, \ldots, s_{n_i}^{\mathcal{A}})$.
Let $W_\tau(X_n)^{\mathcal{A}}$ be the set of all n-ary term operations of $\mathcal{A}$. This set is closed under the superposition operation $S^{n,A}$ and contains by definition all n-ary projections. Therefore $n - clone\mathcal{A} := (W_\tau(X_n)^{\mathcal{A}}; S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$ is a subalgebra of $\mathcal{O}^{(n)}(A) := (O^{(n)}(A); S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$.

Let s, t be n-ary terms of type τ. The algebra $\mathcal{A}$ satisfies $s \approx t$, written as $\mathcal{A} \models s \approx t$, if $s^{\mathcal{A}} = t^{\mathcal{A}}$. If V is a variety of algebras of type τ, then $V \models s \approx t$ means $\mathcal{A} \models s \approx t$ for all algebras $\mathcal{A} \in V$. Let IdV be the set of all identities satisfied in the variety V and let $Id_nV := IdV \cap W_\tau(X_n)^2$ be the set of all *n-ary identities* of V. It turns out that Id_nV is a congruence on the Menger algebra $n - clone\tau$ (see e.g. [7]). Therefore the quotient algebra $n - cloneV := n - clone\tau / Id_nV$ is also a unitary Menger algebra of rank n. Moreover, for a single algebra $\mathcal{A}$ of type τ we have $n - clone\mathcal{A} := n - clone\tau / Id_n\mathcal{A}$.

Essential variables in terms with respect to an algebra of the same type are defined as follows.

Definition 1.1 ([11]) Let $t \in W_\tau(X_m)$ be an m-ary term and let $\mathcal{A}$ be a non-trivial algebra of type τ. Then the variable $x_i, 1 \leq i \leq n$, is called *essential in t with respect to the algebra $\mathcal{A}$* if the term operation $t^A : A^m \to A$ induced by t on the algebra $\mathcal{A}$ depends essentially on its i-th input x_i. By $Ess(t, \mathcal{A})$ we denote the set of all variables which are essential in t with respect to the algebra $\mathcal{A}$.

Remark 1.2

1. The variable x_i is essential in the term x_i with respect to the non-trivial algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$, iff $|A| > 1$.
2. Let $var(t)$ be the set of all variables occurring in t. If the variable x_i does not occur in the term t then x_i is not essential in t with respect to any algebra since $Ess(t, \mathcal{A}) \subseteq var(t)$.

In [11] the following characterization for essential variables in a term was given.

Lemma 1.3 ([11]) *A variable x_i is essential in the m-ary term t with respect to a non-trivial algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$, iff*

$$\mathcal{A} \not\models t \approx \bar{h}(t),$$

where $h : X_m \to W_\tau(X_{m+1})$ is a mapping defined by $h(x_i) = x_{m+1}$ and $h(x_j) = x_j$ for all $j \neq i, j \in \{1, \ldots, m\}$ and where $\bar{h}$ is the extension of h to a mapping defined on terms, i.e.

$$\bar{h} : W_\tau(X_m) \to W_\tau(X_{m+1}).$$

Another characterization is given by:

Lemma 1.4 *A variable x_i for $1 \leq i \leq m$ is essential in the m-ary term t with respect to a non-trivial algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$, iff there exists a mapping $\beta : X_m \to W_\tau(X)$ with $\beta(x_j) = x_j$ for all $j \neq i$ and $\beta(x_i) = t'$ with $t' \in W_\tau(X)$ and $\mathcal{A} \not\models x_i \approx t'$ and such that $\mathcal{A} \not\models \bar{\beta}(t) \approx t$.*

Proof We prove the equivalence to the condition in Lemma 1.3. If x_i is essential in t with respect to $\mathcal{A}$, then there is a mapping $h : X_m \to W_\tau(X_{m+1})$ which is defined by $h(x_i) = x_{m+1}$ and $h(x_j) = x_j$ for all $j \neq i, j \in \{1, \ldots, m\}$ such that

$$\bar{h} : W_\tau(X_m) \to W_\tau(X_{m+1})$$

satisfies $\mathcal{A} \not\models \bar{h}(t) \approx t$. If we choose $\beta = h$ and $t' = x_{m+1}$, then $\mathcal{A} \not\models x_i \approx t'$ since $\mathcal{A}$ is not trivial and $\mathcal{A} \not\models \bar{\beta}(t) \approx t$.
If conversely $\mathcal{A} \models t \approx \bar{h}(t)$ for the mapping h from Lemma 1.3 was satisfied, then by substitution of t' for x_{m+1} we obtain $\mathcal{A} \models \bar{\beta}(t) \approx t$, a contradiction. The contradiction shows that x_i is essential in t with respect to $\mathcal{A}$. $\square$

We notice that because of the freeness of the algebras $\mathcal{F}_\tau(X_m)$ and $\mathcal{F}_\tau(X)$ the extensions $\bar{h}$ and $\bar{\beta}$ exist and are uniquely determined. As a consequence, the variable x_i is essential in the m-ary term t with respect to an algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$, iff x_i is essential in t with respect to any $V(\mathcal{A})$- free algebra with at least $n+1$ free generators. One more corollary is:

Corollary 1.5 ([11]) *Let $s, t \in W_\tau(X_m), n \geq 1, m \geq 1$, and assume that $\mathcal{A}$ is an algebra of type τ. If $s \approx t$ is an identity in $\mathcal{A}$ then $Ess(s, \mathcal{A}) = Ess(t, \mathcal{A})$.*

Considering Lemma 1.3 it is quite natural to define variables which are essential in a term of a given type with respect to a variety of the same type.

Definition 1.6 Let V be a variety of type τ, let $t \in W_\tau(X_m)$. Then a variable $x_i \in X_m$ is called essential in t with respect to the variety V if it is essential in t with respect to the free algebra $\mathcal{F}_V(X)$ with $X = \{x_1, \ldots, x_n, \ldots\}$ as set of free generators. The set of all variables in t which are essential with respect to the variety V is denoted by $Ess(t, V)$.

Clearly, Corollary 1.5 is also satisfied for varieties instead of algebras. The following proposition is obvious.

Proposition 1.7 ([11]) *If $x_i \in X_m$ is essential in the m-ary term t of type τ with respect to the variety V of type τ and if V is a subvariety of W then x_i is essential in t with respect to W.*

2 Essential Variables in Operator Terms with respect to Menger Algebras

Terms over unitary Menger algebras of rank n are also called operator terms and are defined in the usual way, i.e. as terms of the type $\tau' = (n+1, 0, \ldots, 0)$ with an $(n+1)-ary$ operation symbol $\tilde{S}^n$ and n nullary operation symbols $\lambda_1, \ldots, \lambda_n$. We also need a new alphabet of variables, $\mathcal{X} := \{\tilde{X}_1 \ldots, \tilde{X}_n, \ldots\}$, or $\mathcal{X}_n := \{\tilde{X}_1 \ldots, \tilde{X}_n\}$ if finitely many variables are enough.

(i) $\tilde{X}_i$ is a term of type τ' for all variables $\tilde{X}_i$.

(ii) $\lambda_1, \ldots, \lambda_n$ are terms of type τ'.

(iii) If $T, T_1, \ldots, T_n$ are terms and if $\tilde{S}^n$ is $(n+1)-ary$, then $\tilde{S}^n(T, T_1, \ldots, T_n)$ is a term of type τ'.

By $W_{\tau'}(\mathcal{X}_n)$ we denote the set of all $n-ary$ terms of type τ' and let $W_{\tau'}(\mathcal{X})$ be the set of all terms of type τ'.
The set $W_{\tau'}(\mathcal{X})$ is the universe of the absolutely free algebra of type $\tau' =$

$(n+1, 0, \ldots, 0)$. If we factorize this algebra by the fully invariant congruence (equational theory) $\langle\{(C1),(C2),(C3)\}\rangle$ generated by the equations (C1), (C2), (C3), we obtain a free unitary Menger algebra of rank n, freely generated by the alphabet $\mathcal{X}$. With a free generating system $\mathcal{X}_I := \{\tilde{X}_i \mid i \in I\}$ we get a free unitary Menger algebra of rank n which is isomorphic to the algebra $n-clone\tau$. The elements of this free algebra are blocks with respect to the equational theory generated by (C1), (C2), (C3), but instead of $|T|$ we will only use the representative T. Further we consider a type τ_n where every operation symbol f_i has the same arity n. Then we have a bijection $\bar{\eta}$ between the set of all terms over the variety of unitary Menger algebras of rank n and the set $W_{\tau_n}(X_n)$ which can be defined inductively as follows: If $\eta : (\{\tilde{X}_i \mid i \in I\} \cup \{\lambda_1, \ldots, \lambda_n\})/\langle\{(C1),(C2),(C3)\}\rangle \to \{f_i(x_1, \ldots, x_n) \mid i \in I\} \cup \{x_1, \ldots, x_n\}$ is a bijection with $\eta(\tilde{X}_i) = f_i(x_1, \ldots, x_n)$ and $\eta(\lambda_i) = x_i$, then η can be extended to a bijection between $W_{\tau'}(\mathcal{X}_I)/\langle\{(\mathrm{C1}), (\mathrm{C2}), (\mathrm{C3})\}\rangle$ and $W_{\tau_n}(X_n)$. Indeed, if we define $\bar{\eta} : W_{\tau'}(\mathcal{X}_I)/\langle\{(C1),(C2),(C3)\}\rangle \to W_{\tau_n}(X_n)$ by

$$\begin{array}{lll} \bar{\eta}(\tilde{X}_i) & := & f_i(x_1, \ldots, x_n) \quad \text{for all } i \in I \\ \bar{\eta}(\lambda_i) & := & x_i \quad \text{for all } i = 1, 2, \ldots, n \\ \bar{\eta}(\tilde{S}^n(T_0, T_1, \ldots, T_n)) & := & S^n(\bar{\eta}(T_0), \bar{\eta}(T_1), \ldots, \bar{\eta}(T_n)), \end{array}$$

then we show that $\bar{\eta}$ is bijective. Given $t \in W_{\tau_n}(X_n)$. To prove that there exists a term $T \in W_{\tau'}(\mathcal{X}_I)/\langle\{(C1),(C2),(C3)\}\rangle$ such that $\bar{\eta}(T) = t$, we proceed by induction on the complexity of the term t. If $t = x_i$ for some $i \in \{1, 2, \ldots, n\}$, then there exists a $\lambda_i \in W_{\tau'}(\mathcal{X}_I)/\langle\{(C1),(C2),(C3)\}\rangle$ such that $\bar{\eta}(\lambda_i) := x_i$. If $t = f_i(t_1, \ldots, t_n)$ and assume that there exist $T_1, T_2, \ldots, T_n \in W_{\tau'}(\mathcal{X}_I)/\langle\{(\mathrm{C1}), (\mathrm{C2}), (\mathrm{C3})\}\rangle$ such that $\bar{\eta}(T_1) = t_1, \ldots, \bar{\eta}(T_n) = t_n$, then we have $\bar{\eta}(\tilde{S}^n(\tilde{X}_i, T_1, \ldots, T_n)) \in W_{\tau'}(\mathcal{X}_I)/\langle\{(\mathrm{C1}), (\mathrm{C2}), (\mathrm{C3})\}\rangle$ such that

$$\bar{\eta}(\tilde{S}^n(\tilde{X}_i, T_1, \ldots, T_n)) = S^n(\bar{\eta}(\tilde{X}_i), \bar{\eta}(T_1), \ldots, \bar{\eta}(T_n)) = f_i(t_1, \ldots, t_n).$$

This shows that $\bar{\eta}$ is surjective.

Let $T, T' \in W_{\tau'}(\mathcal{X}_I)/\langle\{(\mathrm{C1}), (\mathrm{C2}), (\mathrm{C3})\}\rangle$ with $\bar{\eta}(T) = \bar{\eta}(T')$. We want to prove that $T = T'$ and proceed by induction on the complexity of the term T. If $T = \tilde{X}_i$, then $\bar{\eta}(\tilde{X}_i) = f_i(x_1, \ldots, x_n) = \bar{\eta}(T')$ and so we get $T' = \tilde{X}_i$ (using the freeness of the algebra $\mathcal{F}_{\tau_n}(X_n)$).
If $T = \lambda_i$, then $\bar{\eta}(\lambda_i) = x_i = \bar{\eta}(T')$ and so we get $T' = \lambda_i$.

Assume that $T = \tilde{S}^n(T_0, T_1, \ldots, T_n)$, $T' = \tilde{S}^n(T'_0, T'_1, \ldots, T'_n)$ and $\bar{\eta}(T) = \bar{\eta}(T')$. Further we assume that $\bar{\eta}(T_j) = \bar{\eta}(T'_j)$ implies $T_j = T'_j$ for all $j = 0, 1, \ldots n$. By definition we have
$S^n(\bar{\eta}(T_0), \bar{\eta}(T_1), \ldots, \bar{\eta}(T_n)) = S^n(\bar{\eta}(T'_0), \bar{\eta}(T'_1), \ldots, \bar{\eta}(T'_n))$.
Since the absolutely free algebra $\mathcal{F}_{\tau_n}(X_n)$ satisfies no identities, there follows $\bar{\eta}(T_j) = \bar{\eta}(T'_j)$ for all $j = 0, 1, \ldots n$ and then $T_j = T'_j$ for all $j = 0, 1, \ldots n$ by our hypothesis. Therefore $T = T'$, this means $\bar{\eta}$ is injective. Altogether, $\bar{\eta}$ is bijective.

Now we apply the definition of essential variables in terms to essential variables in operator terms.

Let $\mathcal{A}$ be a non-trivial algebra of type τ and let $T \in W_{\tau'}(\mathcal{X})$. Then by definition, $\tilde{X}_i, 1 \leq i \leq m$, is essential in T with respect to $n - clone\mathcal{A}$ iff

$$n - clone\mathcal{A} \not\models \bar{\beta}(T) = T,$$

where $\beta : \mathcal{X}_m \rightarrow W_{\tau'}(\mathcal{X}_{m+1})$ is a mapping defined by

$\beta(\tilde{X}_j) = \tilde{X}_j$ for all $j \neq i$ and $\beta(\tilde{X}_i) = \tilde{X}_{m+1}$

and where $\bar{\beta}$ is the extension of β to a mapping defined on terms, i.e. $\bar{\beta} : W_{\tau'}(\mathcal{X}_m) \rightarrow W_{\tau'}(\mathcal{X}_{m+1})$.

3 Essential Operation Symbols in Terms

To extend the definiton of essential variables in terms to essential operation symbols in terms, we need the concept of a hypersubstitution (see e.g. [4], [7], [5]).

Hypersubstitutions of type τ are mappings which assign to each $n_i - ary$ operation symbol of type τ an $n_i - ary$ term of the same type. If $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ is a hypersubstitution of type τ then its extension $\hat{\sigma} : W_\tau(X_n) \rightarrow W_\tau(X_n)$ is defined inductively by the following steps:

(i) If $t = x_i$ for some $1 \leq i \leq n$, then $\hat{\sigma}[t] = x_i$.

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ for the $n_i - ary$ operation symbol f_i and some $n_i - ary$ terms t_j, then $\hat{\sigma}[t] = S_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. Together with the identity hypersubstitution σ_{id} mapping f_j to $f_j(x_1, \ldots, x_{n_j})$ for all $j \in I$ we get a monoid $(Hyp(\tau); \circ_h, \sigma_{id})$, where $\circ_h$ is defined by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$.

An identity $s \approx t$ of terms of type τ is called a hyperidentity of a variety V if for every substitution of $n_i - ary$ terms of V for the operation symbols f_i in $s \approx t$ the resulting identity holds in $V (i \in I)$, i.e. if $V \models \hat{\sigma}[s] \approx \hat{\sigma}[t]$ for every $\sigma \in Hpy(\tau)$. If $s \approx t$ is a hyperidentity in V, we will also write $V \underset{hyp}{\models} s \approx t$.

Definition 3.1 *Let $t \in W_\tau(X_n)$ be an $n - ary$ term of type τ, let $\mathcal{A}$ be an algebra of type τ and let $op_s(t)$ be the set of all operation symbols occurring in* *the term t. An operation symbol f_i of arity n_i is essential in t with respect to $\mathcal{A}$ if there is a hypersubstitution σ of type τ and an n_i-ary term $t' \in W_\tau(X_{n_i})$ such that $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ for every $j \neq i, j \in I$ and $\sigma(f_i) = t'$ with $f_i(x_1, \ldots, x_{n_i}) \approx t'$ is not an identity in $\mathcal{A}$ and $\mathcal{A} \not\models \hat{\sigma}[t] \approx t$. In the opposite case, f_i is called fictitious in t with respect to $\mathcal{A}$. Let $Hypess(t, \mathcal{A})$ be the set of all essential operation symbols in t with respect to $\mathcal{A}$.*

This definition corresponds to Lemma 1.4. We have the following simple consequences.

Corollary 3.2 *Let $s, t \in W_\tau(X_n)$ and let $\mathcal{A}$ be an algebra of type τ. Then*

$$\mathcal{A} \underset{hyp}{\models} s \approx t \Longrightarrow Hypess(s, \mathcal{A}) = Hypess(t, \mathcal{A}).$$

Proof Assume that $f_i \in Hypess(t, \mathcal{A})$. Then there exists a hypersubstitution $\sigma \in Hyp(\tau)$ and an $n_i - ary$ term t' such that $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ for every $j \neq i, j \in I$ and $\sigma(f_i) = t'$ with $\mathcal{A} \not\models f_i(x_1, \ldots, x_{n_i}) \approx t'$ and $\mathcal{A} \not\models \hat{\sigma}[t] \approx t$.
If $\mathcal{A} \models \hat{\sigma}[s] \approx s$ then together with $\mathcal{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t]$ we would have $\mathcal{A} \models \hat{\sigma}[t] \approx t$, a contradiction. Therefore $\mathcal{A} \not\models \hat{\sigma}[s] \approx s$ and $f_i \in Hypess(s, \mathcal{A})$. This shows

$$Hypess(t, \mathcal{A}) \subseteq Hypess(s, \mathcal{A})$$

and similarly,

$$Hypess(s, \mathcal{A}) \subseteq Hypess(t, \mathcal{A}).$$

□

For the set of all operation symbols occurring in the term t, $f_i \notin op_s(t)$ implies $f_i \notin Hypess(t, \mathcal{A})$. Further, for variables x_i we have $Hypess(x_i, \mathcal{A}) = \emptyset$.

One more consequence of the definition is:

Proposition 3.3 *Let $\mathcal{A}$ be an algebra of type τ and assume that the type contains one at least binary operation symbol f_i. Further we assume that $\mathcal{A}$ does not satisfy $f_i(x_1, \ldots, x_{n_i}) \approx x_i$ for $1 \leq i \leq n_i$. Then $\mathcal{A}$ is trivial iff for every $t \in W_\tau(X)$ we have $Hypess(t, \mathcal{A}) = \emptyset$.*

Proof The trivial algebra satisfies $\hat{\sigma}[t] \approx t$ for every hypersubstitution σ and for every term t. Therefore, there is no essential operation symbol in t with respect to the trivial algebra $\mathcal{A}$.

Conversely, assume that for every term $t \in W_\tau(X)$ we have $Hypess(t, \mathcal{A}) = \emptyset$. That means, for every hypersubstitution $\sigma \in Hyp(\tau)$ and for every $i \in I$ with $\sigma(f_i) = t'$ where $f_i(x_1, \ldots, x_{n_i}) \approx t'$ is not an identity in $\mathcal{A}$ and $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ if $j \neq i$ we have $\mathcal{A} \models \hat{\sigma}[t] \approx t$. Consider the hypersubstitutions

σ_1 with $\sigma_1(f_i) = x_1$ and $\sigma_1(f_j) = f_j(x_1, \ldots, x_{n_i})$ if $j \neq i$ and $\sigma_2(f_i) = x_2$ and $\sigma_2(f_j) = f_j(x_1, \ldots, x_{n_i})$ if $j \neq i$. Since f_i is at least binary, we get $x_1 \neq x_2$. Let t be an arbitrary term containing at least one operation symbol such that the first variable occurring in t is x_1 and the last one is x_2. Then $\hat{\sigma}_1[t] = x_1 \approx t \in Id\mathcal{A}$ and $\hat{\sigma}_2[t] = x_2 \approx t \in Id\mathcal{A}$ and therefore $x_1 \approx x_2 \in Id\mathcal{A}$ and $\mathcal{A}$ is trivial. $\square$

As in the case of essential variables in terms with respect to algebras we may extend our definition to varieties.

Definition 3.4 *Let V be a variety of type τ and assume that $t \in W_\tau(X_n)$. Then f_i is called essential in t with respect to the variety V if f_i is essential in t with respect to the free algebra $F_V(X)$. Let $Hypess(t, V)$ be the set of all operation symbols which are essential in t with respect to V.*

An easy consequence of this definition is

Corollary 3.5 *If V is a subvariety of W, then $Hypess(t, V) \subseteq Hypess(t, W)$.*

Proof If $V \subseteq W$, we get $IdW \subseteq IdV$ for the sets of all identities satisfied in W and in V respectively. Assume that $f_i \notin Hypess(t, W)$. Then for every $\sigma \in Hyp(\tau)$ such that $\sigma(f_i) = t' \approx f_i(x_1, \ldots, x_{n_i})$ is not an identity in $\mathcal{A}$ where t' is an n_i -ary term of type τ and $\sigma(f_j) = f_j(x_1, \ldots, x_{n_j})$ if $j \neq i$, we have $W \models \hat{\sigma}[t] \approx t$. But then $V \models \hat{\sigma}[t] \approx t$ and $f_i \notin Hypess(t, V)$. $\square$

The definition of an essential operation symbol in a term can be generalized to that of an essential operation symbol in a hypersubstitution of type τ.

Definition 3.6 *Let V be a variety of type τ and let $\sigma \in Hyp(\tau)$. Then f_i is essential in σ with respect to V iff f_i is essential in the term $\sigma(f_i)$ with respect to V, for all $i \in I$.*

For the type $\tau = (n), n \geq 2$, and the $n-ary$ operation symbol f, the operation symbol f is essential in σ with respect to V iff f is essential in $\sigma(f)$ with respect to V.

Now we consider the set of all hypersubstitutions σ for which f is essential in σ with respect to V, i.e.

$$M(V) := \{\sigma \mid f \in Hypess(\sigma(f), V)\}.$$

Since f is not essential in a variable x_i, the hypersubstitution which maps f to x_i, for some $1 \leq i \leq n$, does not belong to $M(V)$.
Let $Pre(n) := Hyp(n) \backslash \{\sigma_{x_i} \mid 1 \leq i \leq n\}$. The set $Pre(n)$ forms a submonoid

of $Hyp(n)$. The set $M(V)$ is included in $Pre(n)$. We ask whether $M(V)$ is a submonoid of $Pre(n)$.

To answer to this question we need the following well-known auxiliary result (for type $\tau = (2)$ see e.g [9] or [10]).

Lemma 3.7 *Let $V_i = Mod\{f(x_1, \ldots, x_n) \approx x_i\}$. Then V_i is a minimal variety.*

Proof Let V be a variety such that $V \subseteq V_i$ and $V \neq V_i$. We have to prove that V is trivial. We will use the fact that $s \approx t \in IdV_i$ iff there exists a variable x_j which occurs in s and in t such that $s \approx x_j \in IdV_i$ and $t \approx x_j \in IdV_i$. Since $V \subseteq V_i$ and $V \neq V_i$, there are terms $s, t \in W_{(n)}(X_n)$ such that $s \approx t \in IdV$ and $s \approx t \notin IdV_i$. Then there exist variables x_j, x_k, with $j \neq k$ such that $s \approx x_j \in IdV_i$ and $t \approx x_k \in IdV_i$. Therefore $s \approx x_j \in IdV$ and $t \approx x_k \in IdV$ and then $x_j \approx x_k \in IdV$ since $s \approx t \in IdV$. That is, V is trivial. $\square$

Proposition 3.8 *Let $\tau = (n), n \geq 2$, and let V be a non-trivial variety of type (n) with $V \neq V_i = Mod\{f(x_1, \ldots, x_n) \approx x_i\}, 1 \leq i \leq n$. Then $M(V) = \{\sigma \mid f \in Hypess(\sigma(f), V)\}$ is a submonoid of $Pre(n)$.*

Proof Since $V \neq V_i$ and since every variety W with $W \subset V_i$ is trivial, the hypersubstitution $\sigma \in Hyp(n)$ with $\sigma(f) = x_i$ satisfies $V \not\models \sigma(f) \approx f(x_1, \ldots, x_n)$ this means $V \not\models \hat{\sigma}[f(x_1, \ldots, x_n)] \approx f(x_1, \ldots, x_n)$ and then $V \not\models \hat{\sigma}[\sigma_{id}(f)] \approx \sigma_{id}(f)$. Therefore $f \in Hypess(\sigma_{id}(f), V)$ and we get $\sigma_{id} \in M(V)$.

Let $\sigma_1, \sigma_2 \in M(V)$. Then $f \in Hypess(\sigma_1(f), V)$ and $f \in Hypess(\sigma_2(f), V)$ with $\sigma_1(f) \neq x_i, \sigma_2(f) \neq x_i$ for all $i = 1, 2, \ldots, n$ and there are hypersubstitutions $\sigma, \sigma' \in Hyp(n)$ with $\sigma(f) = t,\ \sigma'(f) = t'$ where $t, t' \in W_{(n)}(X_n)$ such that $V \not\models t \approx f(x_1, \ldots, x_n), V \not\models t' \approx f(x_1, \ldots, x_n)$, $V \not\models \hat{\sigma}[\sigma_1(f)] \approx \sigma_1(f)$ and $V \not\models \hat{\sigma}'[\sigma_2(f)] \approx \sigma_2(f)$.
Let $(\sigma_1 \circ_h \sigma_2)(f) = \hat{\sigma}_1[\sigma_2(f)] = w$ where w is an $n-ary$ term with $w \neq x_i$ for all $i = 1, 2, \ldots, n$. Therefore, the hypersubstitution $\sigma'' \in Hyp(n)$ with $\sigma''(f) = x_l$ satisfies $V \not\models \sigma(f) \approx f(x_1, \ldots, x_n)$ and $V \not\models \hat{\sigma}''[w] \approx w$ since $V \neq V_i$ and w is not a variable, but $\hat{\sigma}[w]$ is a variable. This means, f is essential in $(\sigma_1 \circ_h \sigma_2)(f)$ with respect to V and so we have $\sigma_1 \circ_h \sigma_2 \in M(V)$. $\square$

4 Essential Operation Symbols in Terms and Essential Variables in Operator terms

Let η be the bijection between the set $(\{\tilde{X}_i \mid i \in I\} \cup \{\lambda_1, \ldots, \lambda_n\})/C$ with $C := \langle\{(C1), (C2), (C3)\}\rangle$ and the set $\{f_i(x_1, \ldots, x_n) \mid i \in I\} \cup \{x_1, \ldots, x_n\}$, let $\bar{\eta}$ be the extension of η introduced in section 2 and let α be the bijection which maps each f_i to $f_i(x_1, \ldots, x_n)$, $(i \in I)$. Since we are interested in essential variables in terms $T \in W_{\tau'}(\mathcal{X})$ with respect to $n-clone\mathcal{A}$ for an algebra $\mathcal{A}$ of type τ_n, we consider a mapping β and its extension $\bar{\beta}$ defined as in Lemma 1.4. Further we assume in addition that our type τ_n is finite that means, we have only m operation symbols. More precisely,
$\beta : (\mathcal{X}_m \cup \{\lambda_1, \ldots, \lambda_n\})/C \to W_{\tau'}(\mathcal{X}_m)/C$ is a mapping defined by $\beta(\tilde{X}_j) = \tilde{X}_j$ for all $j \neq i$ and $\beta(\tilde{X}_i) = T'$ with $T' \in W_{\tau'}(\mathcal{X}_m)/C$ and $n-clone\mathcal{A} \not\models \tilde{X}_i \approx T'$ and $\beta(\lambda_i) = \lambda_i$ for all $i = 1 \ldots, n$ and where $\bar{\beta}$ is the extension of β to a mapping defined on terms, i.e. $\bar{\beta} : W_{\tau'}(\mathcal{X}_m)/C \to W_{\tau'}(\mathcal{X}_m)/C$.

Then we define a hypersubstitution σ by $\sigma = \bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha$ and have

$$\begin{aligned}
\sigma(f_i) &= (\bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha)(f_i) = \bar{\eta} \circ \beta \circ \eta^{-1}(\alpha(f_i)) \\
&= \bar{\eta} \circ \beta \circ \eta^{-1}(f_i(x_1, \ldots, x_n)) \\
&= \bar{\eta} \circ \beta(\eta^{-1}(f_i(x_1, \ldots, x_n))) \\
&= \bar{\eta}(\beta(\tilde{X}_i)) = \bar{\eta}(T')
\end{aligned}$$

and for all $j \neq i$,

$$\begin{aligned}
\sigma(f_j) &= (\bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha)(f_j) = \bar{\eta} \circ \beta \circ \eta^{-1}(\alpha(f_j)) \\
&= \bar{\eta} \circ \beta \circ \eta^{-1}(f_j(x_1, \ldots, x_n)) \\
&= \bar{\eta} \circ \beta(\eta^{-1}(f_j(x_1, \ldots, x_n))) \\
&= \bar{\eta}(\beta(\tilde{X}_j)) = \bar{\eta}(\tilde{X}_j) \\
&= f_j(x_1, \ldots, x_n).
\end{aligned}$$

To prove the main theorem, we need two auxiliary results.

Lemma 4.1 *If* $\sigma = \bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha$, *then* $\hat{\sigma} = \bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}}$.

Proof Let $t \in W_{\tau_n}(X_n)$. It is easy to check by induction on the complexity of the term t that $\hat{\sigma}[t] = (\bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}})(t)$.

Indeed, if $t = x_i$, then $\hat{\sigma}[x_i] = x_i$ and $(\bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}})(x_i) = (\bar{\eta} \circ \bar{\beta})(\overline{\eta^{-1}}(x_i)) = \bar{\eta}(\bar{\beta}(\lambda_i)) = \bar{\eta}(\lambda_i) = x_i$ and so we get $\hat{\sigma}[x_i] = (\bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}})(x_i)$.

If $t = f_i(t_1, \ldots, t_n)$ and assume that $\hat{\sigma}[t_1] = (\bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}})(t_1), \ldots, \hat{\sigma}[t_n] = (\bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}})(t_n)$, then

$$\begin{aligned}
\hat{\sigma}[t] &= \hat{\sigma}[f_i(t_1, \ldots, t_n)] = S^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n((\bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha)(f_i), (\bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}})(t_1), \ldots, (\bar{\eta} \circ \bar{\beta} \circ \overline{\eta^{-1}})(t_n))
\end{aligned}$$

$$
\begin{aligned}
&= S^n(\bar{\eta}((\beta\circ\eta^{-1}\circ\alpha)(f_i)), \bar{\eta}((\bar{\beta}\circ\overline{\eta^{-1}})(t_1)), \ldots, \bar{\eta}((\bar{\beta}\circ\overline{\eta^{-1}})(t_n)))\\
&= \bar{\eta}(S^n((\beta\circ\eta^{-1}\circ\alpha)(f_i), (\bar{\beta}\circ\overline{\eta^{-1}})(t_1), \ldots, (\bar{\beta}\circ\overline{\eta^{-1}})(t_n))) \quad \text{by section 2}\\
&= \bar{\eta}(S^n((\beta\circ\eta^{-1})(\alpha(f_i)), \bar{\beta}(\overline{\eta^{-1}}(t_1)), \ldots, \bar{\beta}(\overline{\eta^{-1}}(t_n))))\\
&= \bar{\eta}(S^n((\beta\circ\eta^{-1})(f_i(x_1,\ldots,x_n)), \bar{\beta}(\overline{\eta^{-1}}(t_1)), \ldots, \bar{\beta}(\overline{\eta^{-1}}(t_n))))\\
&= \bar{\eta}(S^n(\beta(\eta^{-1}(f_i(x_1,\ldots,x_n))), \bar{\beta}(\overline{\eta^{-1}}(t_1)), \ldots, \bar{\beta}(\overline{\eta^{-1}}(t_n))))\\
&= \bar{\eta}(S^n(\beta(\tilde{X}_i), \bar{\beta}(\overline{\eta^{-1}}(t_1)), \ldots, \bar{\beta}(\overline{\eta^{-1}}(t_n))))\\
&= \bar{\eta}(S^n(\bar{\beta}(\tilde{X}_i), \bar{\beta}(\overline{\eta^{-1}}(t_1)), \ldots, \bar{\beta}(\overline{\eta^{-1}}(t_n)))\\
&= (\bar{\eta}\circ\bar{\beta})(S^n(\overline{\eta^{-1}}(f_i(x_1,\ldots,x_n)), \overline{\eta^{-1}}(t_1), \ldots, \overline{\eta^{-1}}(t_n)))\\
&\qquad \text{because of the freeness of } \mathcal{F}_{\tau'}(\mathcal{X})/C\\
&= (\bar{\eta}\circ\bar{\beta})(\overline{\eta^{-1}}(S^n(f_i(x_1,\ldots,x_n), t_1, \ldots, t_n)))\\
&= (\bar{\eta}\circ\bar{\beta}\circ\overline{\eta^{-1}})(f_i(t_1,\ldots,t_n)) = (\bar{\eta}\circ\bar{\beta}\circ\overline{\eta^{-1}})(t).
\end{aligned}
$$

Therefore $\hat{\sigma} = \bar{\eta}\circ\bar{\beta}\circ\overline{\eta^{-1}}$. □

Lemma 4.2 *Let $\mathcal{A}$ be be a non-trivial algebra of type τ_n, let $T \in W_{\tau'}(\mathcal{X}_m)/C$ and let $\bar{\eta}(\beta(\tilde{X}_i)) \approx f_i(x_1,\ldots,x_n) \notin Id\mathcal{A}$. Then*

$$\bar{\beta}(T)^{n-clone\mathcal{A}} \neq T^{n-clone\mathcal{A}} \iff \bar{\eta}(\bar{\beta}(T))^{\mathcal{A}} \neq \bar{\eta}(T)^{\mathcal{A}}.$$

Proof Let $T \in W_{\tau'}(\mathcal{X}_m)/C$. We proceed by induction on the complexity on the term T.

Assume that $T = \tilde{X}_i$, for $i \neq j$, because of our assumption we have $\bar{\eta}(\beta(\tilde{X}_i)) \approx f_i(x_1,\ldots,x_n) \notin Id\mathcal{A}$ and using η we have $\bar{\beta}(\tilde{X}_i)^{n-clone\mathcal{A}} \neq (\tilde{X}_i)^{n-clone\mathcal{A}}$.
By $n-clone\mathcal{A} \not\models \tilde{X}_i \approx T'$ we have $n-clone\mathcal{A} \not\models \tilde{X}_i \approx \bar{\beta}(\tilde{X}_i)$. This means, $\bar{\eta}(\bar{\beta}(\tilde{X}_i))^{\mathcal{A}} \neq \bar{\eta}(\tilde{X}_i)^{\mathcal{A}}$.

Because of the definition of $\bar{\beta}(\tilde{X}_i)$ we may assume that $\bar{\beta}(\tilde{X}_i)$ in not a variable. Assume that $T = \tilde{S}^n(T_0, T_1, \ldots, T_n)$ and let $\bar{\beta}(T) = T' = \tilde{S}^n(T_0', T_1', \ldots, T_n')$ and suppose that $\bar{\eta}(T_j')^{\mathcal{A}} = \bar{\eta}(T_j)^{\mathcal{A}}$ for all $j = 1, 2, \ldots, n$ iff $(T_j')^{n-clone\mathcal{A}}$ $= (T_j)^{n-clone\mathcal{A}}$ for all $j = 1, 2, \ldots, n$. Then we have $\bar{\eta}(\bar{\beta}(T))^{\mathcal{A}}$

$$
\begin{aligned}
&= S^{n,\mathcal{A}}(\bar{\eta}(T_0')^{\mathcal{A}}, \bar{\eta}(T_1')^{\mathcal{A}}, \ldots, \bar{\eta}(T_n')^{\mathcal{A}})\\
&= \bar{\eta}(T)^{\mathcal{A}}\\
&= S^{n,\mathcal{A}}(\bar{\eta}(T_0)^{\mathcal{A}}, \bar{\eta}(T_1)^{\mathcal{A}}, \ldots, \bar{\eta}(T_n)^{\mathcal{A}})\\
\text{iff}\quad & \bar{\beta}(T)^{n-clone\mathcal{A}}\\
&= S^{n,n-clone\mathcal{A}}((T_0')^{n-clone\mathcal{A}}, \ldots, (T_n')^{n-clone\mathcal{A}})\\
&= S^{n,n-clone\mathcal{A}}((T_0)^{n-clone\mathcal{A}}, \ldots, (T_n)^{n-clone\mathcal{A}})\\
&= T^{n-clone\mathcal{A}}
\end{aligned}
$$

and the desired result follows. □

Now we want to prove that essential variables in operator terms with respect to the $n-ary$ clone of an algebra $\mathcal{A}$ and essential operation symbols in terms with respect to the algebra $\mathcal{A}$ are equivalent.

Theorem 4.3 *Let $\mathcal{A}$ be a non-trivial algebra of type τ_n and let $T \in W_{\tau'}(\mathcal{X})/C$. Then*

$$\tilde{X}_i \in Ess(T, n-clone\mathcal{A}) \iff f_i \in Hypess(\bar{\eta}(T), \mathcal{A}).$$

Proof Let $T \in W_{\tau'}(\mathcal{X})/C$ and assume that $\tilde{X}_i \in Ess(T, n-clone\mathcal{A})$. Then there is a natural number m such that $T \in W_{\tau'}(\mathcal{X}_m)/C$ and we have $T \neq \tilde{X}_j$ and $T \neq \lambda_i$ for all $j = 1, 2, \ldots, m$ and for all $i = 1, 2, \ldots, n$. Then there is a mapping $\beta : \mathcal{X}_m/C \to W_{\tau'}(\mathcal{X}_m)/C$ with $\beta(\tilde{X}_j) = \tilde{X}_j$ for all $j \neq i$ and $\beta(\tilde{X}_i) = T'$ with $T' \in W_{\tau'}(\mathcal{X}_m)/C$ such that $n-clone\mathcal{A} \not\models \tilde{X}_i \approx T'$ and $n-clone\mathcal{A} \not\models \bar{\beta}(T) \approx T$.
We want to prove that $f_i \in Hypess(\bar{\eta}(T), \mathcal{A})$. Since $\tilde{X}_i \in var(T)$, we have $f_i \in op_s(\bar{\eta}(T))$. Now we define a hypersubstitution σ of type τ_n with $\sigma = \bar{\eta} \circ \beta \circ \eta^{-1} \circ \alpha$ such that $\sigma(f_j) = f_j(x_1, \ldots, x_n)$ if $j \neq i$ and $\sigma(f_i) = \bar{\eta}(T')$. By Lemma 4.2 we get $\mathcal{A} \not\models f_i(x_1, \ldots, x_n) \approx \bar{\eta}(T')$ and $\mathcal{A} \not\models \bar{\eta}(\bar{\beta}(T)) \approx \bar{\eta}(T)$ and then $\mathcal{A} \not\models \hat{\sigma}[\bar{\eta}(T)] \approx \bar{\eta}(T)$. This means, $f_i \in Hypess(\bar{\eta}(T), \mathcal{A})$.

Conversely, assume that $f_i \in Hypess(\bar{\eta}(T), \mathcal{A})$. Then there exists a hypersubstitution σ of type τ_n with $\sigma(f_j) = f_j(x_1, \ldots, x_n)$ for every $j \neq i, j \in I$ and $\sigma(f_i) = t'$ with $t' \in W_{\tau'}(X_n)$ such that $f_i(x_1, \ldots, x_n) \approx t'$ is not an identity in $\mathcal{A}$ and $\mathcal{A} \not\models \hat{\sigma}[\bar{\eta}(T)] \approx \bar{\eta}(T)$. We show that $\tilde{X}_i \in Ess(T, n-clone\mathcal{A})$. Since $\bar{\eta}(T) \neq x_i$ for all $i = 1, \ldots, n$ and $f_i \in op_s(\bar{\eta})$, we have $\tilde{X}_i \in var(T)$. Now we define a mapping $\beta : \mathcal{X}_m/C \to W_{\tau'}(\mathcal{X}_m)/C$ with $\beta(\tilde{X}_j) = \tilde{X}_j$ for all $j \neq i$ and $\beta(\tilde{X}_i) = \overline{\eta^{-1}}(t')$. By Lemma 4.2 we have $n-clone\mathcal{A} \not\models \tilde{X}_i \approx \bar{\beta}(\tilde{X}_i)$ and $n-clone\mathcal{A} \not\models \bar{\beta}(T) \approx T$. This means, $\tilde{X}_i \in Ess(T, n-clone\mathcal{A})$. □

Remark 4.4 If V is a variety of a finite type τ_n, then we may consider $n-$ *cloneV* consisting of classes of terms of type τ_n with respect to IdV. Then we have

$$\tilde{X}_i \in Ess(T, n-cloneV) \iff f_i \in Hypess(\bar{\eta}(T), V).$$

References

[1] J. Breitbart, *Essential variables of Boolean functions*, (in Russian), Dokl. Akad. Nauk SSSR, **172**(1967), 9-10.

[2] K. N. Chimev, *On some properties of functions, abstracts of lectures of the colloquium on finite algebra and multiple-valued logic*, Szeged 1979, pp. 38 - 40.

[3] K. N. Chimev, *Separable sets of arguments of functions*, Computer and Automation Institute, Hungarian Academy of Sciences Studies 180/1986.

[4] K. Denecke, S. L. Wismath, *Hyperidentities and Clones,* Series Algebra, Logic and Applications, Vol. 14, Gordon and Breach Science Publishers, 2000.

[5] K. Denecke, J. Koppitz, *Essential variables in hypersubstitutions,* Algebra Universalis **46**(2001), 443-454.

[6] K. Denecke, P. Jampachon, S. L. Wismath, *Clones of* $n-ary$ *algebras,* Journal of Applied Algebra and Discrete Structures **1**(2003), 141-158.

[7] K. Denecke, J. Koppitz, *M-solid Varieties*, to appear in Springer-Verlag 2005.

[8] A. Ehrenfeucht, J. Kahn, R. Maddux, J. Mycielski, *On the Dependence of Functions on Their Variables,* J. Comb. Theory, A-33(1982),106-108.

[9] T. Evans, *The lattice of semigroup varieties,* Semigroup Forum, **2**(1971),1-43.

[10] J. A. Gerhard, *The lattice of equational classes of idempotent semigroups*, J. Algebra, **15**(1970), 195-224.

[11] SL. Shtrakov, K. Denecke, *Essential variables and separable sets in Universal Algebra,* Multiple-Valued Logic in Eastern Europe, Mult.-Valued Logic 8 (2002), no. 2, 165-181.

[12] B. Schein B., S. Trochimenko, *Algebras of Multiplace Functions,* Semigroup Forum **17** (1979), 1 - 64.