

THE P-ADIC FIELD CASE OF THE FUNCTIONAL EQUATION $P(f) = Q(g)$

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Abstract

In this paper, we study the existence of non constant meromorphic solutions f and g of the functional equation $P(f) = Q(g)$, where $P(z)$ and $Q(z)$ are two given nonlinear polynomials with coefficients in the non-Archimedean field \mathbb{K} .

1. Introduction

The Tenth problem of Hilbert is asked to establish an algorithm for finding all integer solutions of $F(x, y) = 0$, where $F(x, y)$ is a polynomial with integer coefficients. It is natural to study the analogous of Hilbert's Tenth problem in the field of meromorphic functions. More specifically, we ask what forms of equation $F(x, y) = 0$, where $F(x, y)$ is a polynomial with complex coefficients, may or may not have non constant meromorphic functions f and g that satisfy $F(f, g) = 0$? Earlier in 1920s, as a simple application of his value distribution theory, Nevanlinna proved that a non constant meromorphic function (in the complex plane) is uniquely determined by the inverse image of five distinct values (including infinity), ignoring multiplicity. Gross [11] extended this study by considering pre-images of a set and posed the question: *Is there a finite set A so that an entire (meromorphic) function is uniquely determined by the pre-image of the set A , counting multiplicities?* Let f be a non-constant meromorphic

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function and S be a subset of distinct elements. Define

$$E_f(S) = \bigcup_{a \in S} \{(z, m) \mid f(z) = a \text{ with multiplicity } m\}.$$

Two functions f and g of the same type are said to *share* S , *counting multiplicity*, if $E_f(S) = E_g(S)$. A subset S is called a *unique range set* (a *URS* in short) for entire (or meromorphic) functions if for any two non-constant entire (or meromorphic) functions f and g such that $E_f(S) = E_g(S)$, one has $f=g$. Assume that S be a finite set, we set :

$$P_S(z) = \prod_{a \in S} (x - a).$$

As a connection to the study of the uniqueness problem, Li and Yang [3] introduced the following definition:

Definition 1 A non-constant polynomial $P(z)$ is said to be a *unique polynomial* for entire (or meromorphic) functions if for two non-constant entire (or meromorphic) functions f and g , $P(f) = P(g)$ implies that $f = g$.

$P(z)$ is said to be a *strong uniqueness polynomial* for entire (or meromorphic) functions if for two non-constant entire (or meromorphic) functions f, g and some nonzero constant c , the condition $P(f) = cP(g)$ implies that $c = 1$ and $f = g$.

To demonstrate that the finite set S be a URS for entire (or meromorphic) functions, we prove that the polynomial $P_S(z)$ is a strong uniqueness polynomial. If P is a strong uniqueness polynomial for entire (or meromorphic) functions, then the set of the zeros of P can be a URS.

Recently, H.H. Khoai and C.C. Yang [1] generalized the above studies by considering a pair of two nonlinear polynomials $P(z)$ and $Q(z)$ such that the only meromorphic solutions f, g satisfying $P(f) = Q(g)$ are constants. This problem is considered in the complex plane \mathbb{C} by H.H. Khoai and C.C. Yang [1] as well as by C.C. Yang and P. Li [2].

In this paper, we find the conditions for the existence of non-constant meromorphic function solutions f and g of the functional equation $P(f) = Q(g)$ in \mathbb{K} with \mathbb{K} being an algebraically closed field, complete for a non-trivial non-Archimedean absolute value. To solve the functional equation, we study the hyperbolicity of the algebraic curve $\{P(x) - Q(y) = 0\}$ by estimating its genus. We shall do this by giving sufficiently many linear independent regular 1-forms of Wronskian type on that curve.

2. Main theorems

Theorem 2.1 *Let $P(z)$ and $Q(z)$ be two nonlinear polynomials of degrees n and m , respectively, with $n \geq m$. Suppose that $P(\alpha) \neq Q(\beta)$ for all zeros α*

of P' and β of Q' . Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$, if n and m satisfy the following condition:

$$n \geq m \geq 2 \text{ and } n \geq 3.$$

Theorem 2.2 Let $P(z)$ and $Q(z)$ be two nonlinear polynomials of degrees n and m , respectively. Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$ provided that P and Q satisfy one of the following conditions:

1. There exists a zero β_1 of Q' with multiplicity m_1 at least 2 and $P(\alpha) \neq Q(\beta_1)$, for all zeros α of P' .
2. There exists two simple zeros β_1, β_2 of Q' such that $P(\alpha) \neq Q(\beta_i)$ for all zeros α of P' and $i = 1, 2$.

Definition 2 Let $P(z)$ be a nonlinear polynomial of degree n whose derivative is given by:

$$P'(z) = c(z - \alpha_1)^{n_1} \dots (z - \alpha_k)^{n_k},$$

where $n_1 + \dots + n_k = n - 1$ and $\alpha_1, \dots, \alpha_k$ are distinct zeros of P' . The number k is called the *derivative index* of P .

A polynomial $P(z)$ is said to satisfy the *condition of separating the roots of P'* (separation condition) if $P(\alpha_i) \neq P(\alpha_j)$ for all $k \geq i \neq j \geq 1$.

Theorem 2.3 Let $P(z)$ and $Q(z)$ be two polynomials defined by (1) and let P satisfy the separation condition. Suppose that β_1, \dots, β_J are distinct zeros of Q' with multiplicity m_j , respectively, such that for every β_j , $j = 1, 2, \dots, J$, there exists zeros α_i of P' with $P(\alpha_i) = Q(\beta_j)$. Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$ if

$$m - \sum_{j=1}^J m_j \geq 3.$$

Remark 2.4 In case $\deg P = \deg Q = 2$, the equation $P(f) = Q(g)$ has some non-constant meromorphic functions. Indeed, in this case we can rewrite the equation $P(f) = Q(g)$ in the form:

$$(f - a)^2 = (bg - c)^2 + d,$$

where $a, b, c, d \in \mathbb{K}$ and $b \neq 0$. Hence

$$(f - bg - a + c)(f + bg - a - c) = d.$$

Assume that h is a non-constant meromorphic function, we set

$$f = \frac{1}{2}\left(h + \frac{d}{h}\right) + a, \quad g = \frac{1}{2b}\left(-h + \frac{d}{h}\right) + \frac{c}{b}.$$

Then f and g are non-constant meromorphic solutions of equation $P(f) = Q(g)$.

3. Lemmas and Proofs

Let \mathbb{K} be an algebraically closed field, complete for a non-trivial non-Archimedean absolute value with characteristic zero. Suppose that $H(X, Y, Z)$ is a homogeneous polynomial of degree n and

$$C := \{(X, Y, Z) \in \mathbb{P}^2(\mathbb{K}) \mid H(X, Y, Z) = 0\}.$$

Denote

$$\begin{aligned} W_1 &= W(X, Y) = \begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}. \\ W_2 &= W(Y, Z) = \begin{vmatrix} Y & Z \\ dY & dZ \end{vmatrix}. \\ W_3 &= W(X, Z) = \begin{vmatrix} X & Z \\ dX & dZ \end{vmatrix}. \end{aligned}$$

Assume that $R(X, Y, Z)$ and $S(X, Y, Z)$ are two homogeneous polynomials in $\mathbb{P}^2(\mathbb{K})$. Let

$$\omega_i = \frac{R(X, Y, Z)}{S(X, Y, Z)} W_i,$$

with $i = 1, 2, 3$. If $R(X, Y, Z)$ and $S(X, Y, Z)$ satisfy $\deg S = \deg R + 2$, then ω_i is a well-defined rational 1-form on $\mathbb{P}^2(\mathbb{K})$.

Definition 3 Let C be an algebraic curve in $\mathbb{P}^2(\mathbb{K})$. A 1-form ω on C is said to be *regular* if it is the pull-back of a rational 1-form on $\mathbb{P}^2(\mathbb{K})$ such that the pole set of ω does not intersect C . A well-defined rational regular 1-form on C is said to be an *1-form of Wronskian type*.

Notice that to solve the functional equation $P(f) = Q(g)$, is the same as to find meromorphic functions f, g on \mathbb{K} such that $(f(z), g(z))$ in curve $\{P(x) - Q(y) = 0\}$. On the other hand, if C is hyperbolic on \mathbb{K} and suppose that f, g be meromorphic functions such that $(f(z), g(z)) \in C$, where $z \in \mathbb{K}$, then f and g are constant (see.[6]). Therefore, to show that this equation has no non-constant solutions, we shall prove the hyperbolicity of $\{P(x) - Q(y) = 0\}$. By Picard-Berkovich's theorem in the p -adic case, a curve C in \mathbb{K} is Brody hyperbolic if and only if the genus of the curve C is at least 1.

It is well-known that the genus g of an algebraic curve C is equal to the dimension of the space of regular 1-forms on C . Therefore, to compute the genus, we have to construct a basis of the space of regular 1-forms on C .

Now, let P and Q be two nonlinear polynomials of degrees n and m in \mathbb{K} , respectively, with

$$\begin{aligned} P(x) &= a_n x^n + \dots + a_1 x + a_0 \\ Q(y) &= b_m y^m + \dots + b_1 y + b_0. \end{aligned} \tag{1}$$

Without loss of generality, we assume that $m \geq n$. Set

$$F_1(x, y) := P(x) - Q(y),$$

$$F(X, Y, Z) := Z^n \left\{ P\left(\frac{X}{Z}\right) - Q\left(\frac{Y}{Z}\right) \right\}. \quad (2)$$

$$C := \{(X, Y, Z) \in \mathbb{P}^2(\mathbb{K}) \mid F(X, Y, Z) = 0\}. \quad (3)$$

We conclude

$$P'(x) = na_n x^{n-1} + \dots + 2a_2 x + a_1 = na_n (x - \alpha_1)^{n_1} \dots (x - \alpha_k)^{n_k}, \quad (4)$$

$$Q'(y) = mb_m y^{m-1} + \dots + 2b_2 y + b_1 = mb_m (y - \beta_1)^{m_1} \dots (y - \beta_l)^{m_l}, \quad (5)$$

where $n_1 + \dots + n_k = n - 1$; $m_1 + \dots + m_l = m - 1$, $\alpha_1, \dots, \alpha_k$ are distinct zeros of P' ; and β_1, \dots, β_l are distinct zeros of Q' . Define

$$P'(X, Z) := Z^{n-1} P'\left(\frac{X}{Z}\right); \quad Q'(Y, Z) := Z^{m-1} Q'\left(\frac{Y}{Z}\right).$$

Then

$$\frac{\partial F}{\partial X} = P'(X, Z),$$

$$\frac{\partial F}{\partial Y} = -Z^{n-m} Q'(Y, Z),$$

$$\frac{\partial F}{\partial Z} = \sum_{i=0}^{n-1} (n-i) a_i X^i Z^{n-1-i} - \sum_{j=0}^{m'} (n-j) b_j Y^j Z^{n-1-j},$$

where

$$m' = \begin{cases} n-1 & \text{if } n = m \\ m & \text{if } n > m. \end{cases}$$

Then, by Euler's theorem, for all points $(X, Y, Z) \in C$, we have

$$\frac{\partial F}{\partial X} X + \frac{\partial F}{\partial Y} Y + \frac{\partial F}{\partial Z} Z = 0. \quad (6)$$

The equation of the tangent space of C at the point $(X, Y, Z) \in C$ is defined by

$$\frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial Y} dY + \frac{\partial F}{\partial Z} dZ = 0. \quad (7)$$

From (6) and (7), we obtain

$$\frac{\partial F}{\partial X} = \frac{\begin{vmatrix} Y & Z \\ dY & dZ \end{vmatrix}}{\begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}} \frac{\partial F}{\partial Z},$$

$$\frac{\partial F}{\partial Y} = \frac{\begin{vmatrix} Z & X \\ dZ & dX \end{vmatrix}}{\begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}} \frac{\partial F}{\partial Z}.$$

Hence, we have

$$\frac{\frac{W(Y, Z)}{\frac{\partial F}{\partial X}}}{\frac{\partial F}{\partial X}} = \frac{\frac{W(Z, X)}{\frac{\partial F}{\partial Y}}}{\frac{\partial F}{\partial Y}} = \frac{\frac{W(X, Y)}{\frac{\partial F}{\partial Z}}}{\frac{\partial F}{\partial Z}}. \quad (8)$$

Set

$$\eta := \frac{W(Y, Z)}{\frac{\partial F}{\partial X}} = \frac{W(Z, X)}{\frac{\partial F}{\partial Y}} = \frac{W(X, Y)}{\frac{\partial F}{\partial Z}},$$

we obtain

$$\begin{aligned} \eta &= \frac{W(Y, Z)}{P'(X, Z)} = \frac{W(X, Z)}{Z^{n-m}Q'(Y, Z)} \\ &= \frac{W(X, Y)}{\sum_{i=0}^{n-1} (n-i)a_i X^i Z^{n-1-i} - \sum_{j=0}^{m'} (n-j)b_j Y^j Z^{n-1-j}}. \end{aligned} \quad (9)$$

In order to prove the main results, we need the following lemmas.

Lemma 3.1 *Let P and Q be two nonlinear polynomials of degrees n and m , respectively, (defined by (1)) with $n \geq m$, and let C be a projective curve defined by (3). If $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q' , (defined by (4),(5)), then we have the following assertions:*

1. *If $n = m$ or $n = m + 1$, then C is non-singular in $\mathbb{P}^2(\mathbb{K})$.*
2. *If $n - m \geq 2$, then the point $(0, 1, 0)$ is a unique singular point of C in $\mathbb{P}^2(\mathbb{K})$.*

Proof By the hypothesis of the lemma, $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q' , we conclude that C is non-singular in $\mathbb{P}^2(\mathbb{K}) \setminus \{Z = 0\}$. Now we consider the singularity of C in $\{Z = 0\}$. Assume that $(X, Y, 0)$ is a singular point of C . We obtain

$$\frac{\partial F}{\partial X}(X, Y, 0) = 0; \frac{\partial F}{\partial Y}(X, Y, 0) = 0; \frac{\partial F}{\partial Z}(X, Y, 0) = 0 \text{ and } F(X, Y, 0) = 0.$$

We consider the following three cases:

If $n = m$, then

$$\begin{aligned} na_n X^{n-1} &= 0 \\ nb_n Y^{n-1} &= 0 \\ a_{n-1} X^{n-1} - b_{n-1} Y^{n-1} &= 0 \\ a_n X^n - b_n Y^n &= 0, \end{aligned}$$

has no root in $\mathbb{P}^2(\mathbb{K})$. If $n = m + 1$, then

$$\begin{aligned} na_n X^{n-1} &= 0 \\ a_n X^n - b_{n-1} Y^{n-1} &= 0 \\ a_n X^n &= 0, \end{aligned}$$

has no root in $\mathbb{P}^2(\mathbb{K})$.

If $n - m \geq 2$, then

$$\begin{aligned} na_n X^{n-1} &= 0 \\ a_{n-1} X^{n-1} &= 0 \\ a_n X^n &= 0, \end{aligned}$$

has a unique root $(0, 1, 0)$ in $\mathbb{P}^2(\mathbb{K})$ and C is singular with a unique singular point at $(0, 1, 0)$.

Thus, if $n = m$ or $n = m + 1$, then C is a smooth curve. If $n - m \geq 2$, then C is singular with a unique singular point at $(0, 1, 0)$. \square

Proposition 3.2 *If $n = m$ or $n = m + 1$ and the hypotheses of Lemma 3.1 are satisfied, then C is Brody hyperbolic on \mathbb{K} for all $n \geq 3$.*

Proof Consider the 1-form defined by (9). Since C is smooth, then by Lemma 3.1, if one of the expressions of η is regular, we can conclude that the other expressions are also regular. Let

$$\begin{aligned} \pi : \mathbb{K}^3 \setminus \{(0, 0, 0)\} &\implies \mathbb{P}^2(\mathbb{K}) \\ (X, Y, Z) &\longmapsto [X : Y : Z], \end{aligned}$$

where $[X : Y : Z] := \{(\lambda X, \lambda Y, \lambda Z) \mid \lambda \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}\}$. Note that η is regular on $\pi^{-1}(C)$. Then the 1-form

$$\omega := \frac{W(X, Z)}{Z^{n-m} Q'(Y, Z)} Z^{n-3} = Z^{n-3} \eta,$$

is a well-defined rational regular 1-form on C if $n \geq 3$. Thus, if $n \geq 4$, then for any homogeneous polynomial $R(X, Y, Z)$ of degree $n - 3$, the 1-form $\omega := R(X, Y, Z)\eta$, is regular on C and vanishes along $\{(X, Y, Z) \mid R(X, Y, Z) = 0\}$. Note that the set of monomials of degree $n - 3$ in X, Y, Z is a basis of the vector space of homogeneous polynomials of degree $n - 3$. This basis only has $\frac{(n-1)(n-2)}{2}$ vectors. Hence, the genus of C is

$$g = \frac{(n-1)(n-2)}{2},$$

Therefore, C is Brody hyperbolic if $n \geq 3$. \square

Remark 3.3 We also require that the 1-form defined by (9) is non trivial when it restricts to a component of $\pi^{-1}(C)$. This is equivalent to the condition that the nominators are not identically zero when they restrict to a component of $\pi^{-1}(C)$, i.e. the Wronskians W_i , (with $i = 1, 2, 3$) are not identically zero. It means that the homogeneous polynomial defining C has no linear factor of the

form $aX - bY$, $aY - bZ$ or $aX - bZ$, with $a, b \in \mathbb{K}$. Indeed, we assume on the contrary that, for example, $aX - bZ$ is a factor of curve C defined by (3). Without loss of generality, we can take $a \neq 0$. Since $aX - bZ$ is a factor of $F(X, Y, Z)$, we have

$$0 = F\left(\frac{b}{a}Z, Y, Z\right) = Z^n \left\{ P\left(\frac{\frac{b}{a}Z}{Z}\right) - Q\left(\frac{Y}{Z}\right) \right\} = Z^n \left\{ P\left(\frac{b}{a}\right) - Q\left(\frac{Y}{Z}\right) \right\},$$

and then we obtain $P\left(\frac{b}{a}\right) \equiv Q\left(\frac{Y}{Z}\right)$ for all Y, Z . It is impossible.

Proposition 3.4 *If $n \geq m + 2$ and the hypotheses of Lemma 3.1 are satisfied, then C is Brody hyperbolic on \mathbb{K} for all $m \geq 3$.*

Proof Since $n \geq m + 2$ by Lemma 3.1, we conclude that C is singular with a unique singular point at $(0, 1, 0)$. Considering the 1-form defined by (9)

$$\begin{aligned} \eta &= \frac{W(Y, Z)}{P'(X, Z)} = \frac{W(X, Z)}{Z^{n-m}Q'(Y, Z)} \\ &= \frac{W(X, Y)}{\sum_{i=0}^{n-1} (n-i)a_i X^i Z^{n-1-i} - \sum_{j=0}^{m'} (n-j)b_j Y^j Z^{n-1-j}}. \end{aligned}$$

Due to Remark 3.3, we deduce that η is not identically zero when it restricts to a component of $\pi^{-1}(C)$ and it is not regular on C . We take

$$\begin{aligned} \xi &:= \frac{Z^{n-m}W(Y, Z)}{P'(X, Z)} = \frac{W(X, Z)}{Q'(Y, Z)} \\ &= \frac{Z^{n-m}W(X, Y)}{\sum_{i=0}^{n-1} (n-i)a_i X^i Z^{n-1-i} - \sum_{j=0}^{m'} (n-j)b_j Y^j Z^{n-1-j}} \\ &= Z^{n-m}\eta. \end{aligned}$$

Then, ξ is regular on $\pi^{-1}(C)$ because the denominators of ξ have no common zeros. Therefore

$$\omega := \frac{W(X, Z)}{Q'(Y, Z)} Z^{m-3} = Z^{m-3}\xi,$$

is well-defined on C , regular and vanishing along $(n-3)\{Z=0\}$. This implies that, if $m \geq 3$, then ω is a 1-form of Wronskian type on C .

If $m = 3$ then $\omega = \xi$ is a linear independent regular 1-form of Wronskian type on the curve C .

If $m \geq 4$, we take $R_1, R_2, \dots, R_{\frac{(m-2)(m-1)}{2}}$ as a basis of monomials of degree $m-3$ in $\{X, Y, Z\}$, then

$$\{R_i \omega \mid i = 1, 2, \dots, \frac{(m-2)(m-1)}{2}\},$$

are linearly independent and global regular 1-forms of Wronskian type on the curve C . Thus, the genus g_C of C is

$$g_C \geq \frac{(m-2)(m-1)}{2}.$$

Therefore, C is Brody hyperbolic if $m \geq 3$. □

Next, we recall some notations. Let C be a curve on \mathbb{K} defined by a homogeneous polynomial $F(X, Y, Z) = 0$ and let ρ be a point of C . A holomorphic map

$$\phi = (\phi_1, \phi_2, \phi_3) : \Delta_\epsilon = \{t \in \mathbb{K} \mid |t| < \epsilon\} \longrightarrow C,$$

with $\phi(0) = \rho$, is referred to a *holomorphic parameterization* of C at ρ . Local holomorphic parameterization always exists for sufficiently small ϵ . If ϕ is a local holomorphic parameterization of C at ρ , then the Laurent expansion of $F \circ \phi(t)$ at ρ has the form

$$F \circ \phi(t) = \sum_{i=p}^q c_i t^i, \quad c_p \neq 0, \quad p < q < \infty.$$

The order of F at ρ , (it is also the order of $F \circ \phi(t)$ at $t = 0$) is defined by p and denoted by

$$p := \text{ord}_{\rho, \phi} F = \text{ord}_{t=0} F(\phi(t)).$$

Assume that $\varphi(x, y)$ is an analytic function in x, y and it is singular at $\rho = (a, b)$. The Puiseux expansion of $\varphi(x, y)$ at ρ is given by

$$x - a = \sum_{i=0}^{\infty} a_{\alpha+i} t^{\alpha+i}, \quad y - b = \sum_{j=0}^{\infty} b_{\beta+j} t^{\beta+j},$$

$$\varphi\left(a + \sum_{i=0}^{\infty} a_{\alpha+i} t^{\alpha+i}, b + \sum_{j=0}^{\infty} b_{\beta+j} t^{\beta+j}\right) = 0.$$

The α (respectively, β) is the order (also the multiplicity number) of x at ρ , (respectively, the order of y at ρ) for φ and is denoted by

$$\alpha := \text{ord}_{\rho, \varphi} x \quad (\text{respectively, } \beta := \text{ord}_{\rho, \varphi} y).$$

We now have the following result

Proposition 3.5 *Let P be a nonlinear polynomial of degree $n \geq 4$ and $Q = b_2 y^2 + b_1 y + b_0$, where $b_2 \neq 0$ and C a projective curve defined by (3). If $P(\alpha_i) \neq Q(\beta)$ for all zeros α_i of P' and β of Q' defined by (4), then C is Brody hyperbolic on \mathbb{K} for all $n \geq 4$.*

Proof Since $n \geq 4$ and $m = 2$, by Lemma 3.1, we conclude that C is singular with a unique singular point at $(0, 1, 0)$. Considering polynomial

$$H(X, Z) := F(X, 1, Z) = \sum_{i=0}^n a_{n-i} X^{n-i} Z^i - Z^{n-2}(b_2 + b_1 Z + b_0 Z^2), \text{ with } a_n, b_2 \neq 0.$$

Since polynomial $H(X, Z)$ is singular at $(0, 0)$, by the Newton diagram of the Puiseux expansion of H at $(0, 0)$, we have

$$\text{ord}_{0,H} X = \frac{n-2}{d}, \quad \text{ord}_{0,H} Z = \frac{n}{d},$$

where d is the greatest common divisor of n and $n-2$.

It follows from (9) that

$$\begin{aligned} \eta &= \frac{W(Y, Z)}{P'(X, Z)} \\ &= \frac{W(X, Z)}{Z^{n-2}(2b_2 Y + b_1)} \\ &= \frac{W(X, Y)}{\left\{ \sum_{i=0}^{n-1} (n-i) a_i X^i Z^{n-1-i} \right\} - Z^{n-3}(b_2 Y^2 + b_1 Y Z + b_0 Z^2)}. \end{aligned}$$

By Remark 3.3, η is non trivial on C . The η is regular on $\pi^{-1}(C \setminus \{Z = 0\})$ because the denominators of η have no common zeros for $Z \neq 0$. Let

$$\omega := Z^{n-3} \eta = \frac{W(X, Z)}{Z(2b_2 Y + b_1)}.$$

By the proof above, ω is regular on $\pi^{-1}(C \setminus \{Z = 0\})$. In $C \cap \{Z = 0\}$, the denominators of ω only vanish at $\rho = (0, 1, 0)$. Note that at this point $H(X, Z) = F(X, 1, Z)$ is singular, and therefore

$$\begin{aligned} \text{ord}_{t=0} \omega &= \text{ord}_{t=0} W(X, Z) - \text{ord}_{0,H} Z \\ &= \text{ord}_{t=0} (XdZ - ZdX) - \text{ord}_{0,H} Z \\ &= \frac{n-2-d}{d}. \end{aligned}$$

Since $d = (n, n-2)$, it follows that either $d = 1$ or $d = 2$. If $d = 1$, then $\text{ord}_{t=0} \omega = n-3$. If $d = 2$, then $\text{ord}_{t=0} \omega = \frac{n-4}{2}$.

Consequently, if $n \geq 4$, then ω is a non trivial well-defined regular 1-form of Wronskian type on C and the curve C is Brody hyperbolic. \square

We now consider the case when the curve C is singular in $\mathbb{P}^2(\mathbb{K}) \setminus \{Z = 0\}$.

From Lemma 3.1, the curve C is only singular in $\mathbb{P}^2(\mathbb{K}) \setminus \{Z = 0\}$ at $(\alpha_i, \beta_j, 1)$, with $\alpha_1, \dots, \alpha_k$ are distinct zeros of P' and β_1, \dots, β_l are distinct

zeros of Q' . By (4),(5), the distinct zeros $\alpha_1, \dots, \alpha_k$ of P' have multiplicities n_1, \dots, n_k and the distinct zeros β_1, \dots, β_l of Q' have multiplicities m_1, \dots, m_l . Let

$$\begin{aligned}\Gamma &:= \{(\alpha_i, \beta_j, 1) \text{ be singular points of } C\}, \\ \Delta &:= \{\alpha_i | (\alpha_i, \beta_j, 1) \text{ be singular points of } C\}\end{aligned}$$

and

$$\Lambda := \{\beta_j | (\alpha_i, \beta_j, 1) \text{ be singular points of } C\}.$$

Setting $I = \#\Delta, J = \#\Lambda$, then we have $k \geq I$ and $l \geq J$. Without loss of generality, we can take

$$\Delta = \{\alpha_1, \dots, \alpha_I\}, \Lambda = \{\beta_1, \dots, \beta_J\}.$$

Assume that P satisfies the separation condition. Then, we conclude $J \geq I$ and for every $j, j = 1, 2, \dots, J$, there exists a unique value i_j such that $(\alpha_{i_j}, \beta_j, 1)$ is a singular point of C (these α_{i_j} can be equal to each other). Hence,

$$\Gamma = \{(\alpha_{i_j}, \beta_j, 1) | j = 1, 2, \dots, J\}, \text{ with } l \geq J. \quad (10)$$

Let

$$H_j = a(X - \alpha_{i_j}Z) + b(Y - \beta_jZ), \text{ with } a, b \neq 0.$$

We obtain $H_j(\alpha_{i_j}, \beta_j, 1) = 0$. Putting

$$\xi := \frac{H_j^{s_j}}{(Y - \beta_jZ)^{m_j}} W(X, Z), \text{ for } s_j = \max\{n_{i_j}, m_j\},$$

it follows that ξ is regular on C . Therefore, from (9) we obtain

$$\xi = \frac{mb_m Z^{n-m} H_j^{s_j} \prod_{j \neq t}^l (Y - \beta_t Z)^{m_t}}{na_n \prod_{\nu=1}^k (X - \alpha_\nu Z)^{n_\nu}} W(Y, Z).$$

Note that P satisfies the separation condition, then $(\alpha_\nu, \beta_j, 1) \notin C$ with $\nu \neq i_j$.

On the other hand, $\text{ord}_{\rho, \phi} H_j^{s_j} = s_j \geq n_{i_j}, m_j$ and hence $\text{ord}_{\rho, \phi} \frac{H_j^{s_j}}{(X - \alpha_{i_j} Z)^{n_{i_j}}} \geq 0$, for $\rho = (\alpha_{i_j}, \beta_j, 1)$.

Now, we set

$$\zeta := \frac{\prod_{j=1}^J H_j^{s_j}}{Q'(Y, Z)} W(X, Z).$$

Then ζ is regular on C . Let

$$\omega := \left\{ \prod_{j=1}^J H_j^{s_j} \right\} Z^{n - \sum_{j=1}^J s_j - 3} \eta = Z^{m - \sum_{j=1}^J s_j - 3} \zeta,$$

we can easily see that ω is a non trivial, well-defined regular 1-form of Wronskian type on C and the curve C is Brody hyperbolic if $m - \sum_{j=1}^J s_j \geq 3$.

We can summarize these facts in the following result.

Proposition 3.6 *Let $P(z)$ and $Q(z)$ (defined by (1)) be two nonlinear polynomials of degrees n, m , respectively, with $n \geq m$, and let P satisfy the separation condition. Suppose that $\Gamma = \{(\alpha_{i_j}, \beta_j, 1) | j = 1, 2, \dots, J\}$ (defined by (10)) is the set of singular points of C . Then the curve C is Brody hyperbolic if $m - \sum_{j=1}^J s_j \geq 3$, with $s_j = \max(n_{i_j}, m_j), j = 1, 2, \dots, J$.*

Lemma 3.7 *Let $P(z)$ and $Q(z)$ be two polynomials defined by (1) and P satisfy the separation condition. Let $\Gamma := \{(\alpha_i, \beta_j, 1) | j = 1, 2, \dots, J\}$ be the set of singular points of C . If $m_j - n_i \geq 2$ at all points $\rho_{ij} = (\alpha_i, \beta_j, 1)$, then C is Brody hyperbolic.*

Proof The condition of Lemma 3.7 implies that the curve C is singular at any point ρ_{ij} and $m_j \geq 3$. Let

$$\zeta := \frac{(X - \alpha_i Z)^{m_j - 2}}{(Y - \beta_j Z)^{m_j}} W(X, Z)$$

be a well-defined rational 1-form on C . It follows from (9) that

$$\begin{aligned} \zeta &= \frac{mb_m Z^{n-m} (X - \alpha_i Z)^{m_j - 2} \prod_{1=t \neq j}^l (Y - \beta_t Z)^{m_t}}{na_n \prod_{\nu=1}^k (X - \alpha_\nu Z)^{n_\nu}} W(Y, Z) \\ &= \frac{mb_m Z^{n-m} (X - \alpha_i Z)^{m_j - n_i - 2} na_n \prod_{i \neq \nu=1}^k (X - \alpha_\nu Z)^{n_\nu}}{na_n \prod_{i \neq \nu=1}^k (X - \alpha_\nu Z)^{n_\nu}} W(Y, Z). \end{aligned}$$

The first expression implies that all poles of ζ must belong to $C \cap \{Y = \beta_j\}$. The second expression implies that all poles of ζ must belong to $C \cap \{X = \alpha_\nu\}$ for all $\nu \neq i$. Since P satisfies the separation condition, we can conclude that $Q(\beta_j) = P(\alpha_i) \neq P(\alpha_\nu)$ for all $\nu \neq i$. This shows that ζ is regular on C if $m_j - n_i \geq 2$. The proof of the lemma is now complete. \square

Proof of Theorem 2.1 It follows immediately from Proposition 3.2, 3.4 and 3.5. \square

Proof of Theorem 2.2 By (9), we see that

$$\eta = \frac{W(Y, Z)}{na_n \prod_{\nu=1}^k (X - \alpha_\nu Z)^{n_\nu}} = \frac{W(X, Z)}{mb_m Z^{n-m} \prod_{t=1}^l (Y - \beta_t Z)^{m_t}}.$$

Set

$$\theta := \frac{W(X, Z)}{(Y - \beta_1 Z)^2}.$$

We then have

$$\theta = \frac{mb_m Z^{n-m} (Y - \beta_1 Z)^{m_1-2} \prod_{t=2}^l (Y - \beta_t Z)^{m_t}}{na_n \prod_{\nu=1}^k (X - \alpha_\nu Z)^{n_\nu}} W(Y, Z).$$

Since $(\alpha_i, \beta_1, 1) \notin C$ with for all α_i , we see that θ is a well-defined regular 1-form of Wronskian type on C , proving the first assertion of the theorem.

Similary, let

$$\xi := \frac{W(X, Z)}{(Y - \beta_1 Z)(Y - \beta_2 Z)}.$$

Then ξ is a well-defined regular 1-form of Wronskian type on C and therefore the second assertion of theorem is proved. \square

Proof of Theorem 2.3 By applying Proposition 3.6 and Lemma 3.7, the Theorem is followed. \square

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