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THE P-ADIC FIELD CASE OF THE FUNCTIONAL EQUATION $P(f) = Q(g)$

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Abstract

In this paper, we study the existence of non constant meromorphic solutions f and g of the functional equation $P(f) = Q(g)$, where $P(z)$ and $Q(z)$ are two given nonlinear polynomials with coefficients in the non-Archimedean field $\mathbb{K}$.

1. Introduction

The Tenth problem of Hilbert is asked to establish an algorithm for finding all integer solutions of $F(x, y) = 0$, where $F(x, y)$ is a polynomial with integer coefficients. It is natural to study the analogous of Hilbert's Tenth problem in the field of meromorphic functions. More specifically, we ask what forms of equation $F(x, y) = 0$, where $F(x, y)$ is a polynomial with complex coefficients, may or may not have non constant meromorphic functions f and g that satisfy $F(f, g) = 0$? Earlier in 1920s, as a simple application of his value distribution theory, Nevanlinna proved that a non constant meromorphic function (in the complex plane) is uniquely determined by the inverse image of five distinct values (including infinity), ignoring multiplicity. Gross [11] extended this study by considering pre-images of a set and posed the question: *Is there a finite set A so that an entire (meromorphic) function is uniquely determined by the pre-image of the set A, counting multiplicities?* Let f be a non-constant meromorphic

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function and S be a subset of distinct elements. Define

$$E_f(S) = \bigcup_{a \in S} \{(z, m) | \quad f(z) = a \text{ with multiplicity } m\}.$$

Two functions f and g of the same type are said to *share* S, *counting multiplicity,* if $E_f(S) = E_g(S)$. A subset S is called a *unique range set* (a *URS* in short) for entire (or meromorphic) functions if for any two non-constant entire (or meromorphic) functions f and g such that $E_f(S) = E_g(S)$, one has f=g. Assume that S be a finite set, we set :

$$P_S(z) = \prod_{a \in S} (x - a).$$

As a connection to the study of the uniqueness problem, Li and Yang [3] introduced the following definition:

Definition 1 A non-constant polynomial $P(z)$ is said to be a *unique polynomial* for entire (or meromorphic) functions if for two non-constant entire (or meromorphic) functions f and g, $P(f) = P(g)$ implies that $f = g$.

P(z) is said to be a *strong uniqueness polynomial* for entire (or meromorphic) functions if for two non-constant entire (or meromorphic) functions f, g and some nonzero constant c, the condition $P(f) = cP(g)$ implies that $c = 1$ and $f = g$.

To demonstrate that the finite set S be a URS for entire (or meromorphic) functions, we prove that the polynomial $P_S(z)$ is a strong uniqueness polynomial. If P is a strong uniqueness polynomial for entire (or meromorphic) functions, then the set of the zeros of P can be a URS.

Recently, H.H. Khoai and C.C. Yang [1] generalized the above studies by considering a pair of two nonlinear polynomials $P(z)$ and $Q(z)$ such that the only meromorphic solutions f, g satisfying $P(f) = Q(g)$ are constants. This problem is considered in the complex plane $\mathbb{C}$ by H.H. Khoai and C.C. Yang [1] as well as by C.C. Yang and P. Li [2].

In this paper, we find the conditions for the existence of non-constant meromorphic function solutions f and g of the functional equation $P(f) = Q(g)$ in $\mathbb{K}$ with $\mathbb{K}$ being an algebraically closed field, complete for a non-trivial non-Archimedean absolute value. To solve the functional equation, we study the hyperbolicity of the algebraic curve $\{P(x) - Q(y) = 0\}$ by estimating its genus. We shall do this by giving sufficiently many linear independent regular 1-forms of Wronskian type on that curve.

2. Main theorems

Theorem 2.1 *Let $P(z)$ and $Q(z)$ be two nonlinear polynomials of degrees n and m, respectively, with $n \geq m$. Suppose that $P(\alpha) \neq Q(\beta)$ for all zeros α*

of P' and β of Q'. Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$, if n and m satisfy the following condition:

$$n \geq m \geq 2 \text{ and } n \geq 3.$$

Theorem 2.2 *Let $P(z)$ and $Q(z)$ be two nonlinear polynomials of degrees n and m, respectively. Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$ provided that P and Q satisfy one of the following conditions:*

1. There exists a zero β_1 of Q' with multiplicity m_1 at least 2 and $P(\alpha) \neq Q(\beta_1)$, for all zeros α of P'.

2. There exists two simple zeros β_1, β_2 of Q' such that $P(\alpha) \neq Q(\beta_i)$ for all zeros α of P' and $i = 1, 2$.

Definition 2 Let $P(z)$ be a nonlinear polynomial of degree n whose derivative is given by:

$$P'(z) = c(z - \alpha_1)^{n_1} \dots (z - \alpha_k)^{n_k},$$

where $n_1 + \dots + n_k = n - 1$ and α_1 , ..., α_k are distinct zeros of P'. The number k is called *the derivative index of P*.

A polynomial $P(z)$ is said to satisfy *the condition of separating the roots of P' (separation condition)* if $P(\alpha_i) \neq P(\alpha_j)$ for all $k \geq i \neq j \geq 1$.

Theorem 2.3 *Let $P(z)$ and $Q(z)$ be two polynomials defined by (1) and let P satisfy the separation condition. Suppose that $\beta_1, \dots, \beta_J$ are distinct zeros of Q' with multiplicity m_j, respectively, such that for every β_j, $j = 1, 2, \dots, J$, there exists zeros α_i of P' with $P(\alpha_i) = Q(\beta_j)$. Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$ if*

$$m - \sum_{j=1}^{J} m_j \geq 3.$$

Remark 2.4 In case $\deg P = \deg Q = 2$, the equation $P(f) = Q(g)$ has some non-constant meromorphic functions. Indeed, in this case we can rewrite the equation $P(f) = Q(g)$ in the form:

$$(f - a)^2 = (bg - c)^2 + d,$$

where $a, b, c, d \in \mathbb{K}$ and $b \neq 0$. Hence

$$(f - bg - a + c)(f + bg - a - c) = d.$$

Assume that h is a non-constant meromorphic function, we set

$$f = \frac{1}{2}(h + \frac{d}{h}) + a, \quad g = \frac{1}{2b}(-h + \frac{d}{h}) + \frac{c}{b}.$$

Then f and g are non-constant meromorphic solutions of equation $P(f) = Q(g)$.

3. Lemmas and Proofs

Let $\mathbb{K}$ be an algebraically closed field, complete for a non-trivial non-Archimedean absolute value with characteristic zero. Suppose that $H(X, Y, Z)$ is a homogeneous polynomial of degree n and

$$C := \{(X, Y, Z) \in \mathbb{P}^2(\mathbb{K}) | H(X, Y, Z) = 0\}.$$

Denote

$$W_1 = W(X, Y) = \begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}.$$

$$W_2 = W(Y, Z) = \begin{vmatrix} Y & Z \\ dY & dZ \end{vmatrix}.$$

$$W_3 = W(X, Z) = \begin{vmatrix} X & Z \\ dX & dZ \end{vmatrix}.$$

Assume that $R(X, Y, Z)$ and $S(X, Y, Z)$ are two homogeneous polynomials in $\mathbb{P}^2(\mathbb{K})$.Let

$$\omega_i = \frac{R(X, Y, Z)}{S(X, Y, Z)} W_i,$$

with $i = 1, 2, 3$. If $R(X, Y, Z)$ and $S(X, Y, Z)$ satisfy $\mathrm{d}egS = \mathrm{d}egR + 2$, then ω_i is a well-defined rational 1-form on $\mathbb{P}^2(\mathbb{K})$.

Definition 3 Let C be an algebraic curve in $\mathbb{P}^2(\mathbb{K})$. An 1-form ω on C is said to be *regular* if it is the pull-back of a rational 1-form on $\mathbb{P}^2(\mathbb{K})$ such that the pole set of ω does not intersect C. A well-defined rational regular 1-form on C is said to be an *1-form of Wronskian type.*

Notice that to solve the functional equation $P(f) = Q(g)$, is the same as to find meromorphic functions f, g on $\mathbb{K}$ such that $(f(z), g(z))$ in curve $\{P(x) - Q(y) = 0\}$. On the other hand, if C is hyperbolic on $\mathbb{K}$ and suppose that f, g be meromorphic functions such that $(f(z), g(z)) \in C$, where $z \in \mathbb{K}$, then f and g are constant (see.[6]). Therefore, to show that this equation has no non-constant solutions, we shall prove the hyperbolicity of $\{P(x) - Q(y) = 0\}$. By Picard-Berkovich's theorem in the p-adic case, a curve C in $\mathbb{K}$ is Brody hyperbolic if and only if the genus of the curve C is at least 1.

It is well-known that the genus g of a algebraic curve C is equal to the dimension of the space of regular 1-forms on C. Therefore, to compute the genus, we have to construct a basis of the space of regular 1-forms on C.

Now, let P and Q be two nonlinear polynomials of degrees n and m in $\mathbb{K}$, respectively, with

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$
$$Q(y) = b_m y^m + \ldots + b_1 y + b_0. \tag{1}$$

Without loss of generality, we assume that $m \geq n$. Set

$$F_1(x, y) := P(x) - Q(y),$$

$$F(X,Y,Z) := Z^n \left\{ P(\frac{X}{Z}) - Q(\frac{Y}{Z}) \right\}. \tag{2}$$

$$C := \{(X,Y,Z) \in \mathbb{P}^2(\mathbb{K}) \ | F(X,Y,Z) = 0\}. \tag{3}$$

We conclude

$$P'(x) = na_n x^{n-1} + \cdots + 2a_2 x + a_1 = na_n (x-\alpha_1)^{n_1} \cdots (x-\alpha_k)^{n_k}, \tag{4}$$

$$Q'(y) = mb_m y^{m-1} + \cdots + 2b_2 y + b_1 = mb_m (y-\beta_1)^{m_1} \cdots (y-\beta_l)^{m_l}, \tag{5}$$

where $n_1 + \ldots + n_k = n-1$; $m_1 + \ldots + m_l = m-1, \alpha_1, \ldots, \alpha_k$ are distinct zeros of P'; and $\beta_1, \ldots, \beta_l$ are distinct zeros of Q'. Define

$$P'(X,Z) := Z^{n-1} P'(\frac{X}{Z}) \ ; \ Q'(Y,Z) := Z^{m-1} Q'(\frac{Y}{Z}).$$

Then

$$\begin{aligned}
\frac{\partial F}{\partial X} &= P'(X,Z), \\
\frac{\partial F}{\partial Y} &= -Z^{n-m} Q'(Y,Z), \\
\frac{\partial F}{\partial Z} &= \sum_{i=0}^{n-1} (n-i) a_i X^i Z^{n-1-i} - \sum_{j=0}^{m'} (n-j) b_j Y^j Z^{n-1-j},
\end{aligned}$$

where

$$m' = \begin{cases} n-1 & \text{if} \quad n = m \\ m & \text{if} \quad n > m. \end{cases}$$

Then, by Euler's theorem, for all points $(X,Y,Z) \in C$, we have

$$\frac{\partial F}{\partial X} X + \frac{\partial F}{\partial Y} Y + \frac{\partial F}{\partial Z} Z = 0. \tag{6}$$

The equation of the tangent space of C at the point $(X,Y,Z) \in C$ is defined by

$$\frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial Y} dY + \frac{\partial F}{\partial Z} dZ = 0. \tag{7}$$

From (6) and (7), we obtain

$$\frac{\partial F}{\partial X} = \frac{\begin{vmatrix} Y & Z \\ dY & dZ \end{vmatrix}}{\begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}} \frac{\partial F}{\partial Z},$$

$$\frac{\partial F}{\partial Y} = \frac{\begin{vmatrix} Z & X \\ dZ & dX \end{vmatrix}}{\begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}} \frac{\partial F}{\partial Z}.$$

Hence, we have

$$\frac{W(Y,Z)}{\dfrac{\partial F}{\partial X}} = \frac{W(Z,X)}{\dfrac{\partial F}{\partial Y}} = \frac{W(X,Y)}{\dfrac{\partial F}{\partial Z}}. \tag{8}$$

Set

$$\eta := \frac{W(Y,Z)}{\dfrac{\partial F}{\partial X}} = \frac{W(Z,X)}{\dfrac{\partial F}{\partial Y}} = \frac{W(X,Y)}{\dfrac{\partial F}{\partial Z}},$$

we obtain

$$\begin{aligned}\eta &= \frac{W(Y,Z)}{P'(X,Z)} = \frac{W(X,Z)}{Z^{n-m}Q'(Y,Z)} \\ &= \frac{W(X,Y)}{\sum_{i=0}^{n-1}(n-i)a_iX^iZ^{n-1-i} - \sum_{j=0}^{m'}(n-j)b_jY^jZ^{n-1-j}}.\end{aligned} \tag{9}$$

In order to prove the main results, we need the following lemmas.

Lemma 3.1 *Let P and Q be two nonlinear polynomials of degrees n and m,respectively, (defined by (1)) with $n \geq m$, and let C be a projective curve defined by (3). If $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q', (defined by (4),(5)), then we have the following assertions:*

1. *If $n = m$ or $n = m+1$, then C is non-singular in $\mathbb{P}^2(\mathbb{K})$.*
2. *If $n - m \geq 2$, then the point $(0,1,0)$ is a unique singular point of C in $\mathbb{P}^2(\mathbb{K})$.*

Proof By the hypothesis of the lemma, $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q',we conclude that C is non-singular in $\mathbb{P}^2(\mathbb{K}) \setminus \{Z = 0\}$. Now we consider the singularity of C in $\{Z = 0\}$.Assume that $(X, Y, 0)$ is a singular point of C. We obtain

$$\frac{\partial F}{\partial X}(X,Y,O) = 0; \frac{\partial F}{\partial Y}(X,Y,O) = 0; \frac{\partial F}{\partial Z}(X,Y,O) = 0 \text{ and } F(X,Y,0) = 0.$$

We consider the following three cases:

If $n = m$, then

$$\begin{aligned} na_nX^{n-1} &= 0 \\ nb_nY^{n-1} &= 0 \\ a_{n-1}X^{n-1} - b_{n-1}Y^{n-1} &= 0 \\ a_nX^n - b_nY^n &= 0, \end{aligned}$$

has no root in $\mathbb{P}^2(\mathbb{K})$. If $n = m+1$, then

$$\begin{aligned} na_nX^{n-1} &= 0 \\ a_nX^n - b_{n-1}Y^{n-1} &= 0 \\ a_nX^n &= 0, \end{aligned}$$

has no root in $\mathbb{P}^2(\mathbb{K})$.

If $n - m \geq 2$, then

$$\begin{aligned} na_n X^{n-1} &= 0 \\ a_{n-1} X^{n-1} &= 0 \\ a_n X^n &= 0, \end{aligned}$$

has a unique root $(0, 1, 0)$ in $\mathbb{P}^2(\mathbb{K})$ and C is singular with a unique singular point at $(0, 1, 0)$.

Thus, if $n = m$ or $n = m + 1$, then C is a smooth curve. If $n - m \geq 2$, then C is singular with a unique singular point at $(0, 1, 0)$. □

Proposition 3.2 *If $n = m$ or $n = m + 1$ and the hypotheses of Lemma 3.1 are satisfied, then C is Brody hyperbolic on $\mathbb{K}$ for all $n \geq 3$.*

Proof Consider the 1-form defined by (9). Since C is smooth, then by Lemma 3.1, if one of the expressions of η is regular, we can conclude that the other expressions are also regular. Let

$$\begin{aligned} \pi : \mathbb{K}^3 \setminus \{(0,0,0)\} &\Longrightarrow \mathbb{P}^2(\mathbb{K}) \\ (X, Y, Z) &\longmapsto [X : Y : Z], \end{aligned}$$

where $[X : Y : Z] := \{ (\lambda X, \lambda Y, \lambda Z) \mid \lambda \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}\}$. Note that η is regular on $\pi^{-1}(C)$. Then the 1-form

$$\omega := \frac{W(X, Z)}{Z^{n-m} Q'(Y, Z)} Z^{n-3} = Z^{n-3}\eta,$$

is a well-defined rational regular 1-form on C if $n \geq 3$. Thus, if $n \geq 4$, then for any homogeneous polynomial $R(X, Y, Z)$ of degree $n - 3$, the 1-form $\omega := R(X, Y, Z)\eta$, is regular on C and vanishes along $\{(X, Y, Z) | R(X, Y, Z) = 0\}$. Note that the set of monomials of degree $n - 3$ in X, Y, Z is a basis of the vector space of homogeneous polynomials of degree $n - 3$. This basis only has $\frac{(n-1)(n-2)}{2}$ vectors. Hence, the genus of C is

$$g = \frac{(n-1)(n-2)}{2},$$

Therefore, C is Brody hyperbolic if $n \geq 3$. □

Remark 3.3 We also require that the 1-form defined by (9) is non trivial when it restricts to a component of $\pi^{-1}(C)$. This is equivalent to the condition that the nominators are not identically zero when they restrict to a component of $\pi^{-1}(C)$, i.e. the Wronskians W_i, (with $i = 1, 2, 3$) are not identically zero. It means that the homogeneous polynomial defining C has no linear factor of the form $aX - bY$, $aY - bZ$ or $aX - bZ$, with $a, b \in \mathbb{K}$. Indeed, we assume on the contrary that, for example, $aX - bZ$ is a factor of curve C defined by (3). Without loss of generality, we can take $a \neq 0$. Since $aX - bZ$ is a factor of $F(X, Y, Z)$, we have

$$0 = F(\frac{b}{a}Z, Y, Z) = Z^n\{P(\frac{\frac{b}{a}Z}{Z}) - Q(\frac{Y}{Z})\} = Z^n\{P(\frac{b}{a}) - Q(\frac{Y}{Z})\},$$

and then we obtain $P(\frac{b}{a}) \equiv Q(\frac{Y}{Z})$ for all Y, Z. It is impossible.

Proposition 3.4 *If $n \geq m + 2$ and the hypotheses of Lemma 3.1 are satisfied, then C is Brody hyperbolic on $\mathbb{K}$ for all $m \geq 3$.*

Proof Since $n \geq m + 2$ by Lemma 3.1, we conclude that C is singular with a unique singular point at $(0, 1, 0)$. Considering the 1-form defined by (9)

$$\begin{aligned} \eta &= \frac{W(Y, Z)}{P'(X, Z)} = \frac{W(X, Z)}{Z^{n-m}Q'(Y, Z)} \\ &= \frac{W(X, Y)}{\sum_{i=0}^{n-1}(n-i)a_iX^iZ^{n-1-i} - \sum_{j=0}^{m'}(n-j)b_jY^jZ^{n-1-j}}. \end{aligned}$$

Due to Remark 3.3, we deduce that η is not identically zero when it restricts to a component of $\pi^{-1}(C)$ and it is not regular on C. We take

$$\begin{aligned} \xi &:= \frac{Z^{n-m}W(Y, Z)}{P'(X, Z)} = \frac{W(X, Z)}{Q'(Y, Z)} \\ &= \frac{Z^{n-m}W(X, Y)}{\sum_{i=0}^{n-1}(n-i)a_iX^iZ^{n-1-i} - \sum_{j=0}^{m'}(n-j)b_jY^jZ^{n-1-j}} \\ &= Z^{n-m}\eta. \end{aligned}$$

Then, ξ is regular on $\pi^{-1}(C)$ because the denominators of ξ have no common zeros. Therefore

$$\omega := \frac{W(X, Z)}{Q'(Y, Z)}Z^{m-3} = Z^{m-3}\xi,$$

is well-defined on C, regular and vanishing along $(n-3)\{Z = 0\}$. This implies that, if $m \geq 3$, then ω is a 1-form of Wronskian type on C.

If $m = 3$ then $\omega = \xi$ is a linear independent regular 1-form of Wronskian type on the curve C.

If $m \geq 4$, we take $R_1, R_2, \ldots, R_{\frac{(m-2)(m-1)}{2}}$ as a basis of monomials of degree $m - 3$ in $\{X, Y, Z\}$, then

$$\{R_i\omega | i = 1, 2, \ldots, \frac{(m-2)(m-1)}{2}\},$$

are linearly independent and global regular 1-forms of Wronskian type on the curve C. Thus, the genus g_C of C is

$$g_C \geq \frac{(m-2)(m-1)}{2}.$$

Therefore, C is Brody hyperbolic if $m \geq 3$. □

Next, we recall some notations. Let C be a curve on $\mathbb{K}$ defined by a homogeneous polynomial $F(X,Y,Z) = 0$ and let ρ be a point of C. A holomorphic map

$$\phi = (\phi_1, \phi_2, \phi_3) : \triangle_\epsilon = \{t \in \mathbb{K} | |t| < \epsilon\} \longrightarrow C,$$

with $\phi(0) = \rho$, is referred to a *holomorphic parameterization* of C at ρ. Local holomorphic parameterization always exists for sufficiently small ϵ. If ϕ is a local holomorphic parameterization of C at ρ, then the Laurent expansion of $F \circ \phi(t)$ at ρ has the form

$$F \circ \phi(t) = \sum_{i=p}^{q} c_i t^i, \quad c_p \neq 0, \quad p < q < \infty.$$

The order of F at ρ, (it is also the order of $F \circ \phi(t)$ at $t = 0$) is defined by p and denoted by

$$p := \mathrm{ord}_{\rho,\phi} F = \mathrm{ord}_{t=0} F(\phi(t)).$$

Assume that $\varphi(x,y)$ is an analytic function in x, y and it is singular at $\rho = (a,b)$. The Puiseux expansion of $\varphi(x,y)$ at ρ is given by

$$x - a = \sum_{i=0}^{\infty} a_{\alpha+i} t^{\alpha+i}, \quad y - b = \sum_{j=0}^{\infty} b_{\beta+j} t^{\beta+j},$$

$$\varphi\left(a + \sum_{i=0}^{\infty} a_{\alpha+i} t^{\alpha+i}, b + \sum_{j=0}^{\infty} b_{\beta+j} t^{\beta+j}\right) = 0.$$

The α (respectively, β) is the order (also the multiplicity number) of x at ρ, (respectively, the order of y at ρ) for φ and is denoted by

$$\alpha := \mathrm{ord}_{\rho,\varphi} x \quad \text{(respectively, } \beta := \mathrm{ord}_{\rho,\varphi} y).$$

We now have the following result

Proposition 3.5 *Let P be a nonlinear polynomial of degree $n \geq 4$ and $Q = b_2y^2 + b_1y + b_0$, where $b_2 \neq 0$ and C a projective curve defined by (3). If $P(\alpha_i) \neq Q(\beta)$ for all zeros α_i of P' and β of Q' defined by (4), then C is Brody hyperbolic on $\mathbb{K}$ for all $n \geq 4$.*

Proof Since $n \geq 4$ and $m=2$, by Lemma 3.1, we conclude that C is singular with a unique singular point at $(0,1,0)$. Considering polynomial

$$H(X,Z) := F(X,1,Z) = \sum_{i=0}^{n} a_{n-i}X^{n-i}Z^{i} - Z^{n-2}(b_2 + b_1 Z + b_0 Z^2), \text{with } a_n, b_2 \neq 0.$$

Since polynomial $H(X,Z)$ is singular at $(0,0)$, by the Newton diagram of the Puiseux expansion of H at $(0,0)$, we have

$$\mathrm{ord}_{0,H}X = \frac{n-2}{d}, \quad \mathrm{ord}_{0,H}Z = \frac{n}{d},$$

where d is the greatest common divisor of n and $n-2$.

It follows from (9) that

$$\begin{aligned}\eta &= \frac{W(Y,Z)}{P'(X,Z)}\\ &= \frac{W(X,Z)}{Z^{n-2}(2b_2Y + b_1)}\\ &= \frac{W(X,Y)}{\left\{\sum_{i=0}^{n-1}(n-i)a_iX^iZ^{n-1-i}\right\} - Z^{n-3}(b_2Y^2 + b_1YZ + b_0Z^2)}.\end{aligned}$$

By Remark 3.3, η is non trivial on C. The η is regular on $\pi^{-1}(C\backslash\{Z=0\})$ because the denominators of η have no common zeros for $Z \neq 0$. Let

$$\omega := Z^{n-3}\eta = \frac{W(X,Z)}{Z(2b_2Y + b_1)}.$$

By the proof above, ω is regular on $\pi^{-1}(C\backslash\{Z=0\})$. In $C\bigcap\{Z=0\}$, the denominators of ω only vanish at $\rho=(0,1,0)$. Note that at this point $H(X,Z)=F(X,1,Z)$ is singular, and therefore

$$\begin{aligned}\mathrm{ord}_{t=0}\omega &= \mathrm{ord}_{t=0}W(X,Z) - \mathrm{ord}_{0,H}Z\\ &= \mathrm{ord}_{t=0}(XdZ - ZdX) - \mathrm{ord}_{0,H}Z\\ &= \frac{n-2-d}{d}.\end{aligned}$$

Since $d=(n,n-2)$, it follows that either $d=1$ or $d=2$. If $d=1$, then $\mathrm{ord}_{t=0}\omega = n-3$. If $d=2$, then $\mathrm{ord}_{t=0}\omega = \frac{n-4}{2}$.

Consequently, if $n \geq 4$, then ω is a non trivial well-defined regular 1-form of Wronskian type on C and the curve C is Brody hyperbolic. $\square$

We now consider the case when the curve C is singular in $\mathbb{P}^2(\mathbb{K})\backslash\{Z=0\}$.

From Lemma 3.1, the curve C is only singular in $\mathbb{P}^2(\mathbb{K})\backslash\{Z=0\}$ at $(\alpha_i, \beta_j, 1)$, with $\alpha_1, \ldots, \alpha_k$ are distinct zeros of P' and $\beta_1, \ldots, \beta_l$ are distinct zeros of Q'. By (4),(5), the distinct zeros $\alpha_1, \ldots, \alpha_k$ of P' have multiplicities $n_1, \ldots, n_k$ and the distinct zeros $\beta_1, \ldots, \beta_l$ of Q' have multiplicities $m_1, \ldots, m_l$. Let

$$\Gamma := \{(\alpha_i, \beta_j, 1) \text{ be singular points of } C\},$$

$$\Delta := \{\alpha_i | (\alpha_i, \beta_j, 1) \text{be singular points of } C\}$$

and

$$\Lambda := \{\beta_j | (\alpha_i, \beta_j, 1) \text{be singular points of } C\}.$$

Setting $\mathrm{I} = \#\Delta$, $\mathrm{J} = \#\Lambda$, then we have $k \geq I$ and $l \geq J$. Without loss of generality, we can take

$$\Delta = \{\alpha_1, \ldots, \alpha_I\}, \Lambda = \{\beta_1, \ldots, \beta_J\}.$$

Assume that P satisfies the separation condition. Then, we conclude $J \geq I$ and for every j, $j = 1, 2, \ldots, J$, there exists a unique value i_j such that $(\alpha_{i_j}, \beta_j, 1)$ is a singular point of C (these α_{i_j} can be equal to each other). Hence,

$$\Gamma = \{(\alpha_{i_j}, \beta_j, 1) | j = 1, 2, \ldots, J\}, \text{with } \ l \geq J. \tag{10}$$

Let

$$H_j = a(X - \alpha_{i_j} Z) + b(Y - \beta_j Z), \quad \text{with } \ a, b \neq 0.$$

We obtain $H_j(\alpha_{i_j}, \beta_j, 1) = 0$. Putting

$$\xi := \frac{H_j^{s_j}}{(Y - \beta_j Z)^{m_j}} W(X, Z), \quad \text{for } \ s_j = max\{n_{i_j}, m_j\},$$

it follows that ξ is regular on C. Therefore, from (9) we obtain

$$\xi = \frac{m b_m Z^{n-m} H_j^{s_j} \prod_{j \neq t}^{l} (Y - \beta_t Z)^{m_t}}{n a_n \prod_{\nu=1}^{k} (X - \alpha_\nu Z)^{n_\nu}} W(Y, Z).$$

Note that P satisfies the separation condition, then $(\alpha_\nu, \beta_j, 1) \notin C$ with $\nu \neq i_j$. On the other hand, $\mathrm{ord}_{\rho,\phi} H_j^{s_j} = s_j \geq n_{i_j}, m_j$ and hence $\mathrm{ord}_{\rho,\phi} \frac{H_j^{s_j}}{(X - \alpha_{i_j} Z)^{n_{i_j}}} \geq 0$, for $\rho = (\alpha_{i_j}, \beta_j, 1)$.

Now, we set

$$\zeta := \frac{\prod_{j=1}^{J} H_j^{s_j}}{Q'(Y, Z)} W(X, Z).$$

Then ζ is regular on C. Let

$$\omega := \{\prod_{j=1}^{J} H_j^{s_j}\} Z^{n - \sum_{j=1}^{J} s_j - 3} \eta = Z^{m - \sum_{j=1}^{J} s_j - 3} \zeta,$$

we can easily see that ω is a non trivial, well-defined regular 1-form of Wronskian type on C and the curve C is Brody hyperbolic if $m - \sum_{j=1}^{J} s_j \geq 3$.

We can summarize these facts in the following result.

Proposition 3.6 *Let $P(z)$ and $Q(z)$ (defined by (1)) be two nonlinear polynomials of degrees n, m, respectively, with $n \geq m$, and let P satisfy the separation condition. Suppose that $\Gamma = \{(\alpha_{i_j}, \beta_j, 1) | j = 1, 2, \ldots, J\}$ (defined by (10)) is the set of singular points of C. Then the curve C is Brody hyperbolic if $m - \sum_{j=1}^{I} s_j \geq 3$, with $s_j = \max(n_{i_j}, m_j), j = 1, 2, \ldots, J$.*

Lemma 3.7 Let $P(z)$ and $Q(z)$ be two polynomials defined by (1) and P satisfy the separation condition. Let $\Gamma := \{(\alpha_i, \beta_j, 1) | j = 1, 2, \ldots, J\}$ be the set of singular points of C. If $m_j - n_i \geq 2$ at all points $\rho_{ij} = (\alpha_i, \beta_j, 1)$, then C is Brody hyperbolic.

Proof The condition of Lemma 3.7 implies that the curve C is singular at any point ρ_{ij} and $m_j \geq 3$. Let

$$\zeta := \frac{(X - \alpha_i Z)^{m_j - 2}}{(Y - \beta_j Z)^{m_j}} W(X, Z)$$

be a well-defined rational 1-form on C. It follows from (9) that

$$\begin{aligned}
\zeta &= \frac{m b_m Z^{n-m} (X - \alpha_i Z)^{m_j - 2} \prod_{1=t\neq j}^{l} (Y - \beta_t Z)^{m_t}}{n a_n \prod_{\nu=1}^{k} (X - \alpha_\nu Z)^{n_\nu}} W(Y, Z) \\
&= \frac{m b_m Z^{n-m} (X - \alpha_i Z)^{m_j - n_i - 2} n a_n \prod_{i\neq\nu=1}^{k} (X - \alpha_\nu Z)^{n_\nu}}{n a_n \prod_{i\neq\nu=1}^{k} (X - \alpha_\nu Z)^{n_\nu}} W(Y, Z).
\end{aligned}$$

The first expression implies that all poles of ζ must belong to $C \bigcap \{Y = \beta_j\}$. The second expression implies that all poles of ζ must belong to $C \bigcap \{X = \alpha_\nu\}$ for all $\nu \neq i$. Since P satisfies the separation condition, we can conclude that $Q(\beta_j) = P(\alpha_i) \neq P(\alpha_\nu)$ for all $\nu \neq i$. This shows that ζ is regular on C if $m_j - n_i \geq 2$. The proof of the lemma is now complete. □

Proof of Theorem 2.1 It follows immediately from Proposition 3.2, 3.4 and 3.5. □

Proof of Theorem 2.2 By (9), we see that

$$\eta = \frac{W(Y, Z)}{n a_n \prod_{\nu=1}^{k} (X - \alpha_\nu Z)^{n_\nu}} = \frac{W(X, Z)}{m b_m Z^{n-m} \prod_{t=1}^{l} (Y - \beta_t Z)^{m_t}}.$$

Set

$$\theta := \frac{W(X, Z)}{(Y - \beta_1 Z)^2}.$$

We then have

$$\theta = \frac{mb_m Z^{n-m}(Y-\beta_1 Z)^{m_1-2}\prod_{t=2}^{l}(Y-\beta_t Z)^{m_t}}{na_n\prod_{\nu=1}^{k}(X-\alpha_\nu Z)^{n_\nu}}W(Y,Z).$$

Since $(\alpha_i, \beta_1, 1) \notin C$ with for all α_i, we see that θ is a well-defined regular 1-form of Wronskian type on C, proving the first assertion of the theorem.

Similary, let

$$\xi := \frac{W(X,Z)}{(Y-\beta_1 Z)(Y-\beta_2 Z)}.$$

Then ξ is a well-defined regular 1-form of Wronskian type on C and therefore the second assertion of theorem is proved. $\square$

Proof of Theorem 2.3 By applying Proposition 3.6 and Lemma 3.7, the Theorem is followed. $\square$

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