East-West J. of Mathematics: Vol. 6, No 1 (2004) pp. 1-14

DIAGONALITY AND POWERS OF GENERAL ALGEBRAIC SYSTEMS

Josef Šlapal

Department of Mathematics,
Brno University of Technology, 616 69 Brno,
Czech Republic

Abstract

This note is a short survey consisting of some results on the topic published by the author in a series of papers during the last fifteen years. But, in contrast to these papers in which general algebraic systems of different kinds are investigated independently, the note uses a unified, relational approach in which these systems are considered and investigated as special relational systems. The results show that diagonality plays an important role when studying powers (i.e., function spaces) of general algebraic systems because it ensures well behaviour of the powers.

0 Introduction

Given ordered sets $\mathbf{G} = (G, \leq)$ and $\mathbf{H} = (H, \leq)$, the Birkhoff's *direct* (i.e., *cardinal*) *power* of $\mathbf{G}$ and $\mathbf{H}$ is the ordered set $\mathbf{G}^{\mathbf{H}} = (F, \leq)$ where F is the set of all homomorphisms of $\mathbf{H}$ into $\mathbf{G}$ and, for any pair $f, g \in F$, $f \leq g$ if and only if $f(y) \leq g(y)$ for each $y \in H$ (see [4]). Direct powers of ordered sets behave analogously to powers of natural numbers. In particular, they fulfill the so-called *first exponential law*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H}\times\mathbf{K}}$$

Key words: Relational system, partial algebra, hyperalgebra, total algebra, diagonality, mediality, power, first exponential law.
2000 Mathematics Subject Classification: 08A02, 08A55.
Partially supported by Ministry of Education of the Czech Republic, project no. CEZ J22/98 260000013.

and the so-called *second exponential law*

$$\prod_{i\in I}\mathbf{G}_i^{\mathbf{H}} \cong (\prod_{i\in I}\mathbf{G}_i)^{\mathbf{H}}$$

where $\cong$ means an isomorphism and $\times$ and $\prod$ denote direct products.

Now, the following questions arise: Is it possible to extend direct powers, under the validity of the usual exponential laws, to relational systems which are more general than ordered sets? And is it possible to introduce such powers also for partial algebras, hyperalgebras and (total) algebras? In this note we give positive answers to these questions by describing the corresponding powers. We will also mention the categorical meaning and possible applications of the powers investigated.

1 Relational systems

For the proofs of the results of this paragraph see [16], [18] and [28].

Let n be a positive integer. By an *n-ary relational system* we understand a pair $\mathbf{G} = (G, p)$ where G is a set - the so-called *underlying set* of $\mathbf{G}$ - and p is an *n-ary relation* on G, i.e., $p \subseteq G^n (= \underbrace{G \times G \times \ldots \times G}_{n-\text{times}})$. If the underlying set of an n-ary relational system $\mathbf{G}$ is not given explicitly, we denote it by $|\mathbf{G}|$.

Let $\mathbf{G} = (G, p)$, $\mathbf{H} = (H, q)$ be a pair of n-ary relational systems. A map $f : G \to H$ is called a *homomorphism* of $\mathbf{G}$ into $\mathbf{H}$ provided that $(x_1, \ldots, x_n) \in p \Rightarrow (f(x_1), \ldots, f(x_n)) \in q$. If, moreover, $f : G \to H$ is a bijection and f^{-1} is a homomorphism of $\mathbf{H}$ into $\mathbf{G}$, then f is called an *isomorphism* of $\mathbf{G}$ onto $\mathbf{H}$. If there exists an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$, then we write $\mathbf{G} \cong \mathbf{H}$. We denote by $\mathrm{Hom}(\mathbf{G}, \mathbf{H})$ the set of all homomorphisms of $\mathbf{G}$ into $\mathbf{H}$.

The n-ary relational system $\mathbf{G}$ is said to be a *relational subsystem* of $\mathbf{H}$ provided that $G \subseteq H$ and $p = q \cap G^n$.

The direct product of a family $\mathbf{G}_i = (G_i, p_i)$, $i \in I$, of n-ary relational systems is the n-ary relational system $\prod_{i\in I}\mathbf{G}_i = (\prod_{i\in I} G_i, q)$ where, for any $f_1, \ldots, f_n \in \prod_{i\in I} G_i$, $(f_1, \ldots, f_n) \in q$ if and only if $(f_1(i), \ldots, f_n(i)) \in p_i$ for each $i \in I$. If the set I has just two elements, say $I = \{i_1, i_2\}$, then we write $\mathbf{G}_{i_1} \times \mathbf{G}_{i_2}$ instead of $\prod_{i\in I}\mathbf{G}_i$. If $\mathbf{G}_i = \mathbf{G}$ for each $i \in I$, then we write $\mathbf{G}^I$ instead of $\prod_{i\in I}\mathbf{G}_i$.

An n-ary relational system (G, p) is said to be *reflexive* provided that $(x_1, \ldots, x_n) \in p$ whenever $x_1 = x_2 = \ldots = x_n \in G$.

Definition 1.1 Let $\mathbf{G} = (G, p)$, $\mathbf{H} = (H, q)$ be n-ary relational systems. The *power* of $\mathbf{G}$ and $\mathbf{H}$ is the n-ary relational systems $\mathbf{G}^{\mathbf{H}} = (\mathrm{Hom}(\mathbf{H}, \mathbf{G}), r)$ where, for any $f_1, \ldots, f_n \in\mathrm{Hom}(\mathbf{H}, \mathbf{G})$, $(f_1, \ldots, f_n) \in r$ if and only if $(y_1, \ldots, y_n) \in q$ implies $(f_1(y_1), \ldots, f_n(y_n)) \in p$ whenever $y_1, \ldots, y_n \in H$. The power $\mathbf{G}^{\mathbf{H}} =$

$(\mathrm{Hom}(\mathbf{H}, \mathbf{G}), r)$ is said to have *point-wise structure* provided that $(f_1, ..., f_n) \in r$ if and only if $(f_1(y), ..., f_n(y)) \in p$ for each $y \in H$.

Clearly, the power $\mathbf{G}^{\mathbf{H}}$ of a pair of n-ary relational systems $\mathbf{G}$ and $\mathbf{H}$ has point-wise structure if and only if it is a relational subsystem of the direct product $\mathbf{G}^{|\mathbf{H}|}$. It is also evident that the power $\mathbf{G}^{\mathbf{H}}$ is always reflexive. The map $e : |\mathbf{H}| \times |\mathbf{G}|^{|\mathbf{H}|} \to |\mathbf{G}|$ defined by $e(y, f) = f(y)$ is called the *evaluation map* for $\mathbf{G}^{\mathbf{H}}$.

Proposition 1.2 *For any pair* $\mathbf{G}$, $\mathbf{H}$ *of* n*-ary relational systems, the evaluation map for* $\mathbf{G}^{\mathbf{H}}$ *is a homomorphism of* $\mathbf{H} \times \mathbf{G}^{\mathbf{H}}$ *into* $\mathbf{G}$.

An arbitrary set G can be considered to be the n-ary relational system $(G, \emptyset)$ (for every positive integer n). The power G^H of G and H is then the power of $(G, \emptyset)$ and $(H, \emptyset)$, hence the set of all maps of H into G. Given sets G, H, K, the map $f : (G^H)^K \to G^{H \times K}$ given by $f(g)(y, z) = g(z)(y)$ is said to be *canonical*.

Powers of relational structures fulfill, among others, the second exponential law
$$\prod_{i \in I} \mathbf{G}_i^{\mathbf{H}} \cong (\prod_{i \in I} \mathbf{G}_i)^{\mathbf{H}}$$
but they in general fail to fulfill the first one
$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \times \mathbf{K}}.$$
But the first exponential law is very important because it allows many important applications of powers. We will therefore give a sufficient condition for the validity of the first exponential law for n-ary relational systems.

Theorem 1.3 *Let* $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$ *be* n*-ary relational systems. If* $\mathbf{H}$, $\mathbf{K}$ *are reflexive, then*
$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \times \mathbf{K}}$$
and the corresponding isomorphism is given by the canonical map.

Definition 1.4 An n-ary relational system (G, p) is said to be *diagonal* provided that, whenever (x_{ij}) is an $n \times n$-matrix over G, from $(x_{i1}, ..., x_{in}) \in p$ for each $i = 1, ..., n$ and $(x_{1j}, ..., x_{nj}) \in p$ for each $j = 1, ..., n$ it follows that $(x_{11}, x_{22}, ..., x_{nn}) \in p$.

Theorem 1.5 *Let* $\mathbf{G}$, $\mathbf{H}$ *be* n*-ary relational systems. If* $\mathbf{G}$ *is diagonal and* $\mathbf{H}$ *is reflexive, then the power* $\mathbf{G}^{\mathbf{H}}$ *is diagonal and has point-wise structure.*

Example 1.6 It is evident that a binary relational system (G, p) is diagonal if and only if the binary relation p is transitive. Let $\mathbf{G}$, $\mathbf{H}$ be binary relational systems, $\mathbf{G}$ diagonal and $\mathbf{H}$ reflexive. Then the power $\mathbf{G}^{\mathbf{H}}$ is a preordered set and has point-wise structure. If, moreover, $\mathbf{K}$ is a reflexive binary relational system, then we have

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H}\times\mathbf{K}}.$$

Especially, if $\mathbf{G}$ and $\mathbf{H}$ are ordered sets, then $\mathbf{G}^{\mathbf{H}}$ coincides with the Birkhoff's direct power (and hence $\mathbf{G}^{\mathbf{H}}$ is an ordered set too).

2 Partial algebras

Most of the statements of this paragraph immediately follow from results of [27], [30] and [31].

An *n-ary partial algebra* is an $(n+1)$-ary relational system $\mathbf{G} = (G, p)$ such that from $(x_1, ..., x_n, y) \in p$ and $(x_1, ..., x_n, z) \in p$ it follows that $y = z$. The $(n+1)$-ary relation p on G is then called an *n-ary partial operation* on G and we write $y = p(x_1, ..., x_n)$ instead of $(x_1, ..., x_n, y) \in p$.

Let $\mathbf{G} = (G, p)$, $\mathbf{H} = (H, q)$ be a pair of n-ary partial algebras. Then, clearly, a map $f : G \to H$ is a homomorphism of $\mathbf{G}$ into $\mathbf{H}$ if and only if $p(x_1, ..., x_n) = x \Rightarrow q(f(x_1), ..., f(x_n)) = f(x)$ whenever $x_1, ..., x_n, x \in G$.

The n-ary partial algebra $\mathbf{G}$ is called a *partial subalgebra* of $\mathbf{H}$ provided that $G \subseteq H$ and, whenever $x_1, ..., x_n \in G$ and $x \in H$, $p(x_1, ..., x_n) = x \Leftrightarrow q(x_1, ..., x_n) = x$ (i.e., provided that $\mathbf{G}$ is a relational subsystem of $\mathbf{H}$ and, whenever $x_1, ..., x_n \in G$ and $x \in H$, from $q(x_1, ..., x_n) = x$ it follows that $x \in G$).

Of course, the direct product $\prod_{i\in I} \mathbf{G}_i = (\prod_{i\in I} G_i, q)$ of a family $(\mathbf{G}_i, p_i)$, $i \in I$, of n-ary partial algebras is an n-ary partial algebra (where, for any $f_1, ..., f_n$, $f \in \prod_{i\in I} \mathbf{G}_i$, $q(f_1, ..., f_n) = f$ if and only if $p_i(f_1(i), ..., f_n(i)) = f(i)$ for each $i \in I$).

For n-ary partial algebras we use the term *idempotent* instead of reflexive. So, an n-ary partial algebra (G, p) is idempotent provided that $p(x, ..., x) = x$ whenever $x \in G$.

Let $\mathbf{G} = (G, p)$, $\mathbf{H} = (H, q)$ be n-ary partial algebras and let $\mathbf{H}$ be idempotent. Then $\mathbf{G}^{\mathbf{H}} = (\mathrm{Hom}(\mathbf{H}, \mathbf{G}), r)$ is an n-ary partial algebra where, for any $f_1, ..., f_n, f \in \mathrm{Hom}(\mathbf{H}, \mathbf{G})$, $r(f_1, ..., f_n) = f$ if and only if $q(y_1, ..., y_n) = y$ implies $p(f_1(y_1), ..., f_n(y_n)) = f(y)$ whenever $y_1, ..., y_n, y \in H$. It is evident that $\mathbf{G}^{\mathbf{H}}$ is idempotent. It is also clear that the power $\mathbf{G}^{\mathbf{H}} = (\mathrm{Hom}(\mathbf{H}, \mathbf{G}), r)$ has point-wise structure if and only if, for any $f_1, ..., f_n, f \in \mathrm{Hom}(\mathbf{H}, \mathbf{G})$, the condition $r(f_1, ..., f_n) = f$ is equivalent to $p(f_1(y), ..., f_n(y)) = f(y)$ for each $y \in H$.

Theorem 1.3 immediately results in

Theorem 2.1 *Let* $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$ *be* n*-ary partial algebras. If* $\mathbf{H}$, $\mathbf{K}$ *are idempotent, then*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H}\times\mathbf{K}}.$$

By Definition 1.4, an n-ary partial algebra (G, p) is diagonal provided that, whenever (x_{ij}) is an $n \times n$-matrix over G and $x \in G$, from $p(p(x_{11}, ..., x_{1n}), ..., p(x_{n1}, ..., x_{nn})) = p(p(x_{11}, ..., x_{n1}), ..., p(x_{n1}, ..., x_{nn})) = x$ it follows that $p(x_{11}, x_{22}, ..., x_{nn}) = x$.

Theorem 1.5 immediately results in

Theorem 2.2 *Let* $\mathbf{G}$, $\mathbf{H}$ *be* n*-ary partial algebras. If* $\mathbf{G}$ *is diagonal and* $\mathbf{H}$ *is idempotent, then the power* $\mathbf{G}^{\mathbf{H}}$ *is diagonal and has point-wise structure.*

Of course, if $\mathbf{G}$, $\mathbf{H}$ are n-ary partial algebras, $\mathbf{G}$ diagonal and $\mathbf{H}$ idempotent, then the n-ary partial algebra $\mathbf{G}^{\mathbf{H}}$ is a relational subsystem but not necessarily a partial subalgebra of the direct product $\mathbf{G}^{|\mathbf{H}|}$. We will find a condition under which $\mathbf{G}^{\mathbf{H}}$ is a partial subalgebra of $\mathbf{G}^{|\mathbf{H}|}$.

Definition 2.3 An n-ary partial algebra (G, p) is called *medial* provided that, whenever (x_{ij}) is an $n \times n$-matrix over G and $x, x_1, ..., x_n \in G$, from $p(p(x_{11}, ..., x_{1n}), ..., p(x_{n1}, ..., x_{nn})) = x$ and $p(x_{1j}, ..., x_{nj}) = x_j$ for each $j = 1, ..., n$ it follows that $p(x_1, ..., x_n) = x$.

Theorem 2.4 *Let* $\mathbf{G}$, $\mathbf{H}$ *be* n*-ary partial algebras and let* $\mathbf{G}$ *be medial. Then there exists a partial subalgebra* $\mathbf{K}$ *of the direct product* $\mathbf{G}^{|\mathbf{H}|}$ *such that* $|\mathbf{K}| = \mathrm{Hom}(\mathbf{H}, \mathbf{G})$.

Corollary 2.5 *Let* $\mathbf{G}$, $\mathbf{H}$ *be* n*-ary partial algebras. If* $\mathbf{G}$ *is both diagonal and medial and* $\mathbf{H}$ *is idempotent, then the power* $\mathbf{G}^{\mathbf{H}}$ *is a medial* n*-ary partial subalgebra of the direct product* $\mathbf{G}^{|\mathbf{H}|}$.

Example 2.6 (1) Let (G, r) be a binary relational system and (G, p) be the binary partial algebra defined by

$$p(x, y) = z \Leftrightarrow (x, y) \in r \text{ and z = x}$$

whenever $x, y, z \in G$. Then we have:
a) (G, p) is idempotent if and only if (G, r) is reflexive,
b) (G, p) is diagonal if and only if (G, r) is transitive,
c) (G, p) is medial.

(2) Let $(X \times Y, p)$ be a *partial rectangular band*, i.e., a partial binary algebra where X, Y are sets and, whenever $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times Y$, $p((x_1, y_1), (x_2, y_2)) = (z_1, z_2)$ if and only if $x_1 = z_1,\ x_2 = y_1$ and $y_2 = z_2$. Then $(X \times Y, p)$ is diagonal and medial.

3 Hyperalgebras

This paragraph is a summary of [29].

An *n-ary hyperalgebra* is an $(n+1)$-ary relational system $\mathbf{G} = (G, p)$ such that for each $(x_1, ..., x_n) \in G$ there exists $y \in G$ with $(x_1, ..., x_n, y) \in p$. The $(n+1)$-ary relation p on G is then called an *n-ary hyperoperation* on G and we put $p(x_1, ..., x_n) = \{y \in G;\ (x_1, ..., x_n, y) \in p\}$. Further, if (G, p) is an n-ary hyperalgebra and $A_1, ...A_n$ are subsets of G, we put $p(A_1, ..., A_n) = \bigcup\{p(x_1, ..., x_n);\ x_i \in A_i \text{ for each } i = 1, ..., n\}$.

Let $\mathbf{G} = (G, p)$, $\mathbf{H} = (H, q)$ be a pair of n-ary hyperalgebras. Then, clearly, a map $f : G \to H$ is a homomorphism of $\mathbf{G}$ into $\mathbf{H}$ if and only if $f(p(x_1, ..., x_n)) \subseteq q(f(x_1), ..., f(x_n))$ whenever $x_1, ..., x_n \in G$.

The n-ary hyperalgebra $\mathbf{G}$ is called a *subhyperalgebra* of $\mathbf{H}$ provided that $G \subseteq H$ and, whenever $x_1, ..., x_n \in G$, $p(x_1, ..., x_n) = q(x_1, ..., x_n)$ (i.e., provided that $\mathbf{G}$ is a relational subsystem of $\mathbf{H}$ and $q(x_1, ..., x_n) \subseteq G$ whenever $x_1, ..., x_n \in G$).

Of course, the direct product $\prod_{i\in I} \mathbf{G}_i = (\prod_{i\in I} G_i, q)$ of a family $(\mathbf{G}_i, p_i)$, $i \in I$, of n-ary hyperalgebras is an n-ary hyperalgebra (where, for any $f_1, ..., f_n \in \prod_{i\in I} \mathbf{G}_i$, $q(f_1, ..., f_n) = \prod_{i\in I} p_i(f_1(i), ..., f_n(i))$.

Similarly to partial algebras, also for hyperalgebra we use the term *idempotent* instead of reflexive. So, an n-ary hyperalgebra (G, p) is idempotent provided that $x \in p(x, ..., x)$ whenever $x \in G$.

Unfortunately, given n-ary hyperalgebras $\mathbf{G} = (G, p)$ and $\mathbf{H} = (H, q)$, the power $\mathbf{G}^{\mathbf{H}}$ need not be an n-ary hyperalgebra even if $\mathbf{H}$ is idempotent. We will find conditions under which this power is an n-ary hyperalgebra (and a subhyperalgebra of the direct product $\mathbf{G}^{|\mathbf{H}|}$).

By Definition 1.4, an n-ary hyperalgebra (G, p) is diagonal provided that, whenever (x_{ij}) is an $n\times n$-matrix over G and $x \in G$, from $x \in p(p(x_{11}, ..., x_{1n}), ..., p(x_{n1}, ..., x_{nn}))$ and $x \in p(p(x_{11}, ..., x_{n1}), ..., p(x_{n1}, ..., x_{nn}))$ it follows that $x \in p(x_{11}, x_{22}, ..., x_{nn})$.

Definition 3.1 An n-ary hyperalgebra (G, p) is called *medial* provided that, whenever (x_{ij}) is an $n\times n$-matrix over G and $y_1, ..., y_n, z_1, ..., z_n \in G$, from $y_i \in p(x_{i1}, ..., x_{in})$ for each $i = 1, ..., n$ and $z_j \in p(x_{1j}, ..., x_{nj})$ for each $j = 1, ..., n$ it follows that $p(y_1, ..., y_n) = p(z_1, ..., z_n)$.

Theorem 3.2 *Let* $\mathbf{G}$, $\mathbf{H}$ *be* n*-ary hyperalgebras and let* $\mathbf{G}$ *be medial. Then there exists a subhyperalgebra* $\mathbf{K}$ *of the direct product* $\mathbf{G}^{|\mathbf{H}|}$ *such that* $|\mathbf{K}| = \mathrm{Hom}(\mathbf{H}, \mathbf{G})$.

Theorem 3.3 *Let* $\mathbf{G}$, $\mathbf{H}$ *be* n*-ary hyperalgebras. If* $\mathbf{G}$ *is both diagonal and medial and* $\mathbf{H}$ *is idempotent, then the power* $\mathbf{G}^{\mathbf{H}}$ *is a medial* n*-ary subhyperalgebra of the direct product* $\mathbf{G}^{|\mathbf{H}|}$.

Of course, the power from the previous Theorem is also idempotent and diagonal.

Corollary 3.4 *Let* $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$ *be* n*-ary hyperalgebras and let* $\mathbf{G}$ *be medial and diagonal and* $\mathbf{H}$, $\mathbf{K}$ *be idempotent. Then* $\mathbf{G}^{\mathbf{H}}$, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$ *and* $\mathbf{G}^{\mathbf{H}\times\mathbf{K}}$ *are* n*-ary hyperalgebras and we have*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H}\times\mathbf{K}}.$$

The following statement gives a useful criterion of diagonality for medial hyperalgebras:

Proposition 3.5 *Let* $\mathbf{G} = (G, p)$ *be a medial* n*-ary hyperalgebra. Then* $\mathbf{G}$ *is diagonal if and only if* $p(p(x_{11}, ..., x_{1n}), ..., p(x_{n1}, ..., x_{nn})) \subseteq p(x_{11}, x_{22}, ..., x_{nn})$ *for each* $n \times n$*-matrix* (x_{ij}) *over* G.

Example 3.6 (1) Let $(G, \leq)$ be an ordered set wit a least element 0 and let A be the set of all atoms of $(G, \leq)$. For any $x, y \in G$ put $x * y = \{z \in A;\ z < x,\ z < y\} \cup \{0\}$. Then $(G, *)$ is a hypergroupoid (i.e., a binary hyperalgebra) which is medial and diagonal.

(2) For any point (x_0, y_0) of the real plane $R \times R$ put $(x_0, y_0)^\neg = \{(x, y) \in R \times R;\ x \leq x_0,\ y \leq y_0\}$ (i.e., $(x_0, y_0)^\neg$ is the left lower quarter of $R \times R$ with the vertex (x_0, y_0)). Further, for any pair (x_1, y_1), (x_2, y_2) of points of $R \times R$ put $(x_1, y_1) * (x_2, y_2) = (x_1, y_2)^\neg$. Then $(R \times R, *)$ is a diagonal hypergroupoid.

4 Total algebras

The results of this paragraph are taken from [19], [21], [23], [24] and [26].

An *n-ary (total) algebra* is an $(n+1)$-ary relational system $\mathbf{G} = (G, p)$ which is both an n-ary partial algebra and an n-ary hyperalgebra or, equivalently, which has the property that p is an n-ary operation on G (i.e., a map $p : G^n \to G$). Then $p(x_1, ..., x_n) = y$ is written instead of $(x_1, ..., x_n, y) \in p$, which coincides with the denotation introduced for partial algebras (and, if we do not distinguish between y and $\{y\}$, also with the denotation introduced for hyperalgebras).

Let $\mathbf{G} = (G, p)$, $\mathbf{H} = (H, q)$ be a pair of n-ary algebras. Clearly, a map $f : G \to H$ is a homomorphism of $\mathbf{G}$ into $\mathbf{H}$ if and only if $f(p(x_1, ..., x_n)) = q(f(x_1), ..., f(x_n))$ whenever $x_1, ..., x_n \in G$.

The n-ary algebra $\mathbf{G}$ is called a *subalgebra* of $\mathbf{H}$ provided that $\mathbf{G}$ is a partial subalgebra of $\mathbf{H}$ (or, equivalently, provided that $\mathbf{G}$ is a subhyperalgebra of $\mathbf{H}$). Thus, $\mathbf{G}$ is a subalgebra of $\mathbf{H}$ if and only if $G \subseteq H$ and $p(x_1, ..., x_n) = q(x_1, ..., x_n)$ whenever $x_1, ..., x_n \in G$ (i.e., if and only if $\mathbf{G}$ is a relational subsystem of $\mathbf{H}$ such that $q(x_1, ..., x_n) \in G$ whenever $x_1, ..., x_n \in G$).

Of course, the direct product $\prod_{i\in I}\mathbf{G}_i=(\prod_{i\in I}G_i,q)$ of a family $(\mathbf{G}_i,p_i)$, $i\in I$, of n-ary algebras is an n-ary algebra (where, for any $f_1,...,f_n,f\in\prod_{i\in I}\mathbf{G}_i$, $q(f_1,...,f_n)=f\Leftrightarrow p_i(f_1(i),...,f_n(i))=f(i)$ for each $i\in I$).

To follow the convention introduced for partial algebras and hyperalgebras, also for algebras the term *idempotent* is used instead of reflexive. So, an n-ary algebra (G,p) is idempotent provided that $x=p(x,...,x)$ whenever $x\in G$.

Given n-ary algebras $\mathbf{G}=(G,p)$ and $\mathbf{H}=(H,q)$, the power $\mathbf{G}^{\mathbf{H}}$ need not be an n-ary algebra because, by the previous paragraph, it need not be an n-ary hyperalgebra. We will find conditions under which this power is an n-ary algebra (and subalgebra of the direct product $\mathbf{G}^{|\mathbf{H}|}$).

An n-ary algebra is said to be *diagonal* (resp. *medial*) provided that it is diagonal (resp. medial) as an n-ary partial algebra or, equivalently, as an n-ary hyperalgebra. Thus, an n-ary algebra (G,p) is

a) diagonal provided that, whenever (x_{ij}) is an $n\times n$-matrix over G and $x\in G$, from $p(p(x_{11},...,x_{1n}),...,p(x_{n1},...,x_{nn}))=p(p(x_{11},...,x_{n1}),...,p(x_{1n},...,x_{nn}))=x$ it follows that $p(x_{11},x_{22},...,x_{nn})=x$,

b) medial provided that, whenever (x_{ij}) is an $n\times n$-matrix over G, we have $p(p(x_{11},...,x_{1n}),...,p(x_{n1},...,x_{nn}))=p(p(x_{11},...,x_{n1}),...,p(x_{1n},...,x_{nn}))$.

Theorem 4.1 *Let $\mathbf{G}$, $\mathbf{H}$ be n-ary algebras and let $\mathbf{G}$ be medial. Then there exists a subalgebra $\mathbf{K}$ of the direct product $\mathbf{G}^{|\mathbf{H}|}$ such that $|\mathbf{K}|=\mathrm{Hom}(\mathbf{H},\mathbf{G})$.*

Theorem 4.2 *Let $\mathbf{G}$, $\mathbf{H}$ be n-ary algebras. If $\mathbf{G}$ is both diagonal and medial and $\mathbf{H}$ is idempotent, then the power $\mathbf{G}^{\mathbf{H}}$ is a medial n-ary subalgebra of the direct product $\mathbf{G}^{|\mathbf{H}|}$.*

Of course, the power from the previous Theorem is also idempotent and diagonal.

Corollary 4.3 *Let $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$ be n-ary algebras and let $\mathbf{G}$ be medial and diagonal and $\mathbf{H}$, $\mathbf{K}$ be idempotent. Then $\mathbf{G}^{\mathbf{H}}$, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$ and $\mathbf{G}^{\mathbf{H}\times\mathbf{K}}$ are n-ary algebras and we have*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}\cong\mathbf{G}^{\mathbf{H}\times\mathbf{K}}.$$

The following statement gives useful criterions of diagonality and mediality for algebras:

Proposition 4.4 *Let $\mathbf{G}=(G,p)$ be an n-ary algebra. Then the following conditions are equivalent:*

(1) $\mathbf{G}$ is diagonal and medial,

(2) $p(p(x_{11},...,x_{1n}),...,p(x_{n1},...,x_{nn}))=p(x_{11},x_{22},...,x_{nn})$ for each $n\times n$-matrix (x_{ij}) over G,

(3) $p(p(x_{11}, ..., x_{1n}), x_{22}, ..., x_{nn}) = p(x_{11}, p(x_{21}, ..., x_{2n}), x_{33}, ..., x_{nn}) = ... = p(x_{11}, ..., x_{n-1,n-1}, p(x_{n1}, ..., x_{nn})) = p(x_{11}, x_{22}, ..., x_{nn})$ *for each* $n \times n$*-matrix* (x_{ij}) *over* G*.*

Remark 4.5 a) Medial groupoids (i.e., binary algebras) are studied in [9].

b) In [13] there are studied n-ary idempotent, diagonal and medial algebras (called briefly *diagonal algebras*) and it is shown that they are, up to isomorphisms, the n-ary algebras $(X_1 \times ... \times X_n, p)$ where $X_1, ..., X_n$ are sets and the operation p is defined by $p((x_1^1, ..., x_n^1), ..., (x_1^n, ..., x_n^n)) = (x_1^1, x_2^2, ..., x_n^n)$. These algebras are special cases of the *rectangular algebras* discussed in [14]. Idempotent, diagonal and medial groupoids are usually called *rectangular bands* (cf. [6]).

Example 4.6 1. Let $\mathbf{G} = (G, \cdot)$ be a groupoid. Then, by Proposition 4.4, $\mathbf{G}$ is diagonal and medial if and only if $x(yz) = (xy)z = xz$ whenever $x, y, z \in G$. In other words, $\mathbf{G}$ is diagonal and medial if and only if it is a semigroup with $xyz = xz$ whenever $x, y, z \in G$. We have:

a) If card $G = 2$, then $\mathbf{G}$ is diagonal and medial if and only if $p : G^2 \to G$ is a constant map or a projection.

b) If $G = \{1, 2, 3\}$, then $\mathbf{G}$ is diagonal and medial if and only if $p : G^2 \to G$ is one of the following operations: the three constant maps, the two projections, the six binary operations given by the following tables

p	1	2	3
1	1	1	1
2	2	2	2
3	2	2	2

p	1	2	3
1	1	1	1
2	3	3	3
3	3	3	3

p	1	2	3
1	1	1	1
2	2	2	2
3	1	1	1

p	1	2	3
1	1	1	1
2	1	1	1
3	3	3	3

p	1	2	3
1	2	2	2
2	2	2	2
3	3	3	3

p	1	2	3
1	3	3	3
2	2	2	2
3	3	3	3

and the six dual operations.

c) If $G = \{1, 2, 3, 4\}$, then $\mathbf{G}$ is idempotent, diagonal and medial if and only if $p : G^2 \to G$ is one of the following eight operations: the two projections, the three binary operations given by the following tables

p	1	2	3	4
1	1	2	1	2
2	1	2	1	2
3	3	4	3	4
4	3	4	3	4

p	1	2	3	4
1	1	2	2	1
2	1	2	2	1
3	4	3	3	4
4	4	3	3	4

p	1	2	3	4
1	1	3	3	1
2	4	2	2	4
3	1	3	3	1
4	4	2	2	4

and the three dual operations.

d) If $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and p is given by the following table

p	1	2	3	4	5	6	7	8	9	10	11	12
1	6	4	5	4	5	6	6	6	4	5	4	5
2	7	9	10	9	10	7	7	7	9	10	9	10
3	8	11	12	11	12	8	8	8	11	12	11	12
4	6	4	5	4	5	6	6	6	4	5	4	5
5	6	4	5	4	5	6	6	6	4	5	4	5
6	6	4	5	4	5	6	6	6	4	5	4	5
7	7	9	10	9	10	7	7	7	9	10	9	10
8	8	11	12	11	12	8	8	8	11	12	11	12
9	7	9	10	9	10	7	7	7	9	10	9	10
10	7	9	10	9	10	7	7	7	9	10	9	10
11	8	11	12	11	12	8	8	8	11	12	11	12
12	8	11	12	11	12	8	8	8	11	12	11	12,

then **G** is diagonal and medial.

2. Recall that a groupoid with a neutral element is called a *loop* provided that it satisfies both the left and right unique division laws. A loop **G** is medial if and only if it has the so-called *Hamiltonian property*, i.e., if and only if each subloop of **G** is a block of a congruence on **G**.

Remark 4.7 All the previous considerations and results can be naturally extended from n-ary algebraic systems, i.e., systems of type $\langle n \rangle$, to universal algebraic systems of an arbitrary type $\langle n_k;\ k \in K \rangle$, i.e., systems $\mathbf{G} = (G, (p_k;\ k \in K))$ where (G, p_k) is an n_k-ary relational system (or an n_k-ary partial algebra or an n_k-ary hyperalgebra or an n_k-ary algebra, respectively) for each $k \in K$. (Homomorphisms and isomorphisms of universal algebraic systems are defined component-wise). To do so, it suffices only to consider the reflexivity (resp. idempotency) and diagonality component-wise and to replace the mediality with the so-called *interchange law*. For example, for a universal algebra $\mathbf{G} = (G, (p_k;\ k \in K))$ the interchange law has the form

$$p_k(p_l(x_{11}, ..., x_{1n_l}), ..., p_l(x_{n_k1}, ..., x_{n_kn_l})) =$$

$$p_l(p_k(x_{11},...,x_{n_k1}),...,p(x_{1n_l},...,x_{n_kn_l}))$$

whenever $k,l \in K$ and (x_{ij}) is an $n_k \times n_l$-matrix over G. (Universal algebras fulfilling the interchange law are also called *commutative* in the literature - see e.g. [10],[11]).

All considerations can also be generalized to the cases when the arities of relations (or partial algebras or hyperalgebras or algebras, respectively) are arbitrary sets, not only positive integers - this is the case of most of the papers [15]-[21]. Let us note also that there are several modifications of the diagonality of n-ary relational systems (resp. partial algebras) which give analogical results as the diagonality - see e.g. [28] (resp. [30]).

5 Categorical point of view and applications

For the categorical terminology used in this paragraph see [1] and [2].

Recall that a category $\mathcal{K}$ with finite products is said to be *cartesian closed* provided that, for any $\mathcal{K}$-object $\mathbf{H}$, the functor $\mathbf{H} \times - : \mathcal{K} \to \mathcal{K}$ has a right adjoint $-^{\mathbf{H}}$, i.e., for each $\mathcal{K}$-object $\mathbf{G}$ there exist a $\mathcal{K}$-object $\mathbf{G}^{\mathbf{H}}$ and a $\mathcal{K}$-morphism $e : \mathbf{H} \times \mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ such that, whenever $\mathbf{K}$ is a $\mathcal{K}$-object and $f : \mathbf{H} \times \mathbf{K} \to \mathbf{G}$ a $\mathcal{K}$-morphism, there exists a unique $\mathcal{K}$-morphism $f^* : \mathbf{K} \to \mathbf{G}^{\mathbf{H}}$ such that $e \circ (id_{\mathbf{H}} \times f^*) = f$.

Let us consider the following seven categories the morphisms in which are given by homomorphisms:
$IRel_n$ - the category of reflexive n-ary relational systems,
$IDRel_n$ - the category of reflexive and diagonal n-ary relational systems,
$IPal_n$ - the category of idempotent n-ary partial algebras,
$IDPal_n$ - the category of idempotent and diagonal n-ary partial algebras,
$IDMPal_n$ - the category of idempotent, diagonal and medial n-ary partial algebras,
$IDMHyp_n$ - the category of idempotent, diagonal and medial n-ary hyperalgebras,
$IDMAlg_n$ - the category of idempotent, diagonal and medial n-ary algebras.

By the previous results, all the categories are cartesian closed: the roles of e and f^* are played by the evaluation map and $\varphi^{-1}(f)$, respectively, where φ is the canonical map. Cartesian closed categories have many applications because they posses well-behaved powers. For example, in logic, categories are interpreted as *deductive systems*: objects are considered to be *formulas* and morphisms to be *deductions*. Cartesian closed categories then represent special cases of the so-called *positive intuitionistic propositional calculus*: products $A \times B$ are identified with conjunctions $A \wedge B$ and powers A^B with implications $B \Rightarrow A$. The first exponential law gives an equivalence between the formulas $C \Rightarrow (B \Rightarrow A)$ and $(B \wedge C) \Rightarrow A$ called the *deduction law*. It is also known that cartesian closed categories are models of the so-called typed λ-calculi (a typed λ-*calculus* is an abstract programming language). So, we get a bridge between intuitionistic logic and λ-calculus.

Let $\mathcal{K}$ be a construct, i.e., a category whose objects are structured sets and whose morphisms are structure-compatible maps. For any $\mathcal{K}$-object $\mathbf{G}$, we denote by $|\mathbf{G}|$ its underlying set. Given $\mathcal{K}$-objects $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$, a map $f : |\mathbf{G}| \times |\mathbf{H}| \to |\mathbf{K}|$ is called a *bimorphism* if for each $x \in |\mathbf{G}|$ the map $f(x, -) : |\mathbf{H}| \to |\mathbf{K}|$ is a $\mathcal{K}$-morphism of $\mathbf{H}$ into $\mathbf{K}$, and for each $y \in |\mathbf{H}|$ the map $f(-, y) : |\mathbf{G}| \to |\mathbf{K}|$ is a $\mathcal{K}$-morphism of $\mathbf{G}$ into $\mathbf{K}$. A *tensor product* of $\mathcal{K}$-objects $\mathbf{G}$ and $\mathbf{H}$ is a $\mathcal{K}$-object $\mathbf{G} \otimes \mathbf{H}$ together with a bimorphism $f : |\mathbf{G}| \times |\mathbf{H}| \to |\mathbf{G} \otimes \mathbf{H}|$ such that, whenever $\mathbf{K}$ is a $\mathcal{K}$-object and $g : |\mathbf{G}| \times |\mathbf{H}| \to |\mathbf{K}|$ a bimorphism, there exists a unique $\mathcal{K}$-morphism $g^* : \mathbf{G} \otimes \mathbf{H} \to \mathbf{K}$ such that $g = g^* \circ f$. By the previous results, in the four (cartesian closed) categories $IDRel_n$, $IDMPal_n$, $IDMHyp_n$ and $IDMAlg_n$ powers $\mathbf{G}^{\mathbf{H}}$ are subobjects of the direct products $\mathbf{G}^{|\mathbf{H}|}$. This fact together with some further properties of these categories (*semifinal completeness* and the existence of *unit objects* - see [1]) imply that in each of the four categories tensor products coincide with direct products - cf. [26]. For example, in $IDMAlg_n$ it means that, whenever $\mathbf{G}, \mathbf{H} \in IDMAlg_n$, $\mathbf{G} \times \mathbf{H}$ is isomorphic to the factor n-ary algebra of the free idempotent, diagonal and medial n-ary algebra $\mathbf{K}$ over the set $|\mathbf{G}| \times |\mathbf{H}|$ with respect to the least congruence on $\mathbf{K}$.

References

[1] J. Adámek, Theory of Mathematical Structures, D. Reidel Publ. Comp., Dordrecht-Boston-Lancaster, 1983.

[2] J. Adámek, H. Herrlich, and G.E. Strecker, Abstract and Concrete Categories, Wiley-Intersci., New York, 1990.

[3] B. Banaschewski and E. Nelson, Tensor products and bimorphisms, Canad. Math. Bull. 19 (1976), 385–402.

[4] G. Birkhoff, Generalized arithmetic, Duke Math. J. 9 (1942), 283–302.

[5] P. Burmeister, A Model Theoretic Oriented Approach to Partial Algebras, Akademie-Verlag, Berlin, 1986.

[6] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vol. I, American Mathematical Society, Providence, Rhode Island, 1964.

[7] P. Corsini (ed.), Algebraic Hyperstructures and Applications, Proceedings of the Sixth International Congress, Prague, 1996, Democritus University of Thrace, Greece, 1997.

[8] G. Grätzer, Universal Algebra, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[9] J. Ježek and T. Kepka, Medial groupoids, Rozpravy ČSAV, Řada Mat. a Přír. Věd. 93/1, Academia, Prague, 1983.

[10] L. Klukovits, On commutative universal algebras, Acta Sci. Math. (Szeged) 34 (1973), 171–174.

[11] F.E. Linton, Autonomous equational categories, J. Math. Mech. 15 (1966), 637–642.

[12] M. Novotný and J. Šlapal, On powers of relational structures, Publ. Math. Debrecen 61 (2002), 341–356.

[13] J. Plonka, Diagonal algebras, Fund. Math. 58 (1966), 309–321.

[14] R. Pöschel and M. Reichel, Projection algebras and rectangular algebras, in: General Algebra and Applications, Heldermann-Verlag, Berlin, 1993.

[15] J. Šlapal, On the direct power of relational systems, Math. Slovaca 39 (1989), 251–255.

[16] J. Šlapal, Cartesian closedness in categories of relational systems, Arch. Math. 52 (1989), 603–606.

[17] J. Šlapal, Directly cartesian closed subcategories of relational categories, Ann. Univ. Sci Budapest., s. Math. 33 (1990), 161–166.

[18] J. Šlapal, Direct arithmetic of relational systems, Publ. Math. Debrecen 38 (1991), 39–48.

[19] J. Šlapal, A note on diagonality of algebras, Math. Nachr. 158 (1992), 195–197.

[20] J. Šlapal, Cardinal arithmetic of general relational systems, Czech. Math. J. 43 (1993), 125–139.

[21] J. Šlapal, A cartesian closed category of universal algebras, Acta Sci. Math. (Szeged) 58 (1993), 93–97.

[22] J. Šlapal, Exponentiality in concrete categories, New Zealand J. of Math. 22 (1993), 87–90.

[23] J. Šlapal, A cartesian closed category of universal algebras, Acta Sci. Math. (Szeged) 58 (1993), 93–97.

[24] J. Šlapal, On exponentiation of n-ary algebras, Acta Math. Hungar. 63 (1994), 313–322.

[25] J. Šlapal, On exponentiation in concrete categories, Publ. Math. Debrecen 44 (1994), 249–253.

[26] J. Šlapal, Categories of universal algebras in which direct products are tensor products, Ann. Math. Silesianae 9 (1995), 7–10.

[27] J. Šlapal, Cartesian closedness in categories of partial algebras, Math. Pannon. 7 (1996), 273-279 .

[28] J. Šlapal, On some cartesian closed relational categories, in: K.Denecke and O.Lüders (eds.), General Algebra and Applications in Discrete Mathematics, Proceedings of Conference on General Algebra and Discrete Mathematics, Potsdam, 1996, Shaker Verlag, Aachen, 1997 .

[29] J. Šlapal, On exponentiation of universal hyperalgebras, Alg. Universalis 44 (2000), 187–193.

[30] J. Šlapal, Another cartesian closed category of partial algebras, Demonstr. Math. 33 (2000), 689–694.

[31] J. Šlapal, Exponentiality in categories of partial algebras, Mat. Pannon. 13 (2002), 201–206.

[32] J. Šlapal, Categories of locally diagonal partial algebras, Contr. to Alg. and Geom. - to appear.