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A 3-$(v, 4, 1)$ PACKING CONSTRUCTION

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Abstract

The packing number $D(v, k, t)$ is the maximum number of blocks for any t-$(v, k, 1)$ packing. A construction for 3-$(v, 4, 1)$ is given, which when applied to $v \equiv 5 (\text{mod } 6)$ produces packings which can improve the packing number for $2v + 1$ in certain cases. In some cases, it can achieve the bound e.g. it is shown that $D(27, 4, 3) = 702$.

1 Introduction

Let $v \geq k \geq t$. A t-(v, k, λ) *packing* is a pair $(X, \mathcal{B})$, where X is a v-set of elements (*points*) and $\mathcal{B}$ is a collection of k-subsets of X (*blocks*), such that every t-subset of points occurs in at most λ blocks in $\mathcal{B}$. The *packing number* $D_\lambda(v, k, t)$ is the maximum number of blocks in any t-(v, k, λ) packing. A t-(v, k, λ) packing $(X, \mathcal{B})$ is *optimal* if $|\mathcal{B}| = D_\lambda(v, k, t)$. If $\lambda = 1$, we usually write $D(v, k, t)$ for $D_1(v, k, t)$.

Upper bounds for $D_\lambda(v, k, t)$ were given by Johnson cited from [1; page 409-410] as stated in the following theorems.

Theorem 1.1. *$D_\lambda(v, k, t) \leq \lfloor \frac{v \cdot D_\lambda(v-1, k-1, t-1)}{k} \rfloor$. Iterating this bound yields $D_\lambda(v, k, t) \leq U_\lambda(v, k, t)$, where $U_\lambda(v, k, t) = \lfloor \frac{v}{k} \lfloor \frac{v-1}{k-1} \cdots \lfloor \frac{\lambda(v-t+1)}{k+t-1} \rfloor \rfloor \rfloor$.*

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Further, if $\lambda(v-1) \equiv 0(\mathrm{mod}\ (k-1))$ *and* $\lambda v(v-1) \equiv -1(\mathrm{mod}\ k)$, *then* $D_\lambda(v,k,2) \leq U_\lambda(v,k,2) - 1$.

Theorem 1.2. *Suppose* $d = D_\lambda(v,k,t) = qv + r$, *where* $0 \leq r \leq v-1$. *Then* $q(q-1)v + 2qr \leq (t-1)d(d-1)$. *From this it follows that* $D_\lambda(v,k,t) \leq \lfloor \frac{v(k+1-t}{k^2 - v(t-1)} \rfloor$.

This note is motivated by a remark in Stinson ([1]; page 410) that for $v \equiv 5(\mathrm{mod}\ 6)$, $v \geq 17$, essentially nothing is known about the number $D(v,4,3)$, the packing number for a 3-$(v,4,1)$ packing and by the well known construction of SQS$(2v)$ from SQS(v), see for example Lindner and Rodger [2].

For the most part, our notation and terminology follows that of Lindner and Rodger [2].

2 Construction from v to $2v$

Observe that the well known $2v$ construction for SQS to produce SQS$(2v)$ from an SQS(v), can be applied for packing and for any even integer v where a packing exists.

Theorem 2.1. *A* 3-$(v,4,1)$ *packing* $(X,\mathcal{B})$ *for even* v *implies the existence of a* 3-$(2v,4,1)$ *packing.*

Proof A 3-$(v,4,1)$ packing $(X,\mathcal{B})$ is given. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_{v-1}\}$ be a 1-factorization of K_v defined on X. Then the following blocks together produce a 3-$(2v,4,1)$ packing on the set of vertices $X \times \{1,2\}$ and set of blocks $\mathcal{B}'$, where $\mathcal{B}'$ is defined as follows.

1. For each block $B \in \mathcal{B}$, $B \times \{1\}, B \times \{2\} \in \mathcal{B}'$.
2. For each $F_j \in \mathcal{F}$, $\{(a,1),(b,1),(c,2),(d,2)\} \in \mathcal{B}'$ if and only if $\{a,b\} \in F_j$ and $\{c,d\} \in F_j$.

□

Corollary 2.2. *The construction produces optimal packing for* $v \equiv 0(\mathrm{mod}\ 6)$

Proof Let $v = 6t$. Then $D(6t,4,3) = \lfloor \frac{18t^3 - 9t^2 - 3t}{2} \rfloor = \frac{18t^3 - 9t^2 - 3t}{2}$. On the other hand the construction produces $\mathcal{B}'$ with $|\mathcal{B}'| = 2 \cdot \frac{18t^3 - 9t^2 - 3t}{2} + (3t) \cdot (3t) \cdot (6t-1) = 72t^3 - 18t^2 - 3t$, the exact number needed for optimal packing for 3-$(v,4,1)$, i.e, $D(12t,4,3)$ is achieved. This means, the construction produces an optimal 3-$(2v,4,1)$ packing, just like in the case of SQS(v). □

For example, when $v = 6$, the packing number $D(6,4,3)$ is 3. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_5\}$ be a set of 1-factorization of K_6. For each $F_j \in \mathcal{F}$, F_i contains 3 independent edges. Therefore we get a packing for $2v = 12$ with $3 + 3 + 3 \cdot 3 \cdot 5 = 51$ blocks. Note that $D(12,4,3)$ is 51.

3 Construction from v to $2v + 1$

A slight modification of above construction is very useful as it can be used to get 3-$(v, 4, 1)$ packing for $v \equiv 5 (\text{mod } 6)$.

Theorem 3.1. *Packings* 3-$(v, 4, 1)$ *and* 3-$(v + 1, 4, 1)$ *implies the existence of a* 3-$(2v + 1, 4, 1)$ *packing.*

Proof Let v be an odd integer. Suppose a packing 3-$(v, 4, 1)$ and a packing 3-$(v + 1, 4, 1)$ with sets of blocks $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, are given. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_v\}$ be a 1-factorization of K_{v+1}. Then the following blocks together produce a 3-$(2v + 1, 4, 1)$ packing on the set of vertices $\{(i, 1) : i = 1, 2, \cdots, v\} \cup \{(i, 2) : i = 1, 2, \cdots, v + 1\}$ with set of blocks $\mathcal{B}'$ according to the following rules.

1. For each block $B \in \mathcal{B}_1$, $B \times \{1\} \in \mathcal{B}'$.
2. For each block $B \in \mathcal{B}_2$, $B \times \{2\} \in \mathcal{B}'$.
3. For each $F_j \in \mathcal{F}$, $\{(a, 1), (b, 1), (c, 2), (d, 2)\} \in \mathcal{B}'$ if and only if $\{a, b\} \in F_j$ and $\{c, d\} \in F_j$.

□

The construction produces $|\mathcal{B}_1| + |\mathcal{B}_2| + \frac{v-1}{2} \cdot \frac{v+1}{2} \cdot v$ blocks. Moreover the construction gives blocks with more structure than the packing obtained from deleting an element from a 3-$(2v + 2, 4, 1)$ and it can produce optimal packing. For example, when $v = 7$, we get $7 + 14 + 3 \cdot 4 \cdot 7 = 105$ blocks, which gives an optimal packing as $D(15, 4, 1) = 105$.

Note that if $v \equiv 5(\text{mod } 6)$, then $2v + 1 \equiv 5(\text{mod } 6)$. Let us first consider in the case when $v = 5$. It is known that $D(5, 4, 3) = 1$ and $D(6, 4, 3) = 3$. Thus by the above construction, we get $D(11, 4, 3) \geq 1 + 3 + 2 \cdot 3 \cdot 5 = 34$. It is also known that $D(11, 4, 3) = 35$. So, the construction achieves one block less than an optimal packing in this case. Now consider when $v = 11$. It is known that $D(11, 4, 3) = 35$ and $D(12, 4, 3) = 51$. Thus by the above construction, we get $D(23, 4, 3) \geq 35 + 51 + 5 \cdot 6 \cdot 11 = 416$. It is known that $D(23, 4, 3) \leq \lfloor \frac{23}{4} \lfloor \frac{22}{3} \lfloor \frac{21}{2} \rfloor \rfloor \rfloor = 419$. Thus $416 \leq D(23, 4, 3) \leq 419$. For $v = 17$, we know that $D(35, 4, 3) \leq 1583$ and by the construction gives that $D(35, 4, 3) \geq 1579$. Finally, it is known that $D(27, 4, 3) = 702$, and the construction of 3-$(27, 4, 1)$ above produces $65 + 91 + 6 \cdot 7 \cdot 13 = 702$ blocks. In fact, we have the following corollary.

Corollary 3.2. *If* $v \equiv 1(\text{mod } 12)$, *say* $v = 12s + 1$, $D(24s + 3, 4, 3) = (24s + 2)(24s + 3)s = 576s^3 + 120s^2 + 6s$ *and the construction also produces the same number of blocks required to give optimal packing.*

4 Generalization of v to $2v$ construction from SQS(v) to $(2t-1)$-$(v,2t,1)$ packing for any t

Suppose large set exists for (v,t), in other words assume that all t-subsets of v-set can be partitioned to BIBD$(v,t,1)$. Note that the set of all $2t$-subsets of a v-set is a $(v, \binom{v}{t}, \binom{v}{t-1}, t, \binom{v}{t-1} \cdot (\frac{t-1}{v-1}))$. So there are $s = \binom{v}{t-1} \cdot (\frac{t-1}{v-1})$ $BIBD(v,t,(\lambda=1)$-designs. Call them $F_1, F_2, \cdots, F_s$. We have the following construction.

Theorem 4.1. *Suppose there exists a $(2t-1)$-$(v,2t,1)$ packing with M blocks, then there exists a $(2t-1)$-$(2v,2t,1)$ packing with* $2M+[\frac{t-1}{v-1}\cdot\binom{v}{t-1}]^2\cdot[\frac{\binom{v}{t}}{\binom{v}{t-1}\cdot\frac{t-1}{v-1}}]$.

Proof As usual let $V=\{1,2,\cdots,v\}$ and the blocks of $(2t-1)$-$(v,2t,1)$ packing be $\mathcal{B}$. The blocks of $(2t-1)$-$(2v,2t,1)$ are in $\mathcal{B}'$ on the set $V\times\{1,2\}$, according to the following rules.

1. If $B\in\mathcal{B}$, then $B\times\{1\}$ and $B\times\{2\}$ are in $\mathcal{B}'$.
2. For each F_j, if $\{a_1,a_2,\cdots,a_t\}$ and $\{b_1,b_2,\cdots,b_t\}$ are any two not necessary distinct blocks in F_j, then $\{(a_1,1),(a_2,1),\cdots,(a_t,1),(b_1,2),(b_2,2),\cdots,(b_t,2)\}$ is in $\mathcal{B}'$.

Note that two blocks of size t in F_j intersect in only 1 point. Therefore the intersection between any two distinct blocks directly obtained from $\mathcal{B}$ and from F_j is at most t and the intersection between the blocks from $\mathcal{B}$ can be $(2t-1)$. Intersection of blocks obtained from the same F_j can be at most $(t+1)$. Intersection of blocks obtained from different F_j can be at most $2(t-1)=2t-2$.

□

For example, let $v=7$, we have $D(7,6,1)=1$. Since $\binom{7}{3}=35$, there is a partition of all subsets of size 3 into 5 2-$(7,3,1)$. So there exists 5-$(14,6,1)$ packing with $1+1+5\times7\times7=247$ blocks. The bound for 5-$(14,6,1)$ is $\lfloor\frac{14}{6}\lfloor\frac{13}{5}\lfloor\frac{12}{4}\lfloor\frac{11}{3}\lfloor\frac{10}{2}\rfloor\rfloor\rfloor\rfloor\rfloor=326$. The construction does not provide close enough number of blocks but notice that block intersections are much lower than $2t-2$ for many pairs.

References

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[2] C. C. Lindner and C. A. Rodger, Design Theory, CRC Press, 1997.