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ON THE NOETHERIAN DIMENSION OF ARTINIAN GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract

In this note, we first prove some bounds for Noetherian dimension of Artinian generalized local cohomology modules $H^j_I(M, N)$ in the cases of small levels. Secondly, in the cases of top generalized local cohomology modules, some bounds of Noetherian dimension for such modules are given.

1 Introduction

Throughout this note, let (R, m) be a commutative Noetherian local ring of dimension d, I an ideal of R, M and N finitely generated R-modules, and A an Artinian $R-$module. The concept of Krull dimension for Artinian modules was introduced by R. N. Roberts [17]. D. Kirby [13] changed the terminology of Roberts [17] and refered to Noetherian dimension (N-dim) to avoid confusion with Krull dimension defined for finitely generated modules. In this note we use the terminology of Kirby [13]. Note that if q is an ideal of R such that $\ell_R(0 : q)_A < +\infty$ then $\ell_R(0 : q^n)_A$ is a polynomial with rational coefficients for large n (cf. [13, Pro. 2]), and furthermore

$$\begin{aligned}\text{N-dim}_R(A) &= \deg(\ell_R(0 : q^n)_A)\\ &= \inf\{t \mid \exists x_1, \dots, x_t \in m, \ell_R(0 : (x_1, \dots, x_t)_A) < +\infty\}\end{aligned}$$

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(cf. [17, Th. 6]). Hence, Noetherian dimension has an important role in the theory of Artinian modules. Many properties of Noetherian dimension of Artinian modules have been given in [17], [13], [21], [8], [7]. The jth generalized local cohomology module $H^j_I(M,N)$ of two modules M and N with respect to I introduced by J. Herzog in [10], which is defined as

$$H^j_I(M,N) = \varinjlim_n \operatorname{Ext}^j_R(M/I^nM, N).$$

There is a number of researching on these modules such as [20], [22], [5], [6], [11], [10], [1], [12]. But until now, little is known about these modules. The purpose of this paper is to study the Noetherian dimension for generalized local cohomology modules. Concretely, we prove the following theorems.

Theorem 1.1. *Let r be a positive integer. If $H^j_I(M,N)$ is Artinian for all $j < r$ then* $\text{N-dim}_R(H^j_I(M,N)) \leq j$ *for all $j < r$.*

Since $H^j_I(R,N) \cong H^j_I(N)$ is usual local cohomology module of N with respect to I, thus, by replacing $M = R$ in Theorem 1.1, we get again a result of Cuong-Nhan in [7, Thm. 3.1] which showed that if $H^j_I(N)$ is Artinian for all $j < t$ for some given integer t then $\text{N-dim}_R(H^j_I(N)) \leq j$ for all $j < t$. Because $H^j_I(N) = 0$ for all $j > 0$ if $\ell(N) < \infty$, but $H^j_I(M,N)$ may not vanish for $j > 0$ in this case (see [5, Rem. 5.5 (i)]). Therefore the proof method in [7] is not valid for our case.

In [7], they also proved that $\text{N-dim}_R(H^n_I(N)) = n$ where $n = \dim_R(N)$ (see [7, Thm. 3.5]). But for the case of generalized local cohomology modules, the similar problem is very complicatetive. We known that in general the module $H^j_I(M,N)$ may not vanish for infinitely many j (for example, see [5, Rem. 5.5 (i)]). We here get the following result in the case of finite homological dimensions of M or N.

Theorem 1.2. *Let $d = \dim(R)$. If $\operatorname{pd}_R(M) < \infty$ or $\operatorname{injd}_R(N) < \infty$ then* $\text{N-dim}_R(H^d_I(M,N)) \leq d - \operatorname{depth}(\operatorname{Ann}_R(M), N)$.

In order to prove Theorem 1.2 we have to establish a key result in Lemma 4.1 which are extensions of Herzog-Zamani ([11, Lem. 3.1]) and Bruns-Herzog ([4, Cor. 3.5.11]). Moreover, we can caculate Noetherian dimension of top non-vanishing generalized local cohomology module.

Theorem 1.3. *Set $d = \dim(R)$. Assume that $H^d_I(M,N) \neq 0$ and $\operatorname{injd}_R(N) < +\infty$. Let $D_0 \subset D_1 \subset \ldots \subset D_{n-1} \subset D_n = \widehat{M}$ be the dimension filtration of $\widehat{M}$, where $D_0 = H^0_m(\widehat{M})$. Set $t = \max\{i \mid H^d_{\widehat{I}}(D_i/D_{i-1}, \widehat{N}) \neq 0, 0 \leq i \leq n\}$. Then* $\text{N-dim}_R(H^d_I(M,N)) = \dim_{\widehat{R}}(D_t)$.

Let us show about the organization of the paper. The next section we recall some basic properties of Noetherian dimension, and recall some known results on generalized local cohomology modules. Section 3 proves Theorem 1.1. The last section devote to prove Theorem 1.2 and 1.3.

2 Preliminaries

First, we recall the notion of Noetherian dimension.

Definition 2.1. ([13], cf. [7, Def. 2.1]). Let A be an Artinian $R-$module. The *Noetherian dimension* of A, denoted by $\text{N-dim}_R(A)$, is defined inductively as follows: when $A = 0$, put $\text{N-dim}_R(A) = -1$. Then by induction, for an integer $t \geq 0$, we put $\text{N-dim}_R(A) = t$ if $\text{N-dim}_R(A) < t$ is false and for every ascending sequence $A_0 \subseteq A_1 \subseteq \ldots$ of submodules of A, there exists n_0 such that $\text{N-dim}_R(A_n/A_{n+1}) < t$ for all $n > n_0$. Therefore $\text{N-dim}_R(A) = 0$ if and only if A is a non-zero Noetherian module.

The next lemma is a basic properties of the Noetherian dimension.

Lemma 2.2. ([7, Lem. 2.2]) *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of Artinian $R-$modules. Then we have*

$$\text{N-dim}_R(A_2) = \max\{\text{N-dim}_R(A_1), \text{N-dim}_R(A_3)\}.$$

The theory of secondary representation is in some sense dual to that of primary decomposition. An $R-$module $S \neq 0$ is said to be *secondary* if for any $x \in R$, the multiplication by x on S is either surjective or nilpotent. The radical of the annihilator of S is then a prime ideal p and we say that S is $p-$secondary. An $R-$module S is said to be *representable* if it has a minimal secondary representation, i.e. it has an statement $S = S_1 + S_2 + \ldots + S_n$ of p_i-secondary modules S_i, where $p_1, \ldots, p_n$ are all distinct and $S_i \nsubseteq S_1 + \ldots + S_{i-1} + S_{i+1} + \ldots + S_n$ for all $i = 1, \ldots, n$. Then the set $\{p_1, p_2, \ldots, p_n\}$ is independent of the choice of minimal representation of S. This set is denoted by $\text{Att}_R(S)$ and called the set of attached prime ideals of S (see [13], [14] for more details).

Recall that the Krull dimension of A, denoted by $\dim_R(A)$ is the Krull dimension of the Noetherian ring $R/\operatorname{ann}_R(A)$, i.e. $\dim_R(A) = \dim(R/\operatorname{ann}_R(A))$ (see [15]). For convention, we stipulate that $\dim_R(A) = -1$ if $A = 0$. Note that every Artinian module A is representable and the set of minimal prime ideals of $\operatorname{ann}_R(A)$ is just the set of minimal elements of $\text{Att}_R(A)$ (see [14]). Therefore

$$\dim_R(A) = \max\{\dim(R/p) \mid p \in \text{Att}_R(A)\}.$$

On the other hand, let $\widehat{R}$ be the $m-$adic completion of R. Then A has a natural structure as an $\widehat{R}-$module as follows: let $(x_n) \in \widehat{R}$, where $x_n \in R$, and let $u \in A$. Then we get $m^n u = 0$ for some $n \in \mathbb{N}$. Thus $x_n u$ is constant for all

$n \gg 0$. So we defined $(x_n)u = x_n u$ for $n \gg 0$. With this structure, a subset of A is an $R-$submodule if and only if it is an $\widehat{R}-$submodule. Thus we have

$$\text{N-dim}_R(A) = \text{N-dim}_{\widehat{R}}(A).$$

Lemma 2.3. ([7, Lem. 2.3, 2.4, Cor. 2.5]) *Let A be an Artinian $R-$module. Then the following statements are true.*

(i) $\text{N-dim}_R(A) \leq \dim_R(A)$*; the equality holds if $R = \widehat{R}$.*

(ii) $\text{N-dim}_R(A) \leq \text{N-dim}_R\big((0:x)_A\big) + 1$ *where $x \in m$.*

Lemma 2.4. *Set $d = \dim R$. The following statements are true.*

i) ([5, Th. 3.1, Lem. 5.4]) If $\text{pd}_R(M) < \infty$ then $H^j_I(M,N) = 0$ for all $j > d$, and $H^d_I(M,N)$ is Artinian.

ii) ([12, Th. 1.2]) Set $s(I,M) = \text{depth}(M/I^nM)$ for $n \gg 0$. If $\text{injd}_R(N) < \infty$, then $H^j_I(M,N) = 0$ for all $j > d - s(I,M)$, and $H^d_I(M,N)$ is Artinian.

3 Noetherian dimension at small levels

Suppose that $H^j_I(M,N)$ is Artinian for all $j < r$. In general, it is not true that $\dim_R(H^j_I(M,N)) \leq j$ for all $j < r$. For example, let (R,m) be an Noetherian local domain of dimension 2 constructed by D. Ferrand-Raynaud in [9] such that $\dim(\widehat{R}/\widehat{q}) = 1$ for some $\widehat{q} \in \text{Ass}(\widehat{R})$. Then $\dim_R(H^1_I(M,N)) = 2$, where $M = N = R$ and $I = m$; while $H^j_I(M,N)$ is Artinian for all $j \geq 0$.

But if we replace Krull dimension "dim" by Noetherian dimension "N-dim" in this case then the situation becomes better as the following result.

Theorem 3.1. *Let r be a positive integer. If $H^j_I(M,N)$ is Artinian for all $j < r$ then $\text{N-dim}_R(H^j_I(M,N)) \leq j$ for all $j < r$.*

Proof. We prove our theorem by induction on r. Note that $H^0_I(M,N) = \Gamma_I(\text{Hom}_R(M,N))$ (cf. [6, Lemma 2.1]). Hence, if $H^0_I(M,N)$ is Artinian, then $\ell_R(H^0_I(M,N)) < \infty$. It implies that $\text{N-dim}_R(H^0_I(M,N)) \leq 0$. Thus the case of $r = 1$ is proved.

We now assume that $r > 1$ and the theorem is true for $r-1$. Since $H^0_I(M,N)$ is Artinian, so we have

$$\{m\} \supseteq \text{Ass}_R(N) \cap V(I_M) = \text{Ass}_R(H^0_I(M,N))$$

where $I_M = \text{ann}_R(M/IM)$. Thus $I_M \nsubseteq \bigcup_{m \neq p \in \text{Ass}\, N} p$. Hence there exists an element $x \in I_M$ such that $x \notin p$ for all $p \in \text{Ass}_R(N) \setminus \{m\}$. Note that we may assume that $I_M \subseteq m$ (and so $x \in m$); because if $I_M \nsubseteq m$ then $H^j_I(M,N) \cong H^j_{I_M}(M,N) = 0$ for all j (see [6, Lem. 2.3]), thus the theorem is trivial.

We consider a commutative diagram

$$\begin{array}{ccccccc} & & N & & & & \\ & & \cdot x \downarrow & \searrow \cdot x & & & \\ 0 \longrightarrow & xN & \longrightarrow & N & \longrightarrow & N/xN & \longrightarrow 0. \end{array}$$

It induces the following commutative diagram

$$\begin{array}{ccccccc} & & \ldots \to & H^j_I(M,N) & \to \ldots & & \\ & & & f_j \downarrow & \searrow \cdot x & & \\ \ldots \to & H^{j-1}_I(M,N/xN) & \to & H^j_I(M,xN) & \to & H^j_I(M,N) & \to \ldots \end{array}$$

for all j, where f_j is induced from the map $N \xrightarrow{\cdot x} xN$. Set

$$A = (0:x)_{H^j_I(M,N)}, \quad \text{i.e.,} \quad A = \operatorname{Ker}\big(H^j_I(M,N) \xrightarrow{\cdot x} H^j_I(M,N)\big).$$

We next claim that

$$\text{N-dim}_R(\operatorname{Ker}(f_j) \cap A) \leq 0 \text{ and } \text{N-dim}_R(f_j(A)) \leq j-1$$

for all $j < r$. Note that $\ell_R((0:x)_N) < \infty$ by the choice of x. Hence we get by [6, Lemma 2.1] that $H^j_I(M,(0:x)_N) \cong \operatorname{Ext}^j_R(M,(0:x)_N)$ for all j. So that $\ell_R\big(H^j_I(M,(0:x)_N)\big) < \infty$ for all j. Thus from the short exact sequence $0 \to (0:x)_N \to N \xrightarrow{\cdot x} xN \to 0$ we get the following exact sequence

$$H^j_I(M,(0:x)_N) \to H^j_I(M,N) \xrightarrow{f_j} H^j_I(M,xN) \to H^{j+1}_I(M,(0:x)_N).$$

Hence $\ell_R(\operatorname{Ker}(f_j) \cap A) < \infty$, and thus $\text{N-dim}_R(\operatorname{Ker}(f_j) \cap A) \leq 0$. Moreover, the above exact sequence and the hypothesis yield that $H^j_I(M,xN)$ is Artinian for all $j < r$. Thus from the exact sequence

$$H^{j-1}_I(M,N) \to H^{j-1}_I(M,N/xN) \to H^j_I(M,xN),$$

it impies that $H^j_I(M,N/xN)$ is Artinian for all $j < r-1$. From this we obtain by induction hypothesis that $\text{N-dim}_R(H^j_I(M,N/xN)) \leq j$ for all $j < r-1$. On the other hand, by the above commutative diagram, we see that $f_j(A)$ is a submodule of $\operatorname{Ker}\big(H^j_I(M,xN) \to H^j_I(M,N)\big)$, and so that $f_j(A)$ is a subquotient of $H^{j-1}_I(M,N/xN)$ for all $j < r$. Hence, we obtain by Lemma 2.2 that

$$\text{N-dim}_R(f_j(A)) \leq \text{N-dim}_R(H^{j-1}_I(M,N/xN)) \leq j-1$$

for all $j < r$, therefore the claim is proved.

Look at the above commutative diagram again, we get the following exact sequence

$$0 \to \operatorname{Ker}(f_j) \cap A \to A \to f_j(A) \to 0$$

for all j. From this, we obtain by Lemma 2.2 and the claim that

$$\text{N-dim}_R(A) = \max\{\text{N-dim}_R(\text{Ker}(f_j) \cap A), \text{N-dim}_R(f_j(A))\} \leq j - 1$$

for all $j < r$. Therefore we get by Lemma 2.3 that

$$\begin{aligned} \text{N-dim}_R(H_I^j(M,N)) &\leq \text{N-dim}_R\left((0:x)_{H_I^j(M,N)}\right) + 1 \\ &= \text{N-dim}_R(A) + 1 \\ &\leq j \end{aligned}$$

for all $j < r$, as required. ☐

By replacing M by R in Theorem 3.1, we get again a result of Cuong-Nhan in [7] as follows.

Corollary 3.2. ([7, Th. 3.1]) *If $H_I^j(N)$ is Artinian for all $j < r$ then* $\text{N-dim}_R(H_I^j(N)) \leq j$ *for all $j < r$.*

Note that the proof method as used in [7] is not valid for Theorem 3.1. Indeed, in [7], they used a fact that if x is a filter regular element of N in m then $H_I^j(N) \cong H_I^j(N/(0:x)_N)$ for all $j \geq 1$; but, for generalized local cohomology modules, it is not true that $H_I^j(M,N) \cong H_I^j(M,N/(0:x)_N)$ for all $j \geq 1$ (because $H_I^j(M,(0:x)_N)$ may not vanish for $j \geq 1$, see example [5, Rem. 5.3 (i)]). Hence the proof of Theorem 3.1 is not trivial.

Note that if $I = m$ then $H_I^j(M,N)$ is Artinian for all j (see [6, Cor. 3.2]). Thus as an immediate consequence of Theorem 3.1 we get the following corollary.

Corollary 3.3. $\text{N-dim}_R(H_m^j(M,N)) \leq j$ *for all j.*

4 Top generalized local cohomology modules

In general the module $H_I^j(M,N)$ may not vanish for infinitely many j (for example, see [5, Rem. 5.5 (i)]). But, if M is of finite projective dimension (i.e., $\text{pd}_R(M) < \infty$) or N is of finite injective dimension (i.e., $\text{injd}_R(N) < \infty$), then $H_I^j(M,N) = 0$ for all $j > d = \dim(R)$ and $H_I^d(M,N)$ is Artinian by Lemma 2.4. Thus, in the next result, we will consider on Noetherian dimension of the module $H_I^d(M,N)$.

We first establish the following lemma which are extensions of Herzog-Zamani (in [11, Lem. 3.1]) and Bruns-Herzog (in [4, Cor. 3.5.11 c)]).

Lemma 4.1. *Let $d = \dim(R)$. The following satements are true.*

i) If $\text{pd}_R(M) < \infty$ *then* $\dim_R(\text{Ext}_R^j(M,N)) \leq d - j$ *for all $j \leq d$.*

ii) If $\operatorname{injd}_R(N) < \infty$ *then* $\dim_R(\operatorname{Ext}^j_R(M,N)) \le d-j$ *for all* $j \le d$.

Proof. i) Since $\operatorname{pd}_R(M) < \infty$, we get by [4, Thm. 1.3.3] that

$$\operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}(M) \le d.$$

We now prove by descending induction on j that $\dim_R(\operatorname{Ext}^j_R(M,N)) \le d-j$ for all $0 \le j \le d$. From the short exact sequence $0 \to U \to R^n \to N \to 0$ for some positive integer n and some finitely generated $R-$module U, we obtain the following exact sequence

$$\operatorname{Ext}^j_R(M,R^n) \to \operatorname{Ext}^j_R(M,N) \to \operatorname{Ext}^{j+1}_R(M,U)$$

for all j. Note that $\operatorname{Ext}^{d+1}_R(M,U) = 0$. Hence, if we replace $j = d$ in the above exact sequence, then we get that

$$\dim_R(\operatorname{Ext}^d_R(M,N)) \le \dim_R(\operatorname{Ext}^d_R(M,R^n)) \le 0$$

because $\dim_R(\operatorname{Ext}^d_R(M,R)) \le 0$ by [11, Lem. 3.1]. Assume that $j < d$ and the required is true for $j+1$ (i.e., $\dim_R(\operatorname{Ext}^{j+1}_R(M,N)) \le d-(j+1)$ for all finitely generated $R-$module N). Thus $\dim_R(\operatorname{Ext}^{j+1}_R(M,U)) \le d-(j+1)$. On the other hand, we obtain by [11, Lem. 3.1] that $\dim_R(\operatorname{Ext}^j_R(M,R^n)) \le d-j$. Therefore we get by the above exact sequence that

$$\begin{aligned}\dim_R(\operatorname{Ext}^j_R(M,N)) &\le \max\{\dim_R(\operatorname{Ext}^j_R(M,R^n)), \dim_R(\operatorname{Ext}^{j+1}_R(M,U))\}\\ &\le d-j.\end{aligned}$$

ii) We may assume that $R = \widehat{R}$. Since $\operatorname{injd}_R(N) < \infty$, R is a Cohen-Macaulay ring by [4, Rem. 9.6.4]. Thus R admits a canonical module ω_R by [4, Cor. 3.3.8]. Then, by [4, Exer. 3.3.28], we get an exact sequence $0 \to V \to \omega_R^n \to N \to 0$ for some positive integer n, where U is a finitely generated $R-$module of finite injective dimension. We now prove by descending induction on $j \le d$ that $\dim_R(\operatorname{Ext}^j_R(M,N)) \le d-j$. If $j = d$ then we have an exact sequence

$$\operatorname{Ext}^d_R(M,\omega_R^n) \to \operatorname{Ext}^d_R(M,N) \to 0.$$

Hence

$$\dim_R(\operatorname{Ext}^d_R(M,N)) \le \dim_R(\operatorname{Ext}^d_R(M,\omega_R^n)) \le 0$$

by [4, Cor. 3.5.11 c)]. Assume that $j < d$. We have the following exact sequence

$$\operatorname{Ext}^j_R(M,\omega_R^n) \to \operatorname{Ext}^j_R(M,N) \to \operatorname{Ext}^{j+1}_R(M,U).$$

Since $\operatorname{injd}_R(U) < \infty$, $\dim_R(\operatorname{Ext}^{j+1}_R(M,U)) \le d-(j+1)$ by inductive assumption. Moreover, by [4, Cor. 3.5.11 c)], $\dim_R(\operatorname{Ext}^j_R(M,\omega_R^n)) \le d-j$. Therefore $\dim_R(\operatorname{Ext}^j_R(M,N)) \le d-j$. This completes the proof of ii). $\square$

We are now ready to show the first main result in this section.

Theorem 4.2. *Let $d = \dim(R)$. If $\mathrm{pd}_R(M) < \infty$ or $\mathrm{injd}_R(N) < \infty$, then*

$$\text{N-dim}_R(H_I^d(M,N)) \le d - \mathrm{depth}(\mathrm{ann}(M), N).$$

Proof. Let $x_1, \ldots, x_m$ be a set of generators of I and $K_\bullet^n$ the Koszul complex of R with respect to $x_1^n, \ldots, x_m^n$. We denote by $C_\bullet^n$ the total complex associated to the double complex $K_\bullet^n \otimes_R F_\bullet$, where $F_\bullet$ is a projective resolution of M. Consider the convergent spectral sequence

$$H^i(\mathrm{Hom}(K_\bullet^n, \mathrm{Ext}_R^j(M,N))) \underset{i}{\Longrightarrow} H^{i+j}(\mathrm{Hom}(C_\bullet^n, N)).$$

Since $H_I^i(M,N) \cong \varinjlim_n H^i(\mathrm{Hom}(C_\bullet^n, N))$ for all $i \ge 0$ by [1, Thm. 4.2], we obtain by passage to direct limits the following convergent spectral sequence

$$E_2^{i,j} := H_I^i(\mathrm{Ext}_R^j(M,N)) \underset{i}{\Longrightarrow} H^{i+j} = H_I^{i+j}(M,N)$$

(This spectral sequence is contructed by similar way as in the proof of [5, Thm. 3.1]). Thus there is a finite filtration of the module $H^d = H_I^d(M,N)$ as follows

$$0 = \phi^{d+1}H^d \subseteq \phi^d H^d \subseteq \ldots \subseteq \phi^1 H^d \subseteq \phi^0 H^d = H^d$$

such that $E_\infty^{i,d-i} \cong \phi^i H^d/\phi^{i+1}H^d$ for all $0 \le i \le d$. Note that $E_\infty^{i,d-i}$ is a subquotient of $E_2^{i,d-i} = H_I^i(\mathrm{Ext}_R^{d-i}(M,N)$ for all $0 \le i \le d$. Since $\mathrm{pd}_R(M) < \infty$, so we get by Lemma 4.1 that $\dim_R(\mathrm{Ext}_R^{d-i}(M,N)) \le i$ for all $0 \le i \le d$. Therefore, for any $i \in \{0, \ldots, d\}$, we consider two cases as follows:

• If $\dim_R(\mathrm{Ext}_R^{d-i}(M,N)) < i$ then $E_2^{i,d-i} = 0$, and so $E_\infty^{i,d-i} = 0$ (here we use the convention that a zero module has dimension $-\infty$);

• If $\dim_R(\mathrm{Ext}_R^{d-i}(M,N)) = i$ then $\text{N-dim}_R(H_I^i(\mathrm{Ext}_R^{d-i}(M,N)) = i$ by [7, Thm. 3.5].

Keep in mind that there exist the following exact sequences

$$0 \to \phi^{i+1}H^d \to \phi^i H^d \to E_\infty^{i,d-i} \to 0$$

for all $0 \le i \le d$. Set $\Omega = \{i \in \{0, \ldots, d\} \mid \dim_R(\mathrm{Ext}_R^{d-i}(M,N)) = i\}$. Thus we get by the above exact sequence and Lemma 2.2 that

$$\begin{aligned} \text{N-dim}_R(H^d) &= \max\{\text{N-dim}_R(E_\infty^{i,d-i}) \mid i \in \Omega\} \\ &\le \max\{\text{N-dim}_R(E_2^{i,d-i}) \mid i \in \Omega\} \\ &= \max\{i \mid i \in \Omega\} = \max(\Omega). \end{aligned}$$

On the other hand, if $i \in \Omega$ then $\dim_R(\mathrm{Ext}_R^{d-i}(M,N)) = i$. Hence $d - i \ge \mathrm{depth}(\mathrm{ann}(M), N)$, so that $d - \mathrm{depth}(\mathrm{ann}(M), N) \ge i$. Thus

$$\text{N-dim}_R(H_I^d(M,N)) \le d - \mathrm{depth}(\mathrm{ann}(M), N),$$

as required. $\square$

The rest of this section devotes to establish a relation between the Noetherian dimension of $H^d_I(M,N)$ with some other invariants provided that $+\infty > \mathrm{injd}_R(N)$. By Lemma 2.4, $H^d_I(M,N)$ is Artinian. Thus $H^d_I(M,N)$ has a minimal secondary representation by [14], so the set of attached primes of $H^d_I(M,N)$ is a finite set. The next result shows that this set is managed by the set of associated primes of M.

Lemma 4.3. *Set $d = \dim(R)$. If $\mathrm{injd}_R(N) < +\infty$, then we have*

$$\mathrm{Att}_R(H^d_I(M,N)) \subseteq \mathrm{Ass}_R(M).$$

Proof. By Lemma 2.4, $H^d_I(M,N)$ is Artinian, so that $H^d_I(M,N)$ can be regarded as $\widehat{R}$-module, where $\widehat{R}$ is the completion of R with respect to m-adic topology. Hence we obtain the following isomorphisms

$$H^d_I(M,N) \cong H^d_I(M,N) \otimes_R \widehat{R} \cong H^d_{\widehat{I}}(\widehat{M},\widehat{N})$$

as $\widehat{R}$-modules. Therefore, by [2, 11.3.7 (iii)], we get

$$\mathrm{Att}_R(H^d_I(M,N)) = \{\widehat{P}\cap R \mid \widehat{P} \in \mathrm{Att}_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M},\widehat{N}))\}.$$

Moreover, by [15, Thm. 23.2], we have $\mathrm{Ass}_R(M) = \{\widehat{P}\cap R \mid \widehat{P} \in \mathrm{Ass}_{\widehat{R}}(\widehat{M})\}$. Therefore we may assume that $R = \widehat{R}$. On the other hand, as $\mathrm{injd}_R(N) < \infty$, R is Cohen-Macaulay by [4, Rem. 9.6.4]. Hence, by [4, Cor. 3.3.8], R admits the canonical module ω_R.

Now, we get by local duality for a Cohen-Macaulay homomorphic image of a Gorenstein local ring the following isomorphisms

$$\mathrm{Ext}^d_R(M/I^nM,\omega_R) \cong \mathrm{Hom}_R(H^0_m(M/I^nM),E(k))$$

for all $n>0$, where $E(k)$ is the injective hull of $k = R/m$. Thus, for each $n>0$, we get $H^0_m(M/I^nM) \cong \mathrm{Hom}(R/m^{t_n}, M/I^nM)$ and $\cap_{i>0} m^i(0:I^n)_{D(M)} = m^{t_n}(0:I^n)_{D(M)}$ for some integer t_n, where $D(M) = \mathrm{Hom}_R(M,E(k))$. It implies by [18, Lem. 3.60] that

$$\mathrm{Ext}^d_R(M/I^nM,\omega_R) \cong \big((0:I^n)_{D(M)}\big)/\big(m^{t_n}(0:I^n)_{D(M)}\big).$$

From this, by passage to direct limits, we have an isomorphism

$$H^d_I(M,\omega_R) \cong D(M)/\sum_{n>0} m^{t_n}(0:I^n)_{D(M)}.$$

On the other hand, by [4, Exer. 3.3.28], there exists an exact sequence $0 \to U \to \omega_R^n \to N \to 0$ for some finitely generated $R-$module U of finite injective dimension. Hence, in view of Lemma 2.4, we get an exact sequence $H^d_I(M,\omega_R^n) \to H^d_I(M,N) \to 0$. Therefore

$$\mathrm{Att}_R(H^d_I(M,N)) \subseteq \mathrm{Att}_R(H^d_I(M,\omega_R)) \subseteq \mathrm{Att}_R(D(M)) = \mathrm{Ass}_R(M),$$

as required. ∎

Lemma 4.4. *Set* $d = \dim(R)$*. If* $\operatorname{injd}_R(N) < \infty$ *then* $\text{N-dim}_R(H^d_I(M,N)) \leq \dim_R(M)$*. Moreover, the equality holds if* $H^d_I(M,N) \neq 0$ *and* $\widehat{M}$ *is unmixed (i.e.,* $\dim(\widehat{R}/P) = \dim_R(M)$ *for all* $P \in \operatorname{Ass}_{\widehat{R}}(\widehat{M})$*), where* $\widehat{M}$ *is the completion of* M *with respect to* m*-adic topology.*

Proof. By Lemma 2.4, $H^d_I(M,N)$ is Artinian, thus $H^d_I(M,N) \cong H^d_{\widehat{I}}(\widehat{M},\widehat{N})$ as $\widehat{R}$-modules. Hence we have

$$\text{N-dim}_R(H^d_I(M,N)) = \text{N-dim}_{\widehat{R}}(H^d_I(M,N)) = \dim_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M},\widehat{N})).$$

Since $H^d_I(M,N) \neq 0$, so that $\operatorname{Att}_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M},\widehat{N})) \neq \emptyset$. By Lemma 4.3, we obtain that $\operatorname{Att}_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M},\widehat{N})) \subseteq \operatorname{Ass}_{\widehat{R}}(\widehat{M})$. Thus

$$\begin{aligned}\dim_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M},\widehat{N})) &= \max\{\dim(\widehat{R}/\widehat{p}) \mid \widehat{p} \in \operatorname{Att}_{\widehat{R}}(H^d_I(M,N))\} \\ &\leq \dim_{\widehat{R}}(\widehat{M}) = \dim_R(M).\end{aligned}$$

Finally, if $\widehat{M}$ is unmixed then $\dim_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M},\widehat{N})) = \dim_{\widehat{R}}(\widehat{M})$. $\square$

Remark 4.5. Let $\widehat{M}$ be the completion of M with respect to m-adic topology. Let

$$H^0_m(\widehat{M}) = D_0 \subset D_1 \subset \ldots \subset D_{n-1} \subset D_n = \widehat{M}$$

be a dimension filtration of $\widehat{M}$, where D_{i-1} is the largest submodule of D_i such that $\dim_{\widehat{R}}(D_{i-1}) < \dim_{\widehat{R}}(D_i)$ for all $i = 1,\ldots,n$. Note that the dimension filtration always exists and to be unique by the Noetherianness of $\widehat{M}$. Moreover, D_i/D_{i-1} is unmixed and $\dim_{\widehat{R}}(D_i/D_{i-1}) = \dim_{\widehat{R}}(D_i)$ for all $i = 1,\ldots,n$. For convention, we put $D_{-1} = 0$.

Lemma 4.6. *Set* $d = \dim(R)$*. If* $\operatorname{injd}_R(N) < +\infty$ *and* $H^d_I(M,N) \neq 0$*, then there exists* $t \in \{0,\ldots,n\}$ *such that* $H^d_{\widehat{I}}(D_t/D_{t-1},\widehat{N}) \neq 0$*, where* D_i *is in Remark 4.5.*

Proof. It is clear that $\operatorname{injd}_{\widehat{R}}(\widehat{N}) < +\infty$ and $H^d_{\widehat{I}}(\widehat{M},\widehat{N}) \neq 0$. Now, for each $i \in \{0,\ldots,n\}$, we have an exact sequence $0 \to D_{i-1} \to D_i \to D_i/D_{i-1} \to 0$. Thus, in view of Lemma 2.4, it induces an exact sequence

$$H^d_{\widehat{I}}(D_i/D_{i-1},\widehat{N}) \to H^d_{\widehat{I}}(D_i,\widehat{N}) \to H^d_{\widehat{I}}(D_{i-1},\widehat{N}) \to 0.$$

Assume that $H^d_{\widehat{I}}(D_i/D_{i-1},\widehat{N}) = 0$ for all $i = 0,\ldots,n$. Then, by applying the above exact sequence many times, we obtain that

$$H^d_{\widehat{I}}(\widehat{M},\widehat{N}) = H^d_{\widehat{I}}(D_n,\widehat{N}) = \ldots = H^d_{\widehat{I}}(D_{-1},\widehat{N}) = 0,$$

this is a contradiction with the fact that $H^d_I(M,N) \neq 0$. Therefore, there exists $t \in \{0,\ldots,n\}$ such that $H^d_{\widehat{I}}(D_t/D_{t-1},\widehat{N}) \neq 0$, as required. $\square$

Theorem 4.7. *Set $d = \dim(R)$. Suppose that $H^d_I(M,N) \neq 0$ and $\operatorname{injd}_R(N) < +\infty$. Let $H^0_m(\widehat{M}) = D_0 \subset D_1 \subset \ldots \subset D_{n-1} \subset D_n = \widehat{M}$ be the dimension filtration of $\widehat{M}$. Set $t = \max\{i \mid H^d_{\widehat{I}}(D_i/D_{i-1}, \widehat{N}) \neq 0, 0 \leq i \leq n\}$. Then* $\text{N-dim}_R(H^d_I(M,N)) = \dim_{\widehat{R}}(D_t)$.

Proof. Set $A = H^d_I(M,N)$. By Lemma 2.4, A is Artinian, thus A can be regarded as $\widehat{R}$-module. Hence $\text{N-dim}_R(A) = \text{N-dim}_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M}, \widehat{N}))$. Therefore, we need only to show that $\text{N-dim}_{\widehat{R}}(H^d_{\widehat{I}}(\widehat{M}, \widehat{N})) = \dim_{\widehat{R}}(D_t)$. For each $i = 0, \ldots, n$, the short exact sequence $0 \to D_{i-1} \to D_i \to D_i/D_{i-1} \to 0$ induces the following exact sequence

$$H^{d-1}_{\widehat{I}}(D_{i-1}, \widehat{N}) \to \\ \to H^d_{\widehat{I}}(D_i/D_{i-1}, \widehat{N}) \to H^d_{\widehat{I}}(D_i, \widehat{N}) \to H^d_{\widehat{I}}(D_{i-1}, \widehat{N}) \to 0. \tag{1}$$

Thus, by the choice of t, we get

$$H^d_{\widehat{I}}(D_t/D_{t-1}, \widehat{N}) \neq 0 \text{ and } H^d_{\widehat{I}}(\widehat{M}, \widehat{N}) = H^d_{\widehat{I}}(D_t, \widehat{N}).$$

Hence, it is enough to prove that $\text{N-dim}_{\widehat{R}}(H^d_{\widehat{I}}(D_t, \widehat{N})) = \dim_{\widehat{R}}(D_t)$. If $t = 0$ then $0 \neq H^d_{\widehat{I}}(\widehat{M}, \widehat{N}) = H^d_{\widehat{I}}(D_0, \widehat{N}) \cong \operatorname{Ext}^d_{\widehat{R}}(D_0, \widehat{N})$ is of finite length; and so that $D_0 \neq 0$. It implies that $\text{N-dim}_{\widehat{R}}(H^d_{\widehat{I}}(D_0, \widehat{N})) = 0 = \dim_{\widehat{R}}(D_0)$. We now assume that $t > 0$. Then, by replacing $i = t$ in (1), we obtain an exact sequence

$$H^{d-1}_{\widehat{I}}(D_{t-1}, \widehat{N}) \xrightarrow{h} \\ \to H^d_{\widehat{I}}(D_t/D_{t-1}, \widehat{N}) \xrightarrow{f} H^d_{\widehat{I}}(D_t, \widehat{N}) \xrightarrow{g} H^d_{\widehat{I}}(D_{t-1}, \widehat{N}) \to 0. \tag{2}$$

We divide the sequence (2) into two the following exact sequences

$$0 \to \operatorname{Im} h \to H^d_{\widehat{I}}(D_t/D_{t-1}, \widehat{N}) \to \operatorname{Im} f \to 0 \tag{3}$$

and

$$0 \to \operatorname{Im} f \to H^d_{\widehat{I}}(D_t, \widehat{N}) \to \operatorname{Im} g \to 0 \tag{4}$$

in which all of the R-modules are Artinian by Lemma 2.4. Since

$$\operatorname{Im} h \cong H^{d-1}_{\widehat{I}}(D_{t-1}, \widehat{N}) / \operatorname{Ker} h,$$

so we get by Lemma 2.2 and Lemma 4.4 that

$$\text{N-dim}_{\widehat{R}}(\operatorname{Im} h) \leq \text{N-dim}_{\widehat{R}}(H^{d-1}_{\widehat{I}}(D_{t-1}, \widehat{N})) \leq \dim_{\widehat{R}}(D_{t-1}) < \dim_{\widehat{R}}(D_t).$$

Note that $\dim_{\widehat{R}}(D_t/D_{t-1}) = \dim_{\widehat{R}}(D_t)$ and D_t/D_{t-1} is unmixed. Hence we get by Lemma 4.4 that

$$\text{N-dim}_{\widehat{R}}(H^d_{\widehat{I}}(D_t/D_{t-1}, \widehat{N})) = \dim_{\widehat{R}}(D_t).$$

It follows by (3) and Lemma 2.2 that $\text{N-dim}_{\widehat{R}}(\operatorname{Im} f) = \dim_{\widehat{R}}(D_t)$. On the other hand, by (2), Lemma 2.2 and Lemma 4.4, we have

$$\text{N-dim}_{\widehat{R}}(\operatorname{Im} g) \leq \text{N-dim}_{\widehat{R}}(H^d_{\widehat{I}}(D_{t-1}, \widehat{N})) \leq \dim_{\widehat{R}}(D_{t-1}) < \dim_{\widehat{R}}(D_t).$$

Therefore, from (4) and Lemma 2.2, we get that

$$\text{N-dim}_{\widehat{R}}(H^d_{\widehat{I}}(D_t, \widehat{N})) = \dim_{\widehat{R}}(D_t),$$

as required. □

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