

## EXISTENCE OF 2-EXCEPTIONAL BERNSTEIN ALGEBRAS

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### Abstract

Given a Bernstein algebra  $\mathbf{A} = Fe \oplus U \oplus V$ , the two ordered pairs of integers  $(1 + \dim U, \dim V)$  and  $(\dim(UV + V^2), \dim U^2)$  are called, respectively, the type and the subtype of  $\mathbf{A}$ . In this paper we determine the minimum and the maximum dimension of the subspace  $UV + V^2$  in 2-exceptional Bernstein algebras (those satisfying  $U(UV) \neq 0$  and  $U((UV)V) = 0$ ) and we introduce an algorithm to construct 2-exceptional Bernstein algebras for some types and subtypes.

### Introduction

This paper is a natural continuation of [2], where the authors investigate under which conditions, given a quadruple of non negative integers  $(r, s, t, z)$ , there exists a  $n$ -exceptional Bernstein algebra of type  $(1 + r, s)$  and subtype  $(t, z)$ , for  $n = 0, 1$ . In this article, we study some properties of  $n$ -exceptional algebras for  $n = 2$  and construct 2-exceptional algebras for some particular types and subtypes.

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We recall that a baric algebra over a field  $F$  is a pair  $(\mathbf{A}, \omega)$ , where  $\mathbf{A}$  is not necessarily associative algebra over  $F$  and  $\omega : \mathbf{A} \rightarrow F$  is nonzero homomorphism. A baric algebra  $(\mathbf{A}, \omega)$  is Bernstein if  $\mathbf{A}$  is commutative and  $(x^2)^2 = \omega(x)^2 x^2$ , for every  $x \in \mathbf{A}$ .

Every Bernstein algebra possesses at least one nonzero idempotent. If  $F$  is a field of characteristic not 2, then for every nonzero idempotent  $e$ ,  $\mathbf{A}$  has a Peirce decomposition  $\mathbf{A} = Fe \oplus U_e \oplus V_e$  relative to  $e$ , where  $U_e = \{x \in \mathbf{A} \mid 2ex = x\}$ ,  $V_e = \{x \in \mathbf{A} \mid ex = 0\}$  and  $\text{Ker}\omega = U_e \oplus V_e$ . Unless necessary, we omit the subscript  $e$  in  $U_e$  and  $V_e$ .

In a Peirce decomposition  $\mathbf{A} = Fe \oplus U \oplus V$ , the subspaces  $U$  and  $V$  verify the relations  $U^2 \subseteq V$ ,  $UV \subseteq U$  and  $V^2 \subseteq U$  and the following identities hold for all  $u \in U$  and  $v \in V$ :

$$u^3 = 0, \quad uv^2 = 0, \quad u(uv) = 0, \quad (uv)^2 = 0, \quad (u^2)^2 = 0. \quad (1)$$

All the linearizations of the equations (1) are also identities in  $\mathbf{A}$ . In particular,  $u_1(u_2v) + u_2(u_1v) = 0$ , for all  $u_1, u_2 \in U$ ,  $v \in V$ .

In this paper, we consider only finite dimensional Bernstein algebras over fields of characteristic not 2. Let  $\mathbf{A} = Fe \oplus U \oplus V$  be a Bernstein algebra. It is known that the dimensions of  $U$ ,  $V$ ,  $UV + V^2$  and  $U^2$  are invariant under change of idempotent. The ordered pairs  $(1 + \dim U, \dim V)$  and  $(\dim(UV + V^2), \dim U^2)$ , which are well defined, are called the *type* and the *subtype* of  $\mathbf{A}$ , respectively.

A Bernstein algebra  $\mathbf{A}$  is said Jordan-Bernstein if it is also Jordan. In [4] it was proved that  $\mathbf{A} = Fe \oplus U_e \oplus V_e$  is Jordan-Bernstein if and only if  $V_e^2 = 0$  and  $(uv)v = 0$ , for all  $u \in U_e, v \in V_e$ . If  $\mathbf{A} = Fe \oplus U_e \oplus V_e$  is a Peirce decomposition of a Bernstein algebra  $\mathbf{A}$ , then the set  $L = \{x \in U_e \mid xu = 0 \text{ for all } u \in U_e\}$  is an ideal of  $\mathbf{A}$  contained in  $U_e$ , which is independent on the idempotent. The quotient algebra  $(\bar{\mathbf{A}}, \bar{\omega})$ , where  $\bar{\mathbf{A}} = \mathbf{A}/L$  and  $\bar{\omega}(a+L) = \omega(a)$ , for all  $a \in \mathbf{A}$ , is Jordan-Bernstein. In the Peirce decomposition  $\bar{\mathbf{A}} = F\bar{e} \oplus \bar{U}_{\bar{e}} \oplus \bar{V}_{\bar{e}}$  relative to the idempotent  $\bar{e} = e + L$ , we have  $\bar{U}_{\bar{e}} = \bar{U}_e := U_e/L$  and  $\bar{V}_{\bar{e}} = \bar{V}_e := (V_e + L)/L$ . For a subspace  $X$  of a Bernstein  $\mathbf{A}$ , we will denote by  $\bar{X}$  the quotient  $(X+L)/L$ . All these facts are well known and can be found in [5], [6], [7] and [8].

If  $X$  and  $Y$  are subspaces of a Bernstein algebra  $\mathbf{A}$ , we define  $XY^{(0)} = X$  and  $XY^{(k)} = (XY^{(k-1)})Y$ ,  $k$  integer  $\geq 1$ , where  $XY = \langle xy \mid x \in X, y \in Y \rangle$ .

A Bernstein algebra  $\mathbf{A} = Fe \oplus U_e \oplus V_e$  is called *exceptional of degree  $n$* , or  *$n$ -exceptional*, if  $n$  is the least non negative integer such that the subspace  $U_e(U_e V_e^{(n)}) = 0$ , for some idempotent  $e$ . The integer  $n$  is called the degree of exceptionality of  $\mathbf{A}$ . If  $\mathbf{A}$  satisfies  $U_e^2 = V_e$  then  $\mathbf{A}$  is said to be *nuclear*. These definitions do not depend on the choice of the idempotent element. It was proved in [2] that every Bernstein algebra of type  $(1+r, s)$  is  $n$ -exceptional for some integer  $n$ , with  $0 \leq n \leq s+1$ .

For Bernstein algebras with degree of exceptionality greater than or equal to 1, we have the following result.

**Proposition 1.** *If  $\mathbf{A} = Fe \oplus U_e \oplus V_e$  is a  $n$ -exceptional Bernstein algebra of finite dimension with  $n \geq 1$ , then for every idempotent element of  $\mathbf{A}$  the chain of subspaces*

$$\overline{U_e} \supseteq \overline{U_e V_e} \supseteq \overline{(U_e V_e) V_e} \supseteq \dots \supseteq \overline{U_e V_e^{(k)}}$$

*is strictly decreasing for every integer  $k < n$  and  $\overline{U_e V_e^{(n)}} = \overline{0}$ .*

**Proof** Let  $e$  be an arbitrary idempotent of  $\mathbf{A}$ . As  $\mathbf{A}$  is  $n$ -exceptional,  $U_e(U_e V_e^{(k)}) \neq 0$ , for every  $k < n$  and  $U_e(U_e V_e^{(n)}) = 0$ , thus  $\overline{U_e V_e^{(n)}} = \overline{0}$  and  $\overline{U_e V_e^{(k)}} \neq \overline{0}$ , for every  $k < n$ . Therefore the chain  $\overline{U_e} \supseteq \overline{U_e V_e} \supseteq \overline{(U_e V_e) V_e} \supseteq \dots \supseteq \overline{U_e V_e^{(n-1)}} \neq \overline{0}$  is strictly decreasing, because if  $\overline{U_e V_e^{(k)}} = \overline{U_e V_e^{(k+1)}}$ , for some integer  $k$ ,  $0 \leq k \leq n-1$ , then  $\overline{0} \neq \overline{U_e V_e^{(k)}} = \overline{U_e V_e^{(k+1)}} = \dots = \overline{U_e V_e^{(n)}} = \overline{0}$ , a contradiction.  $\square$

## 2-exceptional Bernstein algebras

The aim of this section is to study some general properties of 2-exceptional Bernstein algebras.

**Proposition 2.** *If  $\mathbf{A} = Fe \oplus U \oplus V$  is a  $n$ -exceptional Bernstein algebra of type  $(1+r, s)$  with  $n \geq 2$ , then*

- (i)  $r \geq 4$  and  $s \geq 2$ ;
- (ii)  $\dim L \leq r - 4$ ;
- (iii)  $\dim \overline{UV} \geq 2$ .

**Proof** Note that  $\dim \overline{U(UV)} = \dim U(UV) \neq 0$ , since  $n \geq 2$ . Then there exist  $\bar{u}_1, \bar{u}_2 \in \overline{U}$ ,  $\bar{v} \in \overline{V}$ , such that  $\bar{u}_1(\bar{u}_2 \bar{v}) = -\bar{u}_2(\bar{u}_1 \bar{v}) \neq \overline{0}$ . It was proved in [3, Prop. 9] that both sets  $\{\bar{u}_1, \bar{u}_2, \bar{u}_1 \bar{v}, \bar{u}_2 \bar{v}\}$  and  $\{\bar{u}_1(\bar{u}_2 \bar{v}), \bar{v}\}$  are linearly independent. This establishes (i), (ii) and (iii).  $\square$

Now we investigate bounds to the dimension of the subspace  $UV + V^2$  in 2-exceptional algebras.

Let  $\mathbf{A} = Fe \oplus U_e \oplus V_e$  be the Peirce decomposition of a Bernstein algebra with respect to an idempotent  $e$ . For a basis  $B = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_t\}$  of  $\overline{U_e V_e}$ , the set  $\{u_1, u_2, \dots, u_t\}$ , where  $u_i$  ( $i = 1, 2, \dots, t$ ) is a representative of the equivalence class  $\bar{u}_i$ , is a linearly independent set of  $U_e$ . Let  $M = \langle u_1, u_2, \dots, u_t \rangle$  be the subspace of  $\mathbf{A}$  generated by this set. It is clear that  $\dim M = t$ , independent of the basis and of the representatives  $u_i$  chosen. Moreover,  $\dim M = \dim \overline{U_e V_e}$ , which is invariant under change of idempotent. Therefore  $\dim M$  is an invariant of the algebra. By construction,  $M \subset U_e V_e + L$  and  $M \cap L = 0$ . If  $\mathbf{A}$  is a  $n$ -exceptional Bernstein algebra with  $n \leq 2$ , then, for every idempotent,  $M^2 = 0$ , independent of the basis of  $\overline{U_e V_e}$  and of the representatives  $u_i$ .

**Proposition 3.** *If  $\mathbf{A} = Fe \oplus U \oplus V$  is a 2-exceptional Bernstein algebra of type  $(1 + r, s)$ , then  $2 \leq \dim(UV + V^2) \leq r - 2$ .*

**Proof** As  $\mathbf{A}$  is 2-exceptional,  $\dim \overline{UV} \geq 2$ , according to Proposition 2. We consider  $\dim L = r - k$ , with  $4 \leq k \leq r$ . By Proposition 1,  $\dim \overline{UV} \leq \dim \overline{U} - 1 = k - 1$ . Suppose, by contradiction, that  $\dim \overline{UV} = k - 1$ . Let  $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{k-1}\}$  be a basis of  $\overline{UV}$  and let  $M = \langle u_1, u_2, \dots, u_{k-1} \rangle$  be the subspace as previously defined. We have  $M^2 = 0$ , then  $u_1 M = 0 = u_1 L$ . As  $u_1 \notin L$ , since  $\bar{u}_1 \neq \bar{0}$ , there exists  $u \in U$  such that  $u_1 u \neq 0$ . If  $\alpha \in F$ ,  $m \in M$  and  $l \in L$  are such that  $\alpha u = m + l$ , then  $\alpha u_1 u = u_1 m + u_1 l = 0$  and so  $\alpha = 0$ . Therefore,  $U = \langle u \rangle \oplus M \oplus L$ . Thus,  $M \subseteq UV + L = (\langle u \rangle + M + L)V + L \subseteq uV + L$ . Then there exist  $v \in V$ ,  $l \in L$  such that  $u_1 = uv + l$ . This implies that  $0 \neq uu_1 = u(uv) + ul = 0$ , a contradiction. Then  $2 \leq \dim \overline{UV} \leq k - 2$  and consequently  $2 \leq \dim(UV + V^2) \leq r - 2$ .  $\square$

### Some 2-exceptional Bernstein algebras

A natural question is to try to improve the bounds given in Proposition 3 or to investigate whether these values can be reached. Let us show, by construction, that there exist 2-exceptional Bernstein algebras with  $\dim(UV + V^2) = t$ , for  $2 \leq t \leq \dim U - 2$ . An estimate for the maximum dimension of the subspace  $U^2$  is a hard problem. Thus we construct 2-exceptional Bernstein algebras where the dimension of the subspace  $UV + V^2$  reaches the minimum and maximum values given in Proposition 3 for some values of the dimension of  $U^2$ .

In what follows, we denote by  $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$  the mapping defined by  $\lceil x \rceil = n$ , where  $n - 1 < x \leq n$  and  $n$  is an integer.

**Theorem 1.** *Let  $r, s, t$  and  $z$  be integers satisfying any one of the following conditions:*

(i)  $r \geq 4$ ,  $s \geq 2$ ,  $2 \leq t \leq r - 2$  and  $\frac{1}{2}(r - t)(r - t + 1) < z \leq \min\{\frac{1}{2}[(r - t)(r - t + 1) + (r - t - 1)t - r_1(r - t - r_1)], s - \lceil \frac{t}{r - t} \rceil\}$ , where  $r_1$  is the remainder of division of the integer  $t$  by  $r - t$ ;

(ii)  $t = \frac{1}{3}p(p^2 - 1)$ , for some integer  $p \geq 3$ ,  $r = t + p$  and  $s = z = \frac{1}{8}p(p + 1)(p^2 - 3p + 6)$ .

*Then there exists a 2-exceptional Bernstein algebra of type  $(1 + r, s)$  and subtype  $(t, z)$ .*

**Proof** The proof is an algorithm to construct such algebra. Initially we make the following attribution of values for  $m$ ,  $n$  and  $p$ , integers used as bound to the dimension of the some subspace in algorithm: if the integers satisfy the conditions of item (i), take  $m = n = 0$  and  $p = r - t$ . Otherwise, take  $m = t$  and  $n = \frac{1}{8}(p - 2)(p - 1)p(p + 1)$ . Furthermore, let  $m_1 = \frac{1}{2}p(p + 1)$ ,  $q$  and  $r_1$ , respectively, the quotient and remainder of division of  $(t - m)$  by  $p$ . The letters  $i, j, k$  and  $l$ , used as indexes, represent always integer values.

Let  $\mathbf{A}$  be the vector space over  $F$  having  $\{e, u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s\}$  as a basis. We define in  $\mathbf{A}$  the commutative product given by:

- (1).  $e^2 = e$ ;  $eu_i = \frac{1}{2}u_i$ ;  $ev_j = 0$  ( $i = 1, 2, \dots, r$  and  $j = 1, \dots, s$ );
- (2).  $u_i u_j = v_{\epsilon(i,j)}$ , if  $1 \leq i \leq j \leq p$ , where  $\epsilon(i, j) = \frac{1}{2}(2p - i)(i - 1) + j$ ;
- (3).  $u_i^2 u_j = -2u_i(u_i u_j) = 2u_{\rho_1(i,j)}$ ;  
 $u_j^2 u_i = -2u_j(u_i u_j) = 2u_{\rho_2(i,j)}$ ;  
 if  $1 \leq i < j \leq p$  and  $\rho_w(i, j) \leq p + m$  for  $w = 1, 2$ , where  
 $\rho_w(i, j) = p + \sum_{x=1}^{i-1} [(p-x)(p-(x-1))] + (w-1)(p-i) + (j-i)$ ;
- (4).  $(u_i u_j) u_k = u_{\tau_1(i,j,k)}$ ;  
 $(u_j u_k) u_i = u_{\tau_2(i,j,k)}$ ;  
 if  $1 \leq i < j < k \leq p$  and  $\tau_w(i, j, k) \leq p + m$  for  $w = 1, 2$ , where  
 $\tau_w(i, j, k) = p + \sum_{x=1}^{i-1} [(p-x)(p-(x-1))] + 2(p-i) + (2p-(i+j))(j-i-1) + 2(k-j) - 2 + w$ ;
- (5).  $(u_i u_k) u_j = -[(u_i u_j) u_k + (u_j u_k) u_i]$ , if  $1 \leq i < j < k \leq p$ ;
- (6).  $(u_i^2 u_j) u_k = -(u_j^2 u_k) u_i = 2(u_i u_j \cdot u_k) u_i = 2v_{\delta_1(i,j,k)}$ ;  
 $(u_j^2 u_i) u_k = -(u_k^2 u_j) u_i = 2(u_i u_j \cdot u_k) u_j = -2(u_j u_k \cdot u_i) u_j = 2v_{\delta_2(i,j,k)}$ ;  
 $(u_k^2 u_i) u_j = -(u_i^2 u_j) u_k = 2(u_i u_k \cdot u_j) u_k = -2(u_j u_k \cdot u_i) u_k = 2v_{\delta_3(i,j,k)}$ ;  
 if  $1 \leq i < j < k \leq p$  and  $\delta_w(i, j, k) \leq \min\{z, m_1 + n\}$  for  $w = 1, 2, 3$ ,

where

$$\delta_w(i, j, k) = m_1 + \sum_{x=1}^{i-1} \frac{3}{2}[(p-x)(p-(x+1))] + \frac{3}{2}(2p-(i+j))(j-i-1) + 3(k-j) - 3 + w;$$

- (7).  $(u_i u_j \cdot u_k) u_l = -(u_i u_j \cdot u_l) u_k = v_{\gamma_1(i,j,k,l)}$ ;  
 $(u_j u_k \cdot u_i) u_l = -(u_j u_k \cdot u_l) u_i = v_{\gamma_2(i,j,k,l)}$ ;  
 $(u_k u_l \cdot u_i) u_j = -(u_k u_l \cdot u_j) u_i = v_{\gamma_3(i,j,k,l)}$ ;  
 if  $1 \leq i < j < k < l \leq p$  and  $\gamma_w(i, j, k, l) \leq \min\{z, m_1 + n\}$  for  $w = 1, 2, 3$ , where

$$\gamma_w(i, j, k, l) = m_1 + \frac{1}{2}p(p-1)(p-2) + \sum_{x=1}^{i-1} \frac{1}{2}[(p-x)(p-(x+1))(p-(x+2))] +$$

$$\sum_{x=i+1}^{j-1} \frac{3}{2}[(p-x)(p-(x+1))] + 3(2p-(k+j))(k-j-1) + 3(l-k) - 3 + w;$$

- (8).  $(u_i u_k \cdot u_l) u_j = (u_i u_j \cdot u_k) u_l + (u_j u_k \cdot u_i) u_l$ ;  
 $(u_j u_l \cdot u_i) u_k = (u_j u_k \cdot u_l) u_i + (u_k u_l \cdot u_j) u_i$ , if  $1 \leq i < j < k < l \leq p$ ;
- (9).  $u_i v_{z+j} = u_{p+m+(i-1)q+j}$ , if  $1 \leq i \leq p$  and  $1 \leq j \leq q$ ;
- (10).  $u_i v_{z+q+1} = u_{m+p(q+1)+i}$ , if  $1 \leq i \leq r_1$ ;
- (11).  $u_i u_{p+m+(j-1)q+k} = -u_j u_{p+m+(i-1)q+k} = v_{\sigma_1(i,j,k)}$ ;  
 if  $1 \leq i < j \leq p$ ,  $1 \leq k \leq q$  e  $\sigma_1(i, j, k) \leq z$ , where  
 $\sigma_1(i, j, k) = m_1 + n + [\frac{1}{2}(2p-i)(i-1) + (j-i-1)]q + k$ ;
- (12).  $u_i u_{m+p(q+1)+j} = -u_j u_{m+p(q+1)+i} = v_{\sigma_2(i,j)}$ ;  
 if  $1 \leq i < j \leq r_1$  and  $\sigma_2(i, j) \leq z$ , where  
 $\sigma_2(i, j) = m_1 + n + \frac{1}{2}p(p-1)q + \frac{1}{2}(2r_1-i)(i-1) + (j-i)$ ;
- (13). Other products are zero.

Let  $\omega : \mathbf{A} \rightarrow F$ , defined by  $\omega(e) = 1$ ;  $\omega(u_i) = \omega(v_j) = 0$ , for  $i = 1, 2, \dots, r$  and  $j = 1, 2, \dots, s$ . In order to show that  $(\mathbf{A}, \omega)$  is Bernstein, we will use Theorem 3.4.8 of [5]. Consider  $U = \langle u_1, u_2, \dots, u_r \rangle$  and  $V = \langle v_1, v_2, \dots, v_s \rangle$ , then  $\mathbf{A} = Fe \oplus$

$U \oplus V$ . As  $p + m \leq r$ , we may write  $U = \langle u_1, u_2, \dots, u_p, u_{p+1}, \dots, u_{p+m}, \dots, u_r \rangle$ . From commutativity of the product it follows that  $U^2 = \langle u_i u_j \mid 1 \leq i \leq j \leq r \rangle = \langle u_i u_j \mid 1 \leq i \leq j \leq p \rangle + \langle u_i u_j \mid 1 \leq i \leq p, p+1 \leq j \leq p+m \rangle + \langle u_i u_j \mid 1 \leq i \leq p, p+m+1 \leq j \leq r \rangle + \langle u_i u_j \mid p+1 \leq i \leq j \leq r \rangle$ . From items (3), (4) and (5) above it follows that  $\langle u_{p+1}, u_{p+2}, \dots, u_{p+m} \rangle = \langle u_i^2 u_j, u_j^2 u_i \mid 1 \leq i < j \leq p \rangle + \langle (u_i u_j) u_k, (u_j u_k) u_i \mid 1 \leq i < j < k \leq p \rangle$ . Thus  $U^2 = \langle u_i u_j \mid 1 \leq i \leq j \leq p \rangle + \langle (u_i^2 u_j) u_k, (u_j^2 u_i) u_k \mid 1 \leq i < j \leq p, 1 \leq k \leq p \rangle + \langle (u_i u_j \cdot u_k) u_l, (u_j u_k \cdot u_i) u_l \mid 1 \leq i < j < k \leq p, 1 \leq l \leq p \rangle + \langle u_i u_j \mid 1 \leq i \leq p, p+m+1 \leq j \leq r \rangle + \langle u_i u_j \mid p+1 \leq i \leq j \leq r \rangle$ . Using the products given in items (2), (6), (7), (8), (11), (12) and (13) we obtain  $U^2 = \langle v_1, v_2, \dots, v_{m_1}, v_{m_1+1}, \dots, v_{m_1+n}, v_{m_1+n+1}, \dots, v_z \rangle \subseteq V$ . Let  $W = \langle v_{z+1}, v_{z+2}, \dots, v_s \rangle$ , then  $V = U^2 \oplus W$  e  $UV = U^3 + UW$ . From (3), (4) and (5), it follows that  $U^3 = \langle u_i^2 u_j, u_j^2 u_i \mid 1 \leq i < j \leq p \rangle + \langle (u_i u_j) u_k, (u_j u_k) u_i \mid 1 \leq i < j \leq p \rangle = \langle u_{p+1}, u_{p+2}, \dots, u_{p+m} \rangle$  and from (9) and (10) we obtain  $UW = \langle u_i v_{z+j} \mid 1 \leq i \leq p, 1 \leq j \leq q \rangle + \langle u_i v_{z+q+1} \mid 1 \leq i \leq r_1 \rangle = \langle u_{p+m+1}, u_{p+m+2}, \dots, u_{p+t} \rangle$ . Hence  $UV = \langle u_{p+1}, u_{p+2}, \dots, u_{p+t} \rangle \subseteq U$ . Moreover, we have  $V^2 = 0$ . Let  $x = \omega(x)e + u + v \in \mathbf{A}$ , where  $u = \sum_{i=1}^r \alpha_i u_i$ ,  $v = \sum_{j=1}^s \beta_j v_j$ , with  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in F$ . As the product is commutative, from (1) it follows that  $x^2 = \omega(x)^2 e + (\omega(x)u + 2uv + v^2) + u^2$ , with  $\omega(x)u + 2uv + v^2 \in U + UV + V^2 = U$  and  $u^2 \in U^2 \subseteq V$ . It remains to show the identities:

$$uv^2 = (u^2)^2 = u^3 = u(uv) = (uv)^2 = 0$$

for all  $u = \sum_{i=1}^r \alpha_i u_i \in U$  and  $v = \sum_{j=1}^s \beta_j v_j \in V$ .

(i)  $uv^2 \in UV^2 = 0$ , because  $V^2 = 0$ ;

(ii)  $(u^2)^2 \in (U^2)^2 \subseteq V^2 = 0$ ;

(iii)  $u^3 = 0$ :

From commutativity of the product we have

$$u^2 = \sum_{i=1}^r \alpha_i^2 u_i^2 + 2 \sum_{i=1}^{r-1} \sum_{j=i+1}^r \alpha_i \alpha_j u_i u_j \text{ and}$$

$$u^3 = \sum_{i=1}^r \sum_{j=1}^r \alpha_i^2 \alpha_j u_i^2 u_j + 2 \sum_{i=1}^{r-1} \sum_{j=i+1}^r \sum_{k=1}^r \alpha_i \alpha_j \alpha_k (u_i u_j) u_k.$$

Consider a product of the form  $(u_i u_j) u_k$ , with  $1 \leq i, j, k \leq r$ . If  $i$  or  $j \geq p+1$ , then the product  $u_i u_j$  lies in  $\langle v_{m_1+n+1}, v_{m_1+n+2}, \dots, v_z \rangle$ , according to items (11) and (12). Thus  $(u_i u_j) u_k = 0$ , for every  $k$ , because by (9), (10) and (13),  $u_k v_\sigma = 0$ , for any  $\sigma \leq z$ . If  $i, j \leq p$ , then  $u_i u_j \in \langle v_1, v_2, \dots, v_{m_1} \rangle$ , according to (2) and in this case, if  $k \geq p+1$ , then  $(u_i u_j) u_k = 0$ , by the same reason.

Hence, it is enough to consider  $i, j, k \leq p$ . Then  $u^3 = \sum_{i=1}^p \sum_{j=1}^p \alpha_i^2 \alpha_j u_i^2 u_j + 2 \sum_{i=1}^{p-1} \sum_{j=i+1}^p \sum_{k=1}^p \alpha_i \alpha_j \alpha_k (u_i u_j) u_k = \sum_{1 \leq i < j \leq p} \alpha_i^2 \alpha_j u_i^2 u_j + \sum_{i=1}^p \alpha_i^3 u_i^3 + \sum_{1 \leq i < j \leq p} \alpha_i \alpha_j^2 u_i u_j^2 +$

$$2\left(\sum_{1 \leq i < j \leq p} \alpha_i^2 \alpha_j (u_i u_j) u_i + \sum_{1 \leq i < j \leq p} \alpha_i \alpha_j^2 (u_i u_j) u_j + \sum_{1 \leq i < j < k \leq p} \alpha_i \alpha_j \alpha_k (u_j u_k) u_i + \sum_{1 \leq i < j < k \leq p} \alpha_i \alpha_j \alpha_k (u_i u_k) u_j + \sum_{1 \leq i < j < k \leq p} \alpha_i \alpha_j \alpha_k (u_i u_j) u_k\right).$$

Using (3), (4), (5) and (13) we get:

$$u^3 = \sum_{1 \leq i < j \leq p} \left(2\alpha_i^2 \alpha_j u_{\rho_1(i,j)} + 2\alpha_i \alpha_j^2 u_{\rho_2(i,j)} + 2(-\alpha_i^2 \alpha_j u_{\rho_1(i,j)} - \alpha_i \alpha_j^2 u_{\rho_2(i,j)})\right) + \sum_{1 \leq i < j < k \leq p} 2\alpha_i \alpha_j \alpha_k \left(u_{\tau_2(i,j,k)} - (u_{\tau_1(i,j,k)} + u_{\tau_2(i,j,k)}) + u_{\tau_1(i,j,k)}\right) = 0.$$

(iv)  $u(uv) = 0$ :

Initially we calculate the product  $uv$ . Let  $W_1 = \langle v_1, v_2, \dots, v_{m_1} \rangle$ ,  $W_2 = \langle v_{m_1+1}, v_{m_1+2}, \dots, v_z \rangle$  and  $W_3 = \langle v_{z+1}, v_{z+2}, \dots, v_s \rangle$ . Then  $V = W_1 \oplus W_2 \oplus W_3$  and thus  $v = w_1 + w_2 + w_3$  with  $w_i \in W_i$ , for  $i = 1, 2, 3$ . By the rules of the product it follows that  $UW_2 = 0$ , therefore  $uv = uw_1 + uw_3$ . By (2),  $W_1 = \langle u_i u_j \mid 1 \leq i \leq j \leq p \rangle$ , then there exist  $\beta_{ij} \in F$  ( $1 \leq i \leq j \leq p$ ) such that  $w_1 = \sum_{i=1}^p \sum_{j=i}^p \beta_{ij} u_i u_j$ . Thus  $uw_1 = \sum_{1 \leq i \leq j \leq p} \sum_{k=1}^r \alpha_k \beta_{ij} (u_i u_j) u_k = \sum_{i=1}^p \sum_{k=1}^r \alpha_k \beta_{ii} u_i^2 u_k + \sum_{1 \leq i < j \leq p} \sum_{k=1}^p \alpha_k \beta_{ij} (u_i u_j) u_k = \sum_{1 \leq i < j \leq p} \left(\alpha_i \beta_{jj} u_i u_j^2 + \alpha_j \beta_{ii} u_i^2 u_j + \alpha_i \beta_{ij} (u_i u_j) u_i + \alpha_j \beta_{ij} (u_i u_j) u_j\right) + \sum_{1 \leq i < j < k \leq p} \left(\alpha_i \beta_{jk} (u_j u_k) u_i + \alpha_j \beta_{ik} (u_i u_k) u_j + \alpha_k \beta_{ij} (u_i u_j) u_k\right)$ .

Using (3) and (5) we get:

$$uw_1 = \sum_{1 \leq i < j \leq p} \left( (\alpha_i \beta_{jj} - \frac{1}{2} \alpha_j \beta_{ij}) u_i u_j^2 + (\alpha_j \beta_{ii} - \frac{1}{2} \alpha_i \beta_{ij}) u_i^2 u_j \right) + \sum_{1 \leq i < j < k \leq p} \left( (\alpha_i \beta_{jk} - \alpha_j \beta_{ik}) (u_j u_k) u_i + (\alpha_k \beta_{ij} - \alpha_j \beta_{ik}) (u_i u_j) u_k \right). \tag{2}$$

As  $z \leq s - \lceil \frac{t-m}{p} \rceil$ , then  $z + q \leq s$ , thus  $W_3 = \langle v_{z+1}, v_{z+2}, \dots, v_{z+q}, \dots, v_s \rangle$ . Let  $\lambda_{z+j} \in F$  with  $j = 1, 2, \dots, s - z$  such that  $w_3 = \sum_{j=1}^{s-z} \lambda_{z+j} v_{z+j}$ . Then  $uw_3 = \sum_{j=1}^{s-z} \lambda_{z+j} uv_{z+j} = \sum_{j=1}^q \lambda_{z+j} uv_{z+j} + \lambda_{z+q+1} uv_{z+q+1} + \sum_{j=q+2}^{s-(z+1)} \lambda_{z+j} uv_{z+j} + \lambda_s uv_s$ . Using (9) and (10), it follows:

$$uw_3 = \sum_{i=1}^p \sum_{j=1}^q \alpha_i \lambda_{z+j} u_{p+m+(i-1)q+j} + \sum_{i=1}^{r_1} \alpha_i \lambda_{z+q+1} u_{m+p(q+1)+i}. \tag{3}$$

From the identities given in (2) and (3) we get:

$$uv = \sum_{1 \leq i < j \leq p} \left( (\alpha_i \beta_{jj} - \frac{1}{2} \alpha_j \beta_{ij}) u_i u_j^2 + (\alpha_j \beta_{ii} - \frac{1}{2} \alpha_i \beta_{ij}) u_i^2 u_j \right) + \sum_{1 \leq i < j < k \leq p} \left( (\alpha_i \beta_{jk} - \alpha_j \beta_{ik}) (u_j u_k) u_i + (\alpha_k \beta_{ij} - \alpha_j \beta_{ik}) (u_i u_j) u_k \right) + \sum_{i=1}^p \sum_{j=1}^q \alpha_i \lambda_{z+j} u_{p+m+(i-1)q+j} + \sum_{i=1}^{r_1} \alpha_i \lambda_{z+q+1} u_{m+p(q+1)+i}. \tag{4}$$

In order to establish the product  $u(uv)$ , we calculate in separate forms, the products of  $u$  by each of the parts given in (4).

$$\begin{aligned}
1) \quad & \sum_{1 \leq i < j \leq p} (\alpha_i \beta_{jj} - \frac{1}{2} \alpha_j \beta_{ij})(u_i u_j^2) u = \sum_{1 \leq i < j \leq p} \sum_{k=1}^r \alpha_k (\alpha_i \beta_{jj} - \frac{1}{2} \alpha_j \beta_{ij})(u_i u_j^2) u_k \\
& = \sum_{1 \leq i < j < k \leq p} \left( \alpha_i (\alpha_j \beta_{kk} - \frac{1}{2} \alpha_k \beta_{jk})(u_j u_k^2) u_i + \alpha_j (\alpha_i \beta_{kk} - \frac{1}{2} \alpha_k \beta_{ik})(u_i u_k^2) u_j \right. \\
& \left. + \alpha_k (\alpha_i \beta_{jj} - \frac{1}{2} \alpha_j \beta_{ij})(u_i u_j^2) u_k \right) + \sum_{1 \leq i < j \leq p} \left( (\alpha_i^2 \beta_{jj} - \frac{1}{2} \alpha_i \alpha_j \beta_{ij})(u_i u_j^2) u_i + (\alpha_i \alpha_j \beta_{jj} \right. \\
& \left. - \frac{1}{2} \alpha_j^2 \beta_{ij})(u_i u_j^2) u_j \right). \text{ Using the items (6), (7), (8) and (14) we get:}
\end{aligned}$$

$$\sum_{1 \leq i < j \leq p} (\alpha_i \beta_{jj} - \frac{1}{2} \alpha_j \beta_{ij})(u_i u_j^2) u = \tag{5}$$

$$= \sum_{1 \leq i < j < k \leq p} \left( \alpha_i \alpha_k \beta_{jk} - \alpha_j \alpha_k \beta_{ik} \right) v_{\delta_3(i,j,k)} + (2\alpha_i \alpha_k \beta_{jj} - \alpha_j \alpha_k \beta_{ij}) v_{\delta_2(i,j,k)};$$

$$\begin{aligned}
2) \quad & \sum_{1 \leq i < j \leq p} (\alpha_j \beta_{ii} - \frac{1}{2} \alpha_i \beta_{ij})(u_i^2 u_j) u = \sum_{1 \leq i < j \leq p} \sum_{k=1}^r \alpha_k (\alpha_j \beta_{ii} - \frac{1}{2} \alpha_i \beta_{ij})(u_i^2 u_j) u_k \\
& = \sum_{1 \leq i < j < k \leq p} \left( (\alpha_i \alpha_k \beta_{jj} - \frac{1}{2} \alpha_i \alpha_j \beta_{jk})(u_j^2 u_k) u_i + (\alpha_j \alpha_k \beta_{ii} - \frac{1}{2} \alpha_i \alpha_j \beta_{ik})(u_i^2 u_k) u_j \right. \\
& \left. + (\alpha_j \alpha_k \beta_{ii} - \frac{1}{2} \alpha_i \alpha_k \beta_{ij})(u_i^2 u_j) u_k \right). \text{ Using the items (6), (7) and (8) of the product we get:}
\end{aligned}$$

$$\sum_{1 \leq i < j \leq p} (\alpha_j \beta_{ii} - \frac{1}{2} \alpha_i \beta_{ij})(u_i^2 u_j) u = \tag{6}$$

$$\sum_{1 \leq i < j < k \leq p} \left( (\alpha_i \alpha_j \beta_{jk} - 2\alpha_i \alpha_k \beta_{jj}) v_{\delta_2(i,j,k)} + (\alpha_i \alpha_j \beta_{ik} - \alpha_i \alpha_k \beta_{ij}) v_{\delta_1(i,j,k)} \right);$$

$$\begin{aligned}
3) \quad & \sum_{1 \leq i < j < k \leq p} (\alpha_i \beta_{jk} - \alpha_j \beta_{ik})(u_j u_k \cdot u_i) u = \sum_{1 \leq i < j < k \leq p} \sum_{l=1}^r \alpha_l (\alpha_i \beta_{jk} - \alpha_j \beta_{ik})(u_j u_k \cdot u_i) u_l \\
& = \sum_{1 \leq i < j < k < l \leq p} \left( \alpha_i \alpha_j \beta_{kl} - \alpha_i \alpha_k \beta_{jl} \right) (u_k u_l \cdot u_j) u_i + (\alpha_i \alpha_j \beta_{kl} - \alpha_j \alpha_k \beta_{il}) (u_k u_l \cdot u_i) u_j \\
& + (\alpha_i \alpha_k \beta_{jl} - \alpha_j \alpha_k \beta_{il}) (u_j u_l \cdot u_i) u_k + (\alpha_i \alpha_l \beta_{jk} - \alpha_j \alpha_l \beta_{ik}) (u_j u_k \cdot u_i) u_l \\
& + \sum_{1 \leq i < j < k \leq p} \left( (\alpha_i \alpha_j \beta_{jk} - \alpha_j^2 \beta_{ik}) (u_j u_k \cdot u_i) u_j + (\alpha_i \alpha_k \beta_{jk} - \alpha_k \alpha_j \beta_{ik}) (u_j u_k \cdot u_i) u_k \right).
\end{aligned}$$

Using now the items (6), (7) and (8) we have:

$$\begin{aligned} & \sum_{1 \leq i < j < k \leq p} (\alpha_i \beta_{jk} - \alpha_j \beta_{ik})(u_j u_k \cdot u_i) u = \\ & = \sum_{1 \leq i < j < k < l \leq p} \left( \alpha_k \alpha_j \beta_{il} - \alpha_i \alpha_k \beta_{jl} + \alpha_i \alpha_l \beta_{jk} - \alpha_j \alpha_l \beta_{ik} \right) v_{\gamma_2(i,j,k,l)} + \\ & \sum_{1 \leq i < j < k \leq p} \left( (\alpha_j^2 \beta_{ik} - \alpha_i \alpha_j \beta_{jk}) v_{\delta_2(i,j,k)} + (\alpha_j \alpha_k \beta_{ik} - \alpha_i \alpha_k \beta_{jk}) v_{\delta_3(i,j,k)} \right); \end{aligned} \quad (7)$$

$$\begin{aligned} 4) & \sum_{1 \leq i < j < k \leq p} (\alpha_k \beta_{ij} - \alpha_j \beta_{ik})(u_i u_j \cdot u_k) u = \sum_{1 \leq i < j < k \leq p} \sum_{l=1}^r \alpha_l (\alpha_k \beta_{ij} - \alpha_j \beta_{ik})(u_i u_j \cdot u_k) u_l \\ & = \sum_{1 \leq i < j < k < l \leq p} \left( (\alpha_i \alpha_l \beta_{jk} - \alpha_k \alpha_l \beta_{jl})(u_j u_k \cdot u_i) u_i + (\alpha_j \alpha_l \beta_{ik} - \alpha_k \alpha_l \beta_{il})(u_i u_k \cdot u_j) u_j \right. \\ & \left. + (\alpha_k \alpha_l \beta_{ij} - \alpha_j \alpha_k \beta_{il})(u_i u_j \cdot u_l) u_k + (\alpha_k \alpha_l \beta_{ij} - \alpha_j \alpha_l \beta_{ik})(u_i u_j \cdot u_k) u_l \right) \\ & + \sum_{1 \leq i < j < k \leq p} \left( (\alpha_i \alpha_k \beta_{ij} - \alpha_i \alpha_j \beta_{ik})(u_i u_j \cdot u_k) u_i + (\alpha_j \alpha_k \beta_{ij} - \alpha_j^2 \beta_{ik})(u_i u_j \cdot u_k) u_j \right). \end{aligned}$$

From items (6), (7), (8) and (13) it follows that:

$$\begin{aligned} & \sum_{1 \leq i < j < k \leq p} (\alpha_k \beta_{ij} - \alpha_j \beta_{ik})(u_i u_j \cdot u_k) u = \\ & = \sum_{1 \leq i < j < k < l \leq p} \left( \alpha_i \alpha_k \beta_{jl} + \alpha_j \alpha_l \beta_{ik} - \alpha_i \alpha_l \beta_{jk} - \alpha_j \alpha_k \beta_{il} \right) v_{\gamma_2(i,j,k,l)} \\ & + \sum_{1 \leq i < j < k \leq p} \left( (\alpha_i \alpha_k \beta_{ij} - \alpha_i \alpha_j \beta_{ik}) v_{\delta_1(i,j,k)} + (\alpha_j \alpha_k \beta_{ij} - \alpha_j^2 \beta_{ik}) v_{\delta_2(i,j,k)} \right); \end{aligned} \quad (8)$$

$$\begin{aligned} 5) & \sum_{i=1}^p \sum_{j=1}^q \alpha_i \lambda_{z+j} u_{p+m+(i-1)q+j} u = \sum_{i=1}^r \sum_{j=1}^p \sum_{k=1}^q \alpha_i \alpha_j \lambda_{z+k} u_i u_{p+m+(j-1)q+k} \\ & = \sum_{1 \leq i < j \leq p} \sum_{k=1}^q \left( \alpha_i \alpha_j \lambda_{z+k} u_i u_{p+m+(j-1)q+k} + \alpha_j \alpha_i \lambda_{z+k} u_j u_{p+m+(i-1)q+k} \right). \end{aligned}$$

Using the item (11):

$$\sum_{i=1}^p \sum_{j=1}^q \alpha_i \lambda_{z+j} u_{p+m+(i-1)q+j} u = \sum_{1 \leq i < j \leq p} \sum_{k=1}^q \alpha_i \alpha_j \lambda_{z+k} \left( v_{\sigma_1(i,j,k)} - v_{\sigma_1(i,j,k)} \right) = 0; \quad (9)$$

$$\begin{aligned} 6) & \sum_{i=1}^{r_1} \alpha_i \lambda_{z+q+1} u_{m+p(q+1)+i} u = \sum_{i=1}^r \sum_{j=1}^{r_1} \alpha_i \alpha_j \lambda_{z+q+1} u_i u_{p+m+p(q+1)+j} \\ & = \sum_{1 \leq i < j \leq r_1} \left( \alpha_i \alpha_j \lambda_{z+q+1} u_i u_{p+m+p(q+1)+j} + \alpha_j \alpha_i \lambda_{z+q+1} u_j u_{m+p(q+1)+i} \right). \end{aligned}$$

Using the item (12):

$$\sum_{i=1}^{r_1} \alpha_i \lambda_{z+q+1} u_{m+p(q+1)+i} u = \sum_{1 \leq i < j \leq r_1} \alpha_i \alpha_j \lambda_{z+q+1} \left( v_{\sigma_2(i,j)} - v_{\sigma_2(i,j)} \right) = 0; \quad (10)$$





	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$	$u_6$	.....	$u_{20}$	$u_{21}$	$u_{22}$	$u_{23}$	$u_{24}$
$u_{19}$	$-v_{13}$			$v_{21}$								
$u_{20}$	$-v_{16}$		$v_{22}$									
$u_{21}$	$-v_{24}$	$v_{20}$	$v_{21}$									
$u_{22}$	$-v_{25}$		$-v_{21}$	$-v_{22}$								
$u_{23}$	$-v_{18}$	$-v_{21}$										
$u_{24}$	$-v_{19}$	$-v_{22}$										

Table of  $UV + V^2$ :

	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$	....	$u_{24}$	$v_1$	$v_2$	$v_3$	...	$v_{25}$
$v_1$		$2u_5$	$2u_6$	$2u_7$								
$v_2$	$-u_5$	$-u_8$	$u_{11}$	$u_{13}$								
$v_3$	$-u_6$	$-u_{11}$	$-u_{12}$	$-u_9$	$u_{15}$							
$v_4$	$-u_7$	$-u_{13}$	$-u_{14}$	$-u_{15}$	$-u_{16}$	$-u_{10}$						
$v_5$	$2u_8$			$2u_{17}$	$2u_{18}$							
$v_6$	$u_{12}$	$-u_{17}$		$-u_{19}$	$u_{21}$							
$v_7$	$u_{14}$	$-u_{18}$	$-u_{21}$	$-u_{22}$	$-u_{20}$							
$v_8$	$2u_9$	$2u_{19}$			$2u_{23}$							
$v_9$	$u_{16}$	$u_{22}$		$-u_{23}$	$-u_{24}$							
$v_{10}$	$2u_{10}$	$2u_{20}$		$2u_{24}$								
$v_{11}$												
...												
$v_{25}$												

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