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THE LIFTING CONDITION AND FULLY INVARIANT SUBMODULES

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Abstract

A module M is lifting if for every submodule A of M, there exists a direct summand B of M such that $B \leq A$ and A/B small in M/B. Every non-cosingular lifting module has the summand sum property. We call any module M *FI-lifting* if for every fully invariant submodule A of M there exists a direct summand B of M such that $B \leq A$ and A/B is small in M/B. In contrast to lifting modules, any finite direct sum of FI-lifting modules is FI-lifting.

I. Introduction

Throughout this paper R denotes an associative ring with unity and all R-modules are unitial right R-modules.

A submodule N of a module M is called *small*, written $N << M$, if $M \neq N + L$ for every proper submodule L of M. Properties of small submodules are given in [9, Lemma 4.2] and [13, Proposition 19.3]. Let M be a module. M is called *lifting module (or $(D1)$)*, if for every submodule N of M, M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \leq N$ and $M_2 \cap N$ small in M_2, equivalenty if for every submodule A of M there exists a direct summand B of M such that $B \leq A$ and A/B is small in M/B. Let M be a module. M has *summand sum property* if the sum of any two direct summands of M is a direct summand of M and denoted by SSP. M has *summand intersection property* if the intersection of any two direct summands of M is a direct summand of M and denoted by SIP (see [6,7,12]). Let M be an R-module. M is called *small*

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, if $M << E(M)$, where $E(M)$ is the injective hull of M. In [10], Talebi and Vanaja defined $\overline{Z}(M) = \cap\{Ker(g) : g \in Hom(M,N), N \ll E(N)\}$. They call M *cosingular (non-cosingular)* module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Cosingular and non-cosingular modules are studied in [10] and [11].

In Section 2, we prove that (1) for every fully invariant submodule Y of M, M/Y is lifting and (2) every non-cosingular lifting module has the summand sum property.

Following [3], M is called *FI-extending*, every fully invariant submodule of M is essential in a direct summand of M. In Section 3, dually, we called the module M is *FI-lifting* if for every fully invariant submodule A of M, there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B, and shown that

Proposition Let M be a module and X a fully invariant submodule of M. If M is FI-lifting then M/X is FI-lifting.

Theorem Let $M = \oplus_{i=1}^{n} X_i$. If each X_i is FI-lifting, then M is FI- lifting.

We will refer to [1, 9, 13] for all undefined notions used in the text, and also for basic facts concerning coatomic and singular modules.

2. The lifting condition for a factor submodule

In this section we investigate conditions which ensure that a factor submodule of a lifting module will be a lifting module. The following theorem is dual of [2, Theorem 1.1].

Theorem 2.1 *Let M be an $R-$module.*

1. *Assume that M is a lifting module and X a submodule of M. If for every direct summand K of M, $(X+K)/X$ is a direct summand of M/X then M/X is lifting .*

2. *Let D be a submodule of M and X a direct summand of M. Assume that M/X is lifting. If $D/(D \cap X)$ is non-cosingular, then $D + X$ is a direct summand of M.*

3. *If M is non-cosingular and M/X is lifting with X a direct summand of M, then $(X+D)/X$ is a direct summand of M/X for all direct summands D of M.*

Proof (1) Let $A/X \leq M/X$. Since M is lifting, there exists a direct summand D of M such that $D \subseteq A$ and A/D is small in M/D. By hypothesis, $(D+X)/X$

is a direct summand of M/X. Clearly, $(D+X)/X \subseteq A/X$. Now we show that $A/(D+X)$ is small in $M/(D+X)$. Let $M/(D+X) = A/(D+X)+L/(D+X)$ for any submodule $L/(D+X)$ of $M/(D+X)$. Then $M = A + L$ implies that $M/D = A/D + L/D$. Since A/D is small in M/D, $M = L$. Therefore $A/(D+X)$ is small in $M/(D+X)$. Thus M/X is lifting.
(2) Let $D, X \leq M$ with X a direct summand of M. Consider the submodule $(D+X)/X \leq M/X$. Since M/X is lifting, there exists a direct summand C/X of M/X such that $C/X \subseteq (D+X)/X$ and $(D+X)/C$ is small in M/C. Hence $(D+X)/C$ is cosingular. On the other hand $(D+X)/X \cong D/(D\cap X)$ and so $(D+X)/X$ is non-cosingular. Therefore by [10, Proposition 2.4], $(D+X)/C$ is non-cosingular. Hence $D+X = C$.
(3) Let M be non-cosingular module and M/X lifting with X a direct summand of M. Let D be a direct summand of M. Then $D/(D\cap X)$ is non-cosingular by [10, Proposition. 2.4]. By (2) $D+X$ is a direct summand of M and hence $(D+X)/X$ is a direct summand of M/X. □

Let M be a module. A submodule X of M is called *fully invariant* if for every $h \in End_R(M)$, $h(X) \subseteq X$. Some properties of fully invariant submodules are given in Lemma 3.2.

A module M is called *distributive* if its lattice of submodules is a distributive lattice.

Corollary 2.2 *Let M be a lifting module.*

1. *If M is a distributive module, then M/X is lifting for every submodule X of M.*
2. *Let $X \leq M$ and $eX \subseteq X$ for all $e^2 = e \in End(M)$. Then M/X is lifting. In particular, for every fully invariant submodule Y of M, M/Y is lifting.*

Proof (1) Let D be a direct summand of M. Then $M = D \oplus D'$ for some submodule D' of M. Now $M/X = [(D+X)/X] + [(D'+X)/X]$ and $X = X+(D\cap D') = (X+D)\cap(X+D')$. So, $M/X = [(D+X)/X]\oplus[(D'+X)/X]$. By Theorem 2.1.(1), M/X is lifting.
(2) Let D be a direct summand of M. Consider the projection map $e : M \to D$. Then $e^2 = e \in End(M)$. By hypothesis, $eX \subseteq X$ and hence $eX = X\cap D$. There exists a direct summand D' of M such that $M = D \oplus D'$. Therefore $X = (X\cap D)\oplus(X\cap D')$. Now $(D+X)/X = (D\oplus(X\cap D'))/X$ and $(D'+X)/X = (D'\oplus(X\cap D))/X$. Hence $M = D\oplus D' = D+X+D'+X = [D\oplus(X\cap D')]+D'+X$ implies that $M/X = (D\oplus(X\cap D'))/X + (D'+X)/X$. Since $[D\oplus(X\cap D')]\cap(D'+X) = (X\cap D')\oplus(X\cap D)$, $M/X = (D\oplus(X\cap D'))/X \oplus (D'+X)/X$. Thus by Theorem 2.1.(1), M/X is lifting. □

Theorem 2.3 *Let R be a semiperfect ring.*

1. *If R has every idempotent central then, for every right ideal I of R, R/I is right lifting.*

2. *For every ideal I of R, R/I is semiperfect.*

Proof They follows from Corollary 2.2.(2) and [1]. □

Corollary 2.4 *Every non-cosingular lifting module has the summand sum property.*

Proof Let M be a non-cosingular lifting module. Let A and B be two direct summands of M. Let $M = A \oplus A' = B \oplus B'$ for some submodules A', B'. Note that A' and B' are lifting modules. Since $M/A \cong A'$ and $M/B \cong B'$, $(A+B)/A$ is a direct summand of M/A and $(A+B)/B$ is a direct summand of M/B by theorem 2.1.(3). Hence $A+B$ is a direct summand of M. □

We know that there are modules having the SSP and (D1) but not the SIP.

Example 2.5. Let F be a field and R the upper triangular matrix ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. For submodules $A = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $A \oplus (R/B)$ has the SSP by [6] and (D1) by [9]. But has not the SIP.

We consider the following condition:

$(D3)$ If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

Lemma 2.6 *Assume that M is $(D3)$. If M has the SSP then M has the SIP.*

Proof Let M_1 and M_2 be direct summands of M. Since $M_1 + M_2$ is direct summand of M by assumption, we have $(M_1+M_2) \oplus X$ for some submodule X of M. Again by assumption, $M_1 + X$ and $M_2 + X$ are direct summands. Since M is $(D3)$, $M = [(M_1 + X) \cap (M_2 + X)] \oplus Y$ for some submodule Y of M. Now we have $M = (M_1 \cap M_2) \oplus X \oplus Y$. That is $M_1 \cap M_2$ is direct summand of M. □

Corollary 2.7 *Let M be a non-cosingular module with $(D3)$. Then*
M is lifting $\Rightarrow$ M has SSP $\Rightarrow$ M has SIP

Example 2.8 (1) Let $M_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}} \oplus \mathbb{Z}_{\mathbb{Z}}$. $M_{\mathbb{Z}}$ is not lifting. Since $\mathbb{Z}_{\mathbb{Z}} \oplus \mathbb{Z}_{\mathbb{Z}} << \mathbb{Q}_{\mathbb{Z}} \oplus \mathbb{Q}_{\mathbb{Z}}$, we have $M_{\mathbb{Z}}$ is co-singular. Furthermore, M has the SIP and so M has

$(D3)$. Let $N = \mathbb{Z}(2,3)$ and $K = \mathbb{Z}(3,2)$. Since $N \oplus K$ is not direct summand of M, M has not the SSP.
(2) Let $M_{\mathbb{Z}} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$, where p is any prime. $M_{\mathbb{Z}}$ is a lifting module and, since $\mathbb{Z}/p\mathbb{Z} << \mathbb{Q}/p\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$is co-singular and so M is cosingular module. Furthermore $M_{\mathbb{Z}}$ is not $(D3)$ and $M_{\mathbb{Z}}$ has neither the SIP nor the SSP.
(3) The $\mathbb{Z}$-module $\mathbb{Q}$, the set of all rational numbers, is non- cosingular module by [10, Remark 2.11]. We know that $\mathbb{Q}_{\mathbb{Z}}$ has the SIP and so $(D3)$ and has the SSP. But $\mathbb{Q}_{\mathbb{Z}}$ is not a lifting module.

3. FI-lifting modules

Let M be a lifting module. In Corollary 2.2 we proved that, for every fully invariant submodule Y of M, M/Y is lifting. In this section, we determine a generalization of the lifting modules. Let M be any module. Following [3], M is called *FI-extending*, every fully invariant submodule of M is essential in a direct summand of M. FI-extending modules are studied [3], [4] and [5]. Dually, we say the module M is *FI-lifting* if for every fully invariant submodule A of M, there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B.

Clearly, M is FI-lifting if and only if for every fully invariant submodule A of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ is small in M_2.

Lemma 3.1 *Let M be a module.*
(1) *M is FI-lifting.*
(2) *For every fully invariant submodule A of M there is a decomposition*
$A = N \oplus S$ with N a direct summand of M and S small in M.

Proof For the proof, we completely follow the proof of [9, Proposition 4.8].
$(i) \Rightarrow (ii)$ Let A be a fully invariant submodule of M. Since M is FI-lifting, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ small in M_2. Therefore $A = M_1 \oplus (A \cap M_2)$, as required.
$(ii) \Rightarrow (i)$ Assume that every fully invariant submodule has the stated decomposition. Let A be a fully invariant submodule of M. By hypothesis, there exists a direct summand N of M and a small submodule S of M such that $A = N \oplus S$. Now $M = N \oplus N'$ for some submodule N' of M. Consider the natural epimorphism $\pi : M \longrightarrow M/N$. Then $\pi(S) = (S + N)/N = A/N$ small in M/N. Therefore M is FI-lifting. □

Lemma 3.2 *Let M be a module.*

1. *Any sum and intersection of fully invariant submodules of M is again a fully invariant submodule of M.*

2. *If $X \leq Y \leq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.*

3. *If $M = \oplus_{i\in I}X_i$ and S is a fully invariant submodule of M, then $S = \oplus_{i\in I}\pi_i(S) = \oplus_{i\in I}(X_i \cap S)$, where π is the $i-th$ projection homomorphism of M.*

4. *If $X \leq Y \leq M$ such that X is a fully invariant submodule of M and Y/X is a fully invariant submodule of M/X, then Y is a fully invariant submodule of M.*

Proof $(1), (2), (3)$ see [3, Lemma 1.1].
(4) Let $f : M \to M$ be any homomorphism. Then $f(X) \subseteq X$. Now, consider the homomorphism $g : M/X \to M/X$ defined by $g(m+X) = f(m)+X$, $(m \in M)$. Then $g(Y/X) \subseteq Y/X$. Clearly, $g(Y/X) = (f(Y) + X)/X$. Therefore $f(Y) \subseteq Y$. □

Proposition 3.3 *Let M be a module and X a fully invariant submodule of M. If M is FI-lifting then M/X is FI-lifting.*

Proof Let Y be a submodule of M with $X \subseteq Y$ and assume that Y/X is a fully invariant submodule of M/X. By Lemma 3.2, Y is a fully invariant submodule of M. Since M is FI-lifting, there exists a direct summand D of M such that $D \leq Y$ and Y/D is small in M/D. Assume $M = D \oplus D'$ for some submodule D' of M. Let π be the projection with the kernel D and $i : D' \to M$ the inclusion map. Now, $\alpha = i\pi : M \to M$ be a homomorphism of M. Since X and Y are fully invariant submodules of M, $\alpha(X) \subseteq X$ and $\alpha(Y) \subseteq Y$. It is easy to see that $Y = \alpha^{-1}(Y)$. Now, $\alpha^{-1}(X) \subseteq Y = \alpha^{-1}(Y)$. Let K be a submodule of M with $\alpha^{-1}(X) \subseteq K$ and $M/\alpha^{-1}(X) = (Y/\alpha^{-1}(X)) + (K/\alpha^{-1}(X))$. Then $M = Y+K$ and since Y/D is small in M/D, $M = K$. Therefore $Y/\alpha^{-1}(X)$ is small in $M/\alpha^{-1}(X)$, namely $(Y/X)/(\alpha^{-1}(X)/X) << (M/X)/(\alpha^{-1}(X)/X)$. Now, we want to show that $\alpha^{-1}(X)/X$ is a direct summand of M/X. Since $M = D \oplus D'$, then $M = \alpha^{-1}(X) + D'$. Therefore $M/X = (\alpha^{-1}(X)/X) + (D' + X)/X$. Since $\alpha^{-1}(X) \cap (D' + X) = X + (\alpha^{-1}(X) \cap D') = X$, $\alpha^{-1}(X)/X$ is a direct summand of M/X. □

Theorem 3.4 *Let $M = \oplus_{i=1}^{n} X_i$. If each X_i is FI-lifting, then M is FI- lifting.*

Proof Let S be a fully invariant submodule of M. It is easy to see that for every $1 \leq i \leq n$, $S \cap X_i$ is fully invariant in X_i. Since X_i is FI-lifting for every i, there exists a direct summand D_i of X_i such that $D_i \leq S \cap X_i$ and $(S \cap X_i)/D_i$ is small in X_i/D_i for every i. Clearly, $D = \oplus_{i=1}^{n} D_i$ is a direct summand of M and $D \subseteq \oplus_{i=1}^{n}(S \cap X_i)$. We know that $\oplus_{i=1}^{n}(S \cap X_i) = S$ by Lemma 3.2. Now consider the homomorphism $\beta : \oplus_{i=1}^{n}(X_i/D_i) \to (\oplus_{i=1}^{n} X_i)/D$ with $(x_1 + D_1, ..., x_n + D_n) \to (\Sigma_{i=1}^{n} x_i) + D_i$, where $x_i \in X_i$ for $1 \leq i \leq n$.

Then $\beta(\oplus_{i=1}^{n}((S \cap X_i)/D_i)) = (\oplus_{i=1}^{n}(S \cap X_i))/D$. Since any finite sum of small submodules again a small submodule, $\oplus_{i=1}^{n}((S \cap X_i)/D_i)$ is small in $\oplus_{i=1}^{n}(X_i/D_i)$. Then by [9, Lemma 4.2], $(\oplus_{i=1}^{n}(S \cap X_i))/D$ is small in M/D. □

We don't know if any direct sum of FI-lifting module is an FI-lifting module.

Corollary 3.5 *If M is a finite direct sum of lifting (or hollow) modules, then M is FI-lifting.*

Corollary 3.6 *Let R be a PID. Then the torsion submodule of any finitely generated $R-$module M is FI-lifting.*

Proof Let M be a finitely generated $R-$module. Then the torsion submodule $Tor(M)$ of M is a finite direct sum of hollow R-modules. Therefore $Tor(M)$ is FI-lifting by Corollary 3.5. □

Proposition 3.7 *Let M be an FI-lifting module. If M is indecomposable then every proper fully invariant submodule of M is small in M.*

Proof Clear. □

Proposition 3.8 *Let R be any ring and let M be an FI-lifting R-module. Then every fully invariant submodule of the module $M/Rad(M)$ is a direct summand.*

Proof Let $N/Rad(M)$ be any fully invariant submodule of $M/Rad(M)$.Then N is fully invariant submodule of M by Lemma 3.2. By hypothesis, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2$ is small in M_2. Since $N \cap M_2$ is also small in M, $N \cap M_2 \leq Rad(M)$. Thus $M/Rad(M) = (N/Rad(M)) \oplus ((M_2 + Rad(M))/Rad(M))$, as required. □

Example 3.9 (i) Let $M_{\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Then $M_{\mathbb{Z}}$ is FI-lifting by Corollary 3.5. We note that $M_{\mathbb{Z}}$ is not lifting by [8, Example 1] and not non-cosingular module. Furthermore $M_{\mathbb{Z}}$ has the SIP but it is not $(D3)$.
(ii) The $\mathbb{Z}$-module $\mathbb{Q}$, the set of all rational numbers, is non- cosingular module (see example 2.10). $\mathbb{Q}_{\mathbb{Z}}$ is not FI-lifting module.

Example 3.10 Take $R = \mathbb{Z}$ and $M = \mathbb{Z}$. Let $A_i = \mathbb{Z}/2^i\mathbb{Z}$, $foralli \in \mathbb{N}$ and $E = E(A_1)$. Consider $N = \oplus_{i=1}^{n} E_i$, where $E_i = E$ and $n \in \mathbb{N}$. By [11, Example 1.14], N is non-cosingular $\mathbb{Z}$-module and FI-lifting by Corollary 3.5.

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