East-West J. of Mathematics: Vol. 7, No 1 (2005) pp. 93-98

ON LIE IDEALS AND GENERALIZED DERIVATIONS OF PRIME RINGS

Asma Ali, Shakir Ali and **Rekha Rani**

Department of Mathematics
Aligarh Muslim University
Aligarh - 202002, India
e-mail: asma_ali2@rediffmail.com

Abstract

Let R be a ring and S a nonempty subset of R. An additive mapping $F : R \to R$ is called a generalized derivation on S if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in S$. Suppose that U is a Lie ideal of R with the property that $u^2 \in U$, for all $u \in U$. In the present paper, we prove that if R is a prime ring with characteristic different from 2 admitting a generalized derivation F satisfy any one of the properties: (i) $F(uv)-uv \in Z(R)$, (ii) $F(uv)+uv \in Z(R)$, (iii) $F(uv) - vu \in Z(R)$ and (iv) $F(uv) + vu \in Z(R)$, for all $u, v \in U$, then U must be central

1. Introduction

Throughout the present paper R will denote an associative ring with centre $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. For a nonempty subset S of R, we put $C_R(S) = \{x \in R \mid [x, s] = 0, \text{ for all } s \in S\}$. The set of all commutators of elements of S will be written as $[S, S]$. Recall that a ring R is said to be 2-torsion free, if whenever $2x = 0$, with $x \in R$, implies $x = 0$. A ring R is prime if for any $a, b \in R$, $aRb = (0)$, implies that either $a = 0$ or $b = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$, $r \in R$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in S$. Following [8], An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation on R if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$,

Key words: Prime rings, Lie ideals, derivations and generalized derivations.
2000 AMS Mathematics Subject Classification: 16W25, 16N30, 16U80.

holds for all $x, y \in S$. We shall make use of the two basic commutator identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z.$$

There has been a great deal of work concerning the relationship between the commutativity of a ring R and the existance of certain specific types of derivations of R. Recently, many authors viz [1], [2], [3], [5] and [9] etc. have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial constraints. In [1], Ashraf and Nadeem established that a prime ring R with a non-zero ideal I must be commutative if it admits a derivation d satisfying either of the properties $d(xy) + xy \in Z(R)$ or $d(xy) - xy \in Z(R)$, for all $x, y \in I$.

In this paper, we continue the study and attempt to generalize the above mentioned result on a Lie ideal U of the ring R satisfying either of the conditions: (i) $F(uv) - uv \in Z(R)$, (ii) $F(uv) + uv \in Z(R)$, (iii) $F(uv) - vu \in Z(R)$ and (iv) $F(uv) + vu \in Z(R)$, for all $u, v \in U$.

2. Main Results

We begin with the following known results which will be used extensively to prove our theorems.

Lemma 2.1 [4, Lemma 3] *Let R be a 2-torsion free prime ring and U be a Lie ideal of R. If $U \not\subseteq Z(R)$, then $C_R(U) = Z(R)$.*

Lemma 2.2 [4, Lemma 4] *If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = (0)$, then $a = 0$ or $b = 0$.*

Lemma 2.3 [11, Lemma 2.6] *Let R be a 2-torsion free prime ring and U be a Lie ideal of R. If U is a commutative Lie ideal of R i.e., $[u, v] = 0$, for all $u, v \in U$, then $U \subseteq Z(R)$.*

The following lemma is in fact, an extension of a result [9, Lemma 2(a)] due to J. H. Mayne.

Lemma 2.4 *Let R be a 2-torsion free prime ring and U be a Lie ideal of R such that $U \not\subseteq Z(R)$. If R admits a derivation d which is zero on U, then d is* *zero on R.*

Proof By our hypotheses, we have

$$d(u) = 0, \quad \text{for all } u \in U. \tag{2.1}$$

Replacing u by $[u, r]$ in (2.1), we find that $d([u, r]) = ud(r) - d(r)u = 0$ and hence $[u, d(r)] = 0$, for all $u \in U$ and $r \in R$. This yields that $d(r) \in C_R(U)$. Thus, the application of Lemma 2.1 gives $d(r) \in Z(R)$. Hence $[d(r), s] = 0$, for all $r, s \in R$. Replacing r by rr_1 in latter relation and using it, we obtain $d(r)[r_1, s] + [r, s]d(r_1) = 0$, for all $r, r_1, s \in R$. Now replace r_1 by $d(r)$, to get $[r, s]d^2(r) = 0$, for all $r, s \in R$. Again replacing s by us, we find that $[r, u]sd^2(r) = 0$, for all $u \in U$ and $r, s \in R$ i.e., $[r, u]Rd^2(r) = (0)$, for all $u \in U$, $r \in R$. Thus primeness of R implies that either $[r, u] = 0$ or $d^2(r) = 0$. Since $U \not\subseteq Z(R)$, we have $d^2(r) = 0$, for all $r \in R$. Replace r by rs in the above relation, to get $2d(r)d(s) = 0$, for all $r, s \in R$. Since R is 2-torsion free, the latter relation yields that $d(r)d(s) = 0$, for all $r, s \in R$. We conclude that $d(r)d(sr) = (0)$, for all $r, s \in R$. Thus $d(r)Rd(r) = (0)$, for all $r \in R$. The primeness of R forces that $d = 0$. □

Theorem 2.1 *Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) - uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.*

Proof If $F = 0$, then $uv \in Z(R)$, for all $u, v \in U$. Hence $[uv, r] = 0$, for all $u, v \in U$ and $r \in R$. This gives that $u[v, r] + [u, r]v = 0$ for all $u, v \in U$. Replacing u by $2wu$ and using the fact that char$R \neq 2$, we get $[w, r]uv = 0$, for all $u, v, w \in U$ and $r \in R$. Replace r by rs, to get $[w, r]suv = 0$, for all $u, v, w \in U$ and $r, s \in R$ i.e., $[w, r]Ruv = (0)$, for all $u, v, w \in U$ and $r \in R$. Thus primeness of R implies that either $[w, r] = 0$ or $uv = (0)$. If $uv = 0$, for all $u, v \in U$, then replacing v by $[v, r]$, we get $urv = 0$, for all $u, v \in U$ and $r \in R$. Hence $uRv = (0)$, for all $u, v \in U$. Thus primeness of R forces that $U = (0)$, which is not possible. Hence we have $[w, r] = 0$, for all $w \in U$ and $r \in R$ i.e., $U \subseteq Z(R)$.

Hence onward we assume that $F \neq 0$. Suppose on contrary that $U \not\subseteq Z(R)$. Since we have $F(uv) - uv \in Z(R)$, for all $u, v \in U$, $[F(uv) - uv, w] = 0$, for all $u, v, w \in U$. Replacing v by $2vw$ and using the fact that char$R \neq 2$, we get $[(F(uv) - uv)w + uvd(w), w] = 0$, for all $u, v, w \in U$. Hence $[uvd(w), w] = 0$, for all $u, v, w \in U$ i.e.,

$$uv[d(w), w] + u[v, w]d(w) + [u, w]vd(w) = 0, \quad \text{for all } u, v, w \in U. \tag{2.3}$$

Replace u by $2u_1u$ in (2.3) and use (2.3), to obtain $[u_1, w]uvd(w) = 0$, for all $u, u_1, v, w \in U$. Hence $[u_1, w]Uvd(w) = (0)$, for all $u_1, v, w \in U$. Thus by

Lemma 2.2 for each $w \in U$ either $[u_1, w] = 0$ or $vd(w) = 0$. Now, let $U_1 = \{w \in U \mid vd(w) = 0, \text{ for all } v \in U\}$ and $U_2 = \{w \in U \mid [u_1, w] = 0, \text{ for all} u_1 \in U\}$. Then U_1 and U_2 both are additive subgroups of U and $U_1 \cup U_2 = U$. But a group can not be union of its proper subgroups. Thus either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $vd(w] = 0$, for all $v, w \in U$. Replacing v by $[v, r]$ in above relation and using it, we get $vrd(w) = 0$, for all $v, w \in U$ and $r \in R$, i.e. $URd(w) = (0)$, for all $w \in U$. Since R is prime and U is nonzero we conclude that $d(w) = 0$, for all $w \in U$. Hence by Lemma 2.4, we get $d = 0$, a contradiction. On the other hand, if $U_2 = U$, then $[u_1, w] = 0$, for all $u_1, w \in U$. Thus by Lemma 2.3, we get $U \subseteq Z(R)$, again a contradiction. This completes the proof of the theorem. □

Using the same techniques with necessary variations, we get the following:

Theorem 2.2 *Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) + uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.*

Following is the immediate consequence of Theorem 2.1.

Corollary 2.1 *Let R be a prime ring. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(xy) - xy \in Z(R)$, for all $x, y \in R$, then R is commutative.*

Remark 2.1 Since every ideal in a ring R is a Lie ideal of R, conclusion of the above theorem holds even if U is assumed to be an ideal of R. Though the assumption that $u^2 \in U$, for all $u \in U$ seems close to assuming that U is an ideal of the ring, but there exist Lie ideals with this property which are not ideals. For example, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in Z \right\}$. Then it can be easily seen that $U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in Z \right\}$ is a Lie ideal of R satisfying $u^2 \in U$, for all $u \in U$. However, U is not an ideal of R.

Remark 2.2 In conclusion, it is tempting to conjecture as follows:

Conjecture 2.1 Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) - uv \in Z(R)$ or $F(uv) + uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

Theorem 2.3 *Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F*

with associated derivation $d \neq 0$ such that $F(uv) - vu \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

Proof If $F = 0$, then $vu \in Z(R)$, for all $u, v \in U$. Using the same arguments as we have used in the begining of the proof of Theorem 2.1, we get the required result.

Hence, onward we assume that $F \neq 0$. Suppose on contrary that $U \nsubseteq Z(R)$. Since for any $u, v \in U$ we have $F(uv) - vu \in Z(R)$, $[F(uv) - vu, v] = 0$, for all $u, v \in U$. Replacing u by $2uv$ and using the fact that char$R \neq 2$, we get $[(F(uv) - vu)v + uvd(v), v] = 0$, for all $u, v \in U$ and hence $[uvd(v), v] = 0$, for all $u, v \in U$. We have

$$uv[d(v), v] + [u, v]vd(v) = 0, \quad \text{for all } u, v \in U. \tag{2.4}$$

Replace u by $2wu$ in (2.4) and use (2.4), to obtain $[w, v]uvd(v) = 0$, for all $u, v, w \in U$. Hence $[w, v]Uvd(v) = (0)$, for all $v, w \in U$. Thus by Lemma 2.1, for each $v \in U$ either $[w, v] = 0$ or $vd(v) = 0$. Now, let $A = \{v \in U \mid [w, v] = 0, \ \text{for all } w \in U\}$ and $B = \{v \in U \mid vd(v) = 0\}$. Clearly A and B are additive subgroups of U whose union is U. Therefore, either $[w, v] = 0$, for all $v, w \in U$ or $vd(v) = 0$, for all $v \in U$. If $[w, v] = 0$, for all $v, w \in U$, then by Lemma 2.3, we get $U \subseteq Z(R)$, a contradiction. On the other hand, if $vd(v) = 0$, then linearizing the above relation on v, we obtain

$$ud(v) + vd(u) = 0, \quad \text{for all } u, v \in U. \tag{2.5}$$

Again replace v by $2vu$ in (2.5) and use the fact that char$R \neq 2$, to get $ud(vu) + vud(u) = 0$, for all $u, v \in U$. Thus (2.5) yields that $[u, vd(u)] = 0$, for all $u, v \in U$. This gives that $v[u, d(u)] + [u, v]d(u) = 0$, for all $u, v \in U$. Replacing v by $2wv$, we get $[u, w]vd(u) = 0$, for all $u, v, w \in U$ i.e., $[u, w]Ud(u) = (0)$, for all $u, w \in U$. Hence by Lemma 2.2, either $[u, w] = 0$ or $d(u) = 0$. Now, let $U_1 = \{u \in U \mid d(u) = 0\}$ and $U_2 = \{u \in U \mid [u, w] = 0, \text{ for all } w \in U\}$. Then U_1 and U_2 both are additive subgroups of U and $U_1 \cup U_2 = U$. Thus either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $d(u) = 0$, for all $u \in U$ and by Lemma 2.4, we get $d = 0$, a contradiction. On the other hand, if $U_2 = U$, then $[u, w] = 0$, for all $u, w \in U$. Thus by Lemma 2.3, $U \subseteq Z(R)$, again a contradiction. Hence the result is proved. □

Using the same techniques with necessary variations we get the following :

Theorem 2.4 *Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) + vu \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.*

The following example demonstrates that R to be prime is essential in the hypotheses of the above results.

Example 2.1 Consider S as any ring. Let $R=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S\right\}$ and let $I=\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S\right\}$ be a Lie ideal of R. Define $F : R \longrightarrow R$ by $F(x) = 2e_{11}x - xe_{11}$. Then F is a generalized derivation with associated derivation d given by $d(x) = e_{11}x - xe_{11}$. It can be easily seen that R satisfies the properties (i) $F(uv) - uv\ Z(R)$, (ii) $F(uv) + uv\ Z(R)$, (iii) $F(uv) - vu\ Z(R)$ and (iv) $F(uv) + vu\ Z(R)$ for all $u, v \in U$. However, U is not central.

Acknowledgement. The authors would like to thank Dr. M. Ashraf for his helpful suggestions which has improved the contents.

References

[1] Ashraf, M. and N. Rehman, *On derivations and commutativity in prime rings*, East-west J. Math. 3(1) 2001, 87-91.

[2] Bell, H. E. and M. N. Daif, *On commutativity and strong commutativity preserving maps*, Canad. Math. Bull. 37(1994), 443-447.

[3] Bell, H. E. and W. S. Martindale, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. 30(1987), 92-101.

[4] Bergen, J., I. N. Herstien and J. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra 71(1981), 259-267.

[5] Daif, M. N. and H. E. Bell, *Remarks on derivations on semiprime rings*, Internal. J. Math. & Math. Sci. 15(1992), 205-206.

[6] Deng, Q. and M. Ashraf, *On strong commutativity preserving mappings*, Results in Math. 30(1996), 259-263.

[7] Hongan, M., *A note on semiprime rings with derivations*, Internal. J. Math. & Math. Sci. 20(1997), 413-415.

[8] Hvala, B., *Generalized derivations in rings*, Comm. Algebra 26(1998), 1147-1166.

[9] Mayne, J. H., *Centralizing mappings of prime rings*, Canad. Math. Bull. 27(1984), 122-126.

[10] Posner E. C., *Derivations in prime rings*, Proc. Amer. Math. Soc. 8(1957), 1093-1100.

[11] Rehman, N., *On commutativity of prime rings with generalized derivations*, Math. J. Okayama Univ. 44(2002), 43-49.