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FINITE FRACTIONAL-ORDER OF DISTRIBUTIONAL SOLUTIONS OF CERTAIN FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

We establish necessary and sufficient conditions for the existence of solutions of certain fractional differential equations in the form of finite fractional-order of Dirac delta functions. We then give some fractional differential equations whose solutions are in such a form.

1 Introduction

Existence criterion of finite-order of distributional solutions to any homogeneous differential equations was first established by Wiener [1] in 1982. Precisely, his solution is in the form of finite summation of Dirac delta function,

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δ, and its distributional derivatives, that is

$$u(t) = \sum_{k=0}^{m} b_k \delta^{(k)}(t), \quad b_m \neq 0, \tag{1.1}$$

and satisfies the equation

$$\sum_{i=0}^{n} a_i(t) u^{(n-i)}(t) = 0,$$

with coefficients $a_i(t) \in C^{(m+n-i)}$. As for the infinite-order solutions, he considered the r-vector of the type

$$\mathbf{u}(t) = \sum_{k=0}^{\infty} \mathbf{b}_k \delta^{(k)}(t), \tag{1.2}$$

where $\mathbf{b}_k$ are constant r-vectors. He proved the existence of this solutions $\mathbf{u}(t)$ in the space of generalized functions $(S_0^{\beta})'$, $\beta > 1$, for comprehensive systems of any order with countable sets of argument deviations

$$\sum_{i=0}^{\infty} \sum_{j=0}^{m} \mathbf{A}_{ij}(t) \mathbf{u}^{(j)}(\lambda_{ij} t) = 0, \tag{1.3}$$

in which $r \times r$-matrices $\mathbf{A}_{ij}$ and real parameters λ_{ij} satisfy certain conditions. It is well-known that the infinite-order objects in the form (1.2) in general are neither distributions, ultradistributions nor hyperfunctions. Many researchers call them dual Taylor series since in a way they are "dual"to the Taylor series $\sum_{n=0}^{\infty} \mathbf{a}_n t^n$. Using the idea of the proof for the existence of dual Taylor series solutions of system (1.3), Weiner [2] proved existence of entire solutions

$$\mathbf{u}(t) = \sum_{n=0}^{\infty} \mathbf{a}_n t^n$$

of the system

$$\mathbf{u}^{(p)}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{p} \mathbf{Q}_{ij}(t) \mathbf{u}^{(j)}(\lambda_{ij} t)$$
$$\mathbf{u}^{(j)}(0) = \mathbf{u}_j, \quad j = 0, \ldots, p-1$$

under certain conditions of $r \times r$-matrices $\mathbf{Q}_{ij}$ and real parameters λ_{ij}. Cooke and Weiner [3] extended the results in [1] by showing that dual Taylor series solutions in the space $(S_0^\beta)'$, $\beta > 1$, exist for the comprehensive systems of any order with countable sets of variable argument deviations

$$\sum_{i=0}^{\infty}\sum_{j=0}^{m}\mathbf{A}_{ij}(t)\mathbf{u}^{(j)}(\lambda_{ij}(t)) = 0,$$

under suitable conditions on $\mathbf{A}_{ij}(t)$ and functions $\lambda_{ij}(t)$. A thorough overview of finite-order of distributional solutions, dual Taylor series solutions and entire solutions of ordinary differential equations and functional differential equations was given by Shah and Wiener [4].

Dual Taylor series (1.2) has been used frequently in the problem of finding solutions for ordinary differential equations. For example, Littlejohn and Kanwal [5] in 1985 used them to solve the confluent hypergeometric differential equation

$$tu'' + (m-t)u' - pu = 0$$

and the hypergeometric equation

$$t(1-t)u''(t) + [\gamma - (\alpha+\beta+1)t]u'(t) - \alpha\beta u(t) = 0.$$

In particular, for positive integers $m \leq p$, they found that the series solutions reduce to finite-order of distributional solutions (1.1). In 1991 Wiener, Cooke and Shah [6] proved that under certain conditions the equation

$$\sum_{i=0}^{n}(a_i t + b_i)u^{(n-i)}(t) = 0$$

with constant coefficients a_i, b_i and $a_0 = 1$, $b_0 = 0$ admits a sum of finite-order of distributional solutions and a locally integrable function of particular forms. They also proved the existence and non-existence theorems of dual Taylor series solutions in the space $(S_0^\beta)'$, $\beta > 1$ for certain linear equations with polynomial coefficients. Hernandez and Estrada [7] studied connections between dual Taylor series solutions of ordinary differential equations and the asymptotic expansion of classical, distributional, or hyperfunction solutions of the equations. They found through examples that in general the connections are not obvious because there is no relationship between the dimension of the solution spaces of dual Taylor series and others. In 2015, Kanwal [8] reviewed and demonstrated some techniques based on the distributional theory of asymptotic analysis and moment expansion of Dirac delta functions in solving certain differential and integral equations. Next, Nonlaopon et. al. [9] studied the solutions in the space of right-sided distributions of the differential equation

$$tu^{(n)}(t) + mu^{(n-1)}(t) + tu(t) = 0,$$

where m and n are any integers with $n \geq 2$ and $t \in \mathbb{R}$ using Laplace transform technique. They found that the types of Laplace transformable solutions in the space of right-sided distributions depend on the relationship between the values of m and n. Moreover, they found that when $m > n$, all solutions are expressed as finite-order distributional solutions. In 2017, Opio et. al. [10] considered the equation

$$tu^{(n)}(t) + (m - t)u^{(n-1)}(t) - pu(t) = 0,$$

and classified its solutions according the values of m and p. Using Laplace transforms, they found that when $p = 0$ and $m = n$, all solutions are expressed as finite-order distributional solutions.

Morita and Sato [11] discussed a fractional differential equations of the type Laplace's differential equation, a linear fractional differential equation whose coefficients are linear functions of the variable :

$$(a_2 t + b_2) \cdot_0 D_R^{2\sigma} u(t) + (a_1 t + b_1) \cdot_0 D_R^{\sigma} u(t) + (a_0 t + b_0)u(t) = f(t)$$

for $t > 0$, $\sigma = 1$ and $\sigma = 1/2$, where ${}_0D_R^{\sigma}$ is the Riemann-Liouville fractional derivative. Interpreting the equation in the framework of distributional theory and using operational calculus, they were able to solved for its solutions which are locally integrable. Morita and Sato [12] continued their study on Laplace's differential equations and discovered more solutions which are not locally integrable using the method of analytic continuation.

Motivated by the works of Wiener [1], and Morita and Sato [11], we are led to consider the homogeneous fractional differential equations with polynomial coefficients

$$\sum_{i=0}^{nm+r} a_i(t)u^{(i/m)}(t) = 0,$$

where $n, m \in \mathbb{N}$ with $m \geq 2$, $r \in \mathbb{N} \cup \{0\}$ with $r \leq m - 1$, and $u^{(i/m)}$ denotes the i/m-th fractional derivative of u. The polynomial coefficients a_i are of particular form

$$a_i(t) = \sum_{j=0}^{\lfloor i/m \rfloor} a_{ij} t^j,$$

where $\lfloor i/m \rfloor$ denotes the floor function of i/m.

Following Wiener's idea, we derive existence and necessary conditions for its finite fractional-order solution

$$u(t) = \sum_{k=0}^{N} b_k \delta^{(k/m)}(t), \quad b_N \neq 0 \tag{1.4}$$

where $\delta^{(k/m)}$ denotes the k/m-th fractional derivative of the Dirac delta function δ. The proof are presented in section 3. We then provide examples to support our study in section 4.

In order to prove our results, we now introduced the concept of right-sided distributions and its fractional integral and derivatives:

2 Right-sided Distributions and Their Fractional Integrals and Derivatives

A right-sided distribution f is a continuous linear functional on the space of infinitely differentiable functions (test functions) with support on the right, $\mathcal{D}_R$. The space of right-sided distributions is denoted by $\mathcal{D}'_R$. We write a number assigned to each test function $\phi \in \mathcal{D}_R$ from a right-sided distribution f by $\langle f, \phi \rangle$. A regular right-sided distribution f is a locally integrable function on $\mathbb{R}$ which has a support bounded on the left. For a regular distribution, the number $\langle f, \phi \rangle$ is defined by

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)\, dt.$$

A singular right-sided distribution is a right-sided distribution which is not regular.

The fractional integral operator of a test function $\phi \in \mathcal{D}_R$ is denoted by $D_w^{-\nu}\phi$ for $\nu \in (0, \infty)$ and is defined by

$$D_w^{-\nu}\phi(t) = \frac{1}{\Gamma(\nu)} \int_t^{\infty} (x-t)^{\nu-1}\phi(x)\, dx,$$

where $\Gamma(\nu)$ is a gamma function. Moreover, the operator defines the fractional derivative operator. The definition of fractional derivative operator is, let β be a positive real number and $n = \lceil \beta \rceil$, a ceiling function of β,

$$D_w^{\beta}\phi(t) = D_w^{\beta-n}\left[D_w^n \phi(t)\right],$$

where

$$D_w^n \phi(t) = (-1)^n \frac{d^n}{dt^n}\phi(t),$$

and especially for $\nu = 0$, $D_w^0\phi(t) = \phi(t)$.

The fractional integral and fractional derivative operator of a distribution f is denoted by $D^{\beta} f(t)$ for $\beta \in \mathbb{R}$. If β is a positive real number, $D^{\beta} f$ is the fractional derivative of $f(t)$ but if β is a negative real number, it is the fractional integral, and $D^0 f(t) = f(t)$. The definition of $D^{\beta} f(t)$ is defined as follow,

$$\langle D^{\beta} f(t), \phi(t) \rangle = \langle f(t), D_w^{\beta}\phi(t) \rangle,$$

where $\phi \in \mathcal{D}_R$.

One example of a regular right-sided distribution is a Heaviside function $H_a(t)$, which is defined by

$$H_a(t) = \begin{cases} 1, & \text{for} \quad t \geq a; \\ 0, & \text{for} \quad t < a, \end{cases}$$

where $a \in \mathbb{R}$ and in the sense of distribution

$$\langle H_a(t), \phi(t)\rangle = \int_a^\infty \phi(t)\, dt.$$

An example of a singular right-sided distribution is an important element Dirac delta function $\delta(t)$ which is defined by

$$\langle \delta, \phi\rangle = \phi(0).$$

A fractional integral and a fractional derivative of order α of Dirac delta function can be computed as follows

$$\langle D^\alpha \delta(t), \phi(t)\rangle = \langle \delta(t), D_w^\alpha \phi(t)\rangle.$$

In case of $\alpha < 0$, namely, $\alpha = -\nu$, where ν is a positive real number, we have

$$\begin{aligned}\langle \delta(t), D_w^\alpha \phi(t)\rangle &= \langle \delta(t), D_w^{-\nu} \phi(t)\rangle \\ &= \frac{1}{\Gamma(\nu)}\langle \delta(t), \int_t^\infty (x-t)^{\nu-1}\phi(x)\, dx\rangle \\ &= \left(\frac{1}{\Gamma(\nu)}\int_t^\infty (x-t)^{\nu-1}\phi(x)\, dx\right)\Big|_{t=0}.\end{aligned}$$

In case of $\alpha > 0$, we set $n = \lceil \alpha \rceil$ and obtain

$$\begin{aligned}\langle \delta(t), D_w^\alpha \phi(t)\rangle &= \langle \delta(t), D_w^{\alpha-n} D_w^n \phi(t)\rangle \\ &= \langle \delta(t), D_w^{\alpha-n} (-1)^n \phi^{(n)}(t)\rangle \\ &= (-1)^n \langle \delta(t), D_w^{-(n-\alpha)} \phi^{(n)}(t)\rangle \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\langle \delta(t), \int_t^\infty (x-t)^{n-\alpha-1}\phi^{(n)}(x)\, dx\rangle \\ &= \left(\frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^\infty (x-t)^{n-\alpha-1}\phi^{(n)}(x)\, dx\right)\Big|_{t=0}.\end{aligned}$$

Hereafter, in dealing with certain product forms, we use the Pochhammer symbol

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1, 2, 3, \ldots$$

where a is a real number.

We collect certain properties of right-sided distributions and properties of finite double sum needed in our proofs. Lemma 2.1 is proved in [11]. Lemma 2.2 is proved in [11] for the case $n = 1$. However, it can be extended naturally for $n > 1$ by induction.

Lemma 2.1. *For $u \in \mathcal{D}_R^{'}$, the index law*

$$D^\alpha D^\beta u = D^{\alpha+\beta} u,$$

is valid for every $\alpha, \beta \in \mathbb{R}$.

The proof of this lemma is given in [11].

Lemma 2.2. *Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, the formula of $t^n\delta^{(\alpha)}$ is as follow*

$$t^n\delta^{(\alpha)} = (-1)^n(\alpha - n + 1)_n\delta^{(\alpha-n)}. \tag{2.1}$$

The proof of this lemma is given in [11].

Lemma 2.3. *For $n, m \in \mathbb{N}$ with $m \geq 2$, and $r \in \mathbb{N} \cup \{0\}$ with $r \leq m-1$, then*

$$\sum_{i=0}^{nm+r} \sum_{j=0}^{\lfloor i/m \rfloor} a_{i,j} = \sum_{j=0}^{n} \sum_{i=jm}^{nm+r} a_{i,j}.$$

Proof. Let $n, m \in \mathbb{N}$ with $m \geq 2$, and $r \in \mathbb{N} \cup \{0\}$ with $r \leq m-1$. We expand and regroup the summation to get the result:

$$\begin{aligned}
\sum_{i=0}^{nm+r} \sum_{j=0}^{\lfloor i/m \rfloor} a_{i,j} &= \sum_{j=0}^{0} \sum_{i=0}^{m-1} a_{i,j} + \sum_{j=0}^{1} \sum_{i=m}^{2m-1} a_{i,j} + \sum_{j=0}^{2} \sum_{i=2m}^{3m-1} a_{i,j} \\
&\quad + \cdots + \sum_{j=0}^{n-1} \sum_{i=(n-1)m}^{nm-1} a_{i,j} + \sum_{j=0}^{n} \sum_{i=nm}^{nm+r} a_{i,j} \\
&= \sum_{i=0}^{nm+r} a_{i,0} + \sum_{i=m}^{nm+r} a_{i,1} + \sum_{i=2m}^{nm+r} a_{i,2} \\
&\quad + \cdots + \sum_{i=(n-1)m}^{nm+r} a_{i,n-1} + \sum_{i=nm}^{nm+r} a_{i,n} \\
&= \sum_{j=0}^{n} \sum_{i=jm}^{nm+r} a_{i,j}.
\end{aligned}$$

This completes the proof. □

Lemma 2.4. *For $m \in \mathbb{N}$ with $m \geq 2$, and $N_1 \in \mathbb{N} \cup \{0\}$, then*

$$\sum_{l=0}^{m-1} \sum_{j=0}^{N_1} a_{l+jm,j} = \sum_{i=0}^{mN_1+m-1} a_{i,\lfloor i/m \rfloor}.$$

Proof. Let $m \in \mathbb{N}$ with $m \geq 2$, and $N_1 \in \mathbb{N} \cup \{0\}$. We expand and regroup the summation to obtain the result:

$$\begin{aligned}
\sum_{l=0}^{m-1}\sum_{j=0}^{N_1} a_{l+jm,j} &= \sum_{j=0}^{N_1} a_{jm,j} + \sum_{j=0}^{N_1} a_{jm+1,j} + \sum_{j=0}^{N_1} a_{jm+2,j} + \cdots + \sum_{j=0}^{N_1} a_{jm+m-1,j} \\
&= \left(a_{0,0} + a_{m,1} + a_{2m,2} + \cdots + a_{N_1 m,N_1}\right) \\
&\quad + \left(a_{1,0} + a_{m+1,1} + a_{2m+1,2} + \cdots + a_{N_1 m+1,N_1}\right) \\
&\quad + \left(a_{2,0} + a_{m+2,1} + a_{2m+2,2} + \cdots + a_{N_1 m+2,N_1}\right) \\
&\quad + \cdots + \left(a_{m-1,0} + a_{2m-1,1} + a_{3m-1,2} + \cdots + a_{(N_1+1)m-1,N_1}\right) \\
&= \left(a_{0,0} + a_{1,0} + a_{2,0} + \cdots + a_{m-1,0}\right) \\
&\quad + \left(a_{m,1} + a_{m+1,1} + a_{m+2,1} + \cdots + a_{2m-1,1}\right) \\
&\quad + \left(a_{2m,2} + a_{2m+1,2} + a_{2m+2,2} + \cdots + a_{3m-1,2}\right) \\
&\quad + \cdots + \left(a_{N_1 m,N_1} + a_{N_1 m+1,N_1} + a_{N_1 m+2,N_1} + \cdots + a_{(N_1+1)m-1,N_1}\right) \\
&= \sum_{i=0}^{(N_1+1)m-1} a_{i,\lfloor i/m \rfloor}.
\end{aligned}$$

This completes the proof. □

We are now in a position to prove our assertions:

3 Main Results

Theorem 3.1. *Consider an equation*

$$\sum_{i=0}^{nm+r} a_i(t) u^{(i/m)}(t) = 0, \tag{3.1}$$

where $n, m \in \mathbb{N}$ with $m \geq 2$, $r \in \mathbb{N} \cup \{0\}$ with $r \leq m-1$ and each polynomial coefficient $a_i(t)$ is of particular form

$$a_i(t) = \sum_{j=0}^{\lfloor i/m \rfloor} a_{i,j} t^j.$$

If equation (3.1) *has a finite fractional-order distributional solution* (1.4)*, then*

(1) *coefficients $a_{nm+r-p,0} = 0$ for all $p = 0, 1, 2, \ldots, m-1$;*

(2) *the index N in equation* (1.4) *satisfies the relation*

$$a_{(n-1)m+r,0} + a_{nm+r,1}(-1)\left(\frac{N+nm+r}{m}\right) = 0;$$

(3) *there exists a non-trivial solution* $(b_0, b_1, b_2, \ldots, b_N)$ *of the system*

$$\sum_{l=0}^{r} b_{\bar{l}-l} \sum_{j=0}^{n} a_{l+jm,j}(-1)^j \left(\frac{\bar{l}}{m}+1\right)_j + \sum_{\nu=1}^{n} \sum_{l=(\nu-1)m+r+1}^{\nu m+r} b_{\bar{l}-l} \sum_{j=0}^{n-\nu} a_{l+jm,j}(-1)^j \left(\frac{\bar{l}}{m}+1\right)_j = 0, \quad (3.2)$$

for all $\bar{l} = 0,\ 1,\ 2,\ \ldots,\ N+nm+r$.

Proof. We substitute the finite fractional-order distributional solution into (3.1) and interpret the equation in the the sense of distribution as

$$\left\langle \sum_{i=0}^{nm+r} a_i(t) D^{(i/m)} \left(\sum_{k=0}^{N} b_k D^{(k/m)} \delta(t) \right), \phi(t) \right\rangle = 0,$$

where $\phi(t) \in \mathcal{D}_R$. Using Lemma 2.1, we have

$$\left\langle \sum_{i=0}^{nm+r} a_i(t) \sum_{k=0}^{N} b_k D^{(k+i)/m} \delta(t), \phi(t) \right\rangle = 0$$

To reduce the complexity of the notations, we write

$$\sum_{i=0}^{nm+r} a_i(t) \sum_{k=0}^{N} b_k \delta^{((k+i)/m)}(t) = 0.$$

Substituting polynomial a_i yields

$$\sum_{i=0}^{nm+r} \sum_{j=0}^{\lfloor i/m \rfloor} a_{i,j} t^j \sum_{k=0}^{N} b_k \delta^{((i+k)/m)}(t) = 0.$$

Applying Lemma 2.2, we obtain

$$\sum_{i=0}^{nm+r} \sum_{j=0}^{\lfloor i/m \rfloor} \sum_{k=0}^{N} a_{i,j} b_k (-1)^j \left(\frac{i+k}{m} - j + 1\right)_j \delta^{((i+k-jm)/m)} = 0.$$

Swapping double sums by Lemma 2.3, we have

$$\sum_{k=0}^{N} \sum_{j=0}^{n} \sum_{i=jm}^{nm+r} a_{i,j} b_k (-1)^j \left(\frac{i+k}{m} - j + 1\right)_j \delta^{((i+k-jm)/m)} = 0.$$

Next, we let $l = i - jm$ and rearrange the sum as

$$\sum_{k=0}^{N}\Bigg(\sum_{l=0}^{r}\sum_{j=0}^{n} a_{l+jm,j}b_k(-1)^j\left(\frac{l+k}{m}+1\right)_j\delta^{((l+k)/m)}$$
$$+\sum_{l=r+1}^{m+r}\sum_{j=0}^{n-1} a_{l+jm,j}b_k(-1)^j\left(\frac{l+k}{m}+1\right)_j\delta^{((l+k)/m)}$$
$$+\sum_{l=m+r+1}^{2m+r}\sum_{j=0}^{n-2} a_{l+jm,j}b_k(-1)^j\left(\frac{l+k}{m}+1\right)_j\delta^{((l+k)/m)}$$
$$+\cdots+\sum_{l=(n-1)m+r+1}^{nm+r}\sum_{j=0}^{0} a_{l+jm,j}b_k(-1)^j\left(\frac{l+k}{m}+1\right)_j\delta^{((l+k)/m)}\Bigg)=0.$$

We group all terms together except the first term:

$$\sum_{k=0}^{N}\Bigg(\sum_{l=0}^{r}\sum_{j=0}^{n} a_{l+jm,j}b_k(-1)^j\left(\frac{l+k}{m}+1\right)_j\delta^{((l+k)/m)}$$
$$+\sum_{\nu=1}^{n}\sum_{l=(\nu-1)m+r+1}^{\nu m+r}\sum_{j=0}^{n-\nu} a_{l+jm,j}b_k(-1)^j\left(\frac{l+k}{m}+1\right)_j\delta^{((l+k)/m)}\Bigg)=0.$$

Then, we let $\bar{l} = l + k$ and introduce all extra-terms of b_k to be zero, i.e.,

$$b_{-1} = b_{-2} = b_{-3} = \cdots = b_{-(N+nm+r)} = 0$$

and

$$b_{N+1} = b_{N+2} = b_{N+3} = \cdots = b_{N+nm+r} = 0.$$

We thus obtain

$$\sum_{\bar{l}=0}^{N+nm+r}\delta^{(\bar{l}/m)}\Bigg(\sum_{l=0}^{r} b_{\bar{l}-l}\sum_{j=0}^{n} a_{l+jm,j}(-1)^j\left(\frac{\bar{l}}{m}+1\right)_j$$
$$+\sum_{\nu=1}^{n}\sum_{l=(\nu-1)m+r+1}^{\nu m+r} b_{\bar{l}-l}\sum_{j=0}^{n-\nu} a_{l+jm,j}(-1)^j\left(\frac{\bar{l}}{m}+1\right)_j\Bigg)=0.$$

For each $\bar{l} = 0, 1, \ldots, N + nm + r$, we thus have

$$\sum_{l=0}^{r} b_{\bar{l}-l} \sum_{j=0}^{n} a_{l+jm,j} (-1)^j \left(\frac{\bar{l}}{m} + 1\right)_j + \sum_{\nu=1}^{n} \sum_{l=(\nu-1)m+r+1}^{\nu m+r} b_{\bar{l}-l} \sum_{j=0}^{n-\nu} a_{l+jm,j} (-1)^j \left(\frac{\bar{l}}{m} + 1\right)_j = 0,$$

which satisfies condition (3.1). Its last term is $a_{nm+r,0} b_N = 0$. Since $b_N \neq 0$, we have $a_{nm+r,0} = 0$. Notice that $a_{nm+r-p,0} = 0$ for all $p = 1, 2, 3, \ldots, m-1$ which satisfies condition (1). Moreover, N satisfies the follow equation

$$a_{nm+r-m,0} + a_{nm+r,1}(-1)\left(\frac{N + nm + r}{m}\right) = 0,$$

which is the relation (3.1). □

We now give a sufficient condition to obtain the solution of (3.1):

Theorem 3.2. *Equation* (3.1) *has an* N/m *order distributional solution, if the following hypotheses are satisfied:*

(i) *there exists a positive integer* q *with* $m - 1 \leq q < nm$, *such that*

$$a_{nm+r-(pm+s),k-p} = 0$$

for all $k = 0, 1, 2, \ldots, \lfloor q/m \rfloor$, *for all* $s = 0, 1, 2, \ldots, \min\{m-1, q-km\}$, *and for all* $p = 0, 1, 2, \ldots, k$;

(ii) *the index* N *is the smallest positive integer root of the equation*

$$\sum_{j=0}^{\lfloor (q+1)/m \rfloor} a_{nm+r-(q+1-jm),j} (-1)^j \left(\frac{N + nm + r - (q+1)}{m} + 1\right)_j = 0;$$

(iii) *there exists a nonzero solution of system* (3.2).

Proof. By (iii), for any nontrivial solution $(b_0, b_1, \ldots, b_N)$ of system (3.2), we have a finite fractional-order distributional solution (1.4) that satisfies equation (3.1). Assuming conditions (i) and choosing the largest integer q in condition (i), we can reduce system (3.2) to

$$\sum_{\bar{l}=0}^{N+nm+r-(q+1)} \left(\sum_{l=0}^{r} b_{\bar{l}-l} \sum_{j=0}^{n} a_{l+jm,j} (-1)^j \left(\frac{\bar{l}}{m} + 1\right)_j + \sum_{\nu=1}^{n-\lfloor (q+1)/m \rfloor} \sum_{l=(\nu-1)m+r+1}^{\nu m+r} b_{\bar{l}-l} \sum_{j=0}^{n-\nu} a_{l+jm,j} (-1)^j \left(\frac{\bar{l}}{m} + 1\right)_j \right) = 0.$$

Observe that the last equation of the above system is

$$\left(\sum_{j=0}^{\lfloor (q+1)/m \rfloor} a_{nm+r-(q+1-jm),j}(-1)^j \left(\frac{N+nm+r-(q+1)}{m}+1\right)_j\right) b_N = 0.$$

Due to condition (ii), we can set $b_N \neq 0$. Other b_k can be successively written in term of b_N since

$$\sum_{j=0}^{\lfloor (q+1)/m \rfloor} a_{nm+r-(q+1-jm),j}(-1)^j \left(\frac{\bar{l}}{m}+1\right)_j \neq 0,$$

for all $\bar{l} = 0,\, 1,\, \ldots,\, N+nm+r-(q+2)$. □

If each polynomial coefficient $a_i(t)$ is reduced to a monomial $a_{i,\lfloor i/m \rfloor} t^{\lfloor i/m \rfloor}$, then the result in Theorem 3.1 can be reduced as shown in the next theorem. For simplicity, we write $c_i = a_{i,\lfloor i/m \rfloor}$:

Theorem 3.3. *If the equation*

$$\sum_{i=0}^{nm+r} c_i t^{\lfloor i/m \rfloor} u^{(i/m)} = 0, \tag{3.3}$$

where $n, m \in \mathbb{N}$ with $m \geq 2$, $r \in \mathbb{N} \cup \{0\}$ with $r < m$, has a finite fractional-order distributional solution

$$u = \sum_{k=0}^{N} b_k \delta^{(k/m)}(t), \quad b_N \neq 0,$$

then

(i) *the index N satisfies the equation*

$$\sum_{i=1}^{n+1} c_{im-1}(-1)^{\lfloor (im-1)/m \rfloor} \left(\frac{N+m-1}{m}+1\right)_{\lfloor (im-1)/m \rfloor} = 0;$$

(ii) *there exists a nontrivial solution $(b_0,\, b_1,\, b_2,\, \ldots,\, b_N)$ of the system*

$$\sum_{i=0}^{(n+1)m-1} c_i b_{\bar{l}-i+m\lfloor i/m \rfloor}(-1)^{\lfloor i/m \rfloor} \left(\frac{\bar{l}}{m}+1\right)_{\lfloor i/m \rfloor} = 0 \tag{3.4}$$

for all $\bar{l} = 0,\, 1,\, 2,\, \ldots,\, N+m-1$.

Conversely, if the following hypotheses are satisfied: Either there exists the smallest index N which satisfies condition (i) where c_{im-1} are not all zero for all $i = 1, 2, 3, \ldots, n+1$ and there exists a non-trivial solution of system (3.4) *or*

(I) *there exists a positive integer q, $1 \leq q < m - r$ such that*

$$c_{pm-s} = 0,$$

for all $p = 1, 2, \ldots, n+1$, and $s = 1, 2, 3, \ldots, q$;

(II) *the index N is the smallest positive integer root of the equation*

$$\sum_{i=1}^{n+1} c_{im-(q+1)} (-1)^{\lfloor (im-(q+1))/m \rfloor} \left(\frac{N+m-(q+1)}{m} + 1 \right)_{\lfloor (im-(q+1))/m \rfloor} = 0;$$

(III) *there exists a nonzero solution of system* (3.4)*,*

then (3.3) *has an N/m order of distributional solution.*

Proof. Comparing to the polynomial coefficients of the equation in Theorem 3.1, we have

$$a_{i,j} = 0, \text{ for all } j \neq \lfloor i/m \rfloor \text{ and } a_{i,\lfloor i/m \rfloor} = c_i.$$

We only consider all value $j = \lfloor i/m \rfloor$ in system (3.2) of Theorem 3.1. For any $l \geq m$, we have$\lfloor (l+jm)/m \rfloor \neq j$ and only zero terms appear. Thus, we consider only the value $l = 0, 1, 2, \ldots, m-1$. Now system (3.2) is reduced to

$$\sum_{l=0}^{r} b_{\bar{l}-l} \sum_{j=0}^{n} a_{l+jm,j} (-1)^j \left(\frac{\bar{l}}{m} + 1 \right)_j + \sum_{l=r+1}^{m-1} b_{\bar{l}-l} \sum_{j=0}^{n-1} a_{l+jm,j} (-1)^j \left(\frac{\bar{l}}{m} + 1 \right)_j = 0,$$

for all $\bar{l} = 0, 1, 2, \ldots, N + nm + r$. To combine the two double summations, we introduce

$$a_{l+nm,n} = 0, \quad \text{for all} \quad l = r+1, r+2, \ldots, m-1$$

into the previous equation. Combining the two double summations, we have

$$\sum_{l=0}^{m-1} \sum_{j=0}^{n} b_{\bar{l}-l} a_{l+jm,j} (-1)^j \left(\frac{\bar{l}}{m} + 1 \right)_j = 0,$$

for all $\bar{l} = 0, 1, 2, \ldots, N + nm + r$.

Writing new index i, $i = l + jm$ and using Lemma 2.4, we have

$$\sum_{i=0}^{(n+1)m-1} b_{\bar{l}-i+m\lfloor i/m \rfloor} a_{i,\lfloor i/m \rfloor} (-1)^{\lfloor i/m \rfloor} \left(\frac{\bar{l}}{m} + 1 \right)_{\lfloor i/m \rfloor} = 0.$$

Since $a_{i,\lfloor i/m\rfloor} = c_i$, the equation becomes

$$\sum_{i=0}^{(n+1)m-1} b_{\bar{l}-i+m\lfloor i/m\rfloor} c_i (-1)^{\lfloor i/m\rfloor} \left(\frac{\bar{l}}{m}+1\right)_{\lfloor i/m\rfloor} = 0,$$

for all $\bar{l} = 0, 1, 2, \ldots, N+nm+r$. We observe that the equation appeared when $\bar{l} = N+m, N+m+1, N+m+2, \ldots, N+nm+r$, consists of b_k, $k > N$ which are zeros. Therefore, the running index $\bar{l} = 0, 1, 2, \ldots, N+m-1$ is required. Substituting the maximum value of $\bar{l} = N+m-1$ leads to

$$\sum_{i=0}^{(n+1)m-1} c_i b_{N+m-1-i+m\lfloor i/m\rfloor} (-1)^{\lfloor i/m\rfloor} \left(\frac{N+m-1}{m}+1\right)_{\lfloor i/m\rfloor} = 0.$$

The only nonzero term left in the equation is

$$b_N \left\{\sum_{i=1}^{n+1} c_{im-1} (-1)^{\lfloor (im-1)/m\rfloor} \left(\frac{N+2m-1}{m}\right)_{\lfloor (im-1)/m\rfloor}\right\} = 0.$$

Since $b_N \neq 0$, number N satisfies the equation

$$\sum_{i=1}^{n+1} c_{im-1} (-1)^{\lfloor (im-1)/m\rfloor} \left(\frac{N+2m-1}{m}\right)_{\lfloor (im-1)/m\rfloor} = 0.$$

Then we show the proof of the sufficient condition for existence of solution in (3.3).

If there exists the smallest index N which satisfies condition (i) where c_{im-1} are not all zero for all $i = 1, 2, 3, \ldots, n+1$ and there exists a nonzero solution of system (3.4), obviously, we will obtain an N/m order of distributional solution of (3.3). On the other hand, if we use condition (I), then we write the system (3.4) as

$$\sum_{i=0}^{(n+1)m-1} c_i b_{\bar{l}-i+m\lfloor i/m\rfloor} (-1)^{\lfloor i/m\rfloor} \left(\frac{\bar{l}}{m}+1\right)_{\lfloor i/m\rfloor} = 0,$$

for all $\bar{l} = 0, 1, 2, \ldots, N+m-(q+1)$. The last equation in the system is

$$b_N \left(\sum_{i=1}^{n+1} c_{im-(q+1)} (-1)^{\lfloor (im-(q+1))/m\rfloor} \left(\frac{N+m-(q+1)}{m}+1\right)_{\lfloor (im-(q+1))/m\rfloor}\right) = 0.$$

By virtue of (II), we set $b_N \neq 0$. Other b_k can be expressed in term of b_N since

$$\sum_{i=1}^{n+1} c_{im-(q+1)} (-1)^{\lfloor (im-(q+1))/m\rfloor} \left(\frac{\bar{l}}{m}+1\right)_{\lfloor (im-(q+1))/m\rfloor} \neq 0,$$

for all $\bar{l} = 0, 1, 2, \ldots, N+m-(q+2)$. This completes the proof. □

4 Examples

Example 4.1. Consider a fractional differential equation with monomial coefficients

$$
\begin{aligned}
&3m^2t^2u^{(2+6/m)} + 9m(m+7)tu^{(1+6/m)} + 3(m+10)(m+11)u^{(6/m)} \\
&- 3m^2t^2u^{(2+4/m)} - 9m(m+5)tu^{(1+4/m)} - 3(m^2+15m+58)u^{(4/m)} \\
&+ 2m^2t^2u^{(2+2/m)} + 6m(m+3)tu^{(1+2/m)} + 2(m+4)(m+5)u^{(2/m)} = 0,
\end{aligned}
\tag{4.1}
$$

where $m \geq 7$ is an integer. Suppose that $u = \sum_{k=0}^{N} b_k\delta^{(k/m)}$, $b_N \neq 0$, is a solution of the equation. Substituting the solution into the equation, using Lemma 2.1 and 2.2, and combining the like term of fractional-order of Dirac delta function and its derivatives, we find that

$$
\sum_{k=6}^{N+6} 3(k^2-21k+110)b_{k-6}\delta^{(k/m)} + \sum_{k=4}^{N+4} (-3)(k^2-15k+58)b_{k-4}\delta^{(k/m)} + \sum_{k=2}^{N+2} 2(k^2-9k+20)b_{k-2}\delta^{(k/m)} = 0.
$$

Since $b_N \neq 0$, we have $(N+6)^2 - 21(N+6) + 110 = 0$. Therefore, $N = 4$ or $N = 5$. Choosing $N = 4$ and replacing it in the recurrence relation, we have

$$
\begin{aligned}
&6b_3\delta^{(9/m)} + (18b_2 - 6b_4)\delta^{(8/m)} + (36b_1 - 6b_3)\delta^{(7/m)} - 42b_0\delta^{(4/m)} \\
&(-24)b_1\delta^{(5/m)} + 12b_0\delta^{(2/m)} + 4b_1\delta^{(3/m)} + (-12b_2 + 4b_4)\delta^{(6/m)} = 0.
\end{aligned}
$$

Therefore, $b_0 = b_1 = b_3 = 0$ and $b_2 = b_4/3$. The solution is

$$
u = \frac{b_4}{3}\left(2\delta^{(4/m)} + \delta^{(2/m)}\right). \tag{4.2}
$$

By applying (2.1), it is easy to verify that (4.2) satisfies (4.1).

For $N = 5$, we obtain

$$
\begin{aligned}
(6b_3 - 12b_5)\delta^{(9/m)} + (18b_2 - 6b_4)\delta^{(8/m)} - 42b_0\delta^{(4/m)} - 24b_1\delta^{(5/m)} + 12b_0\delta^{(2/m)} \\
+4b_1\delta^{(3/m)} + (-12b_2 + 4b_4)\delta^{(6/m)} + (-6b_3 + 12b_5)\delta^{(7/m)} = 0.
\end{aligned}
$$

Therefore, $b_0 = b_1 = 0$, $b_2 = b_4/3$, $b_3 = 2b_5$. If we set $b_4 = 0$, then we obtain another solution

$$
u = b_5\left(\delta^{(5/m)} + 2\delta^{(3/m)}\right). \tag{4.3}
$$

By applying (2.1), it is easy to verify that (4.3) satisfies (4.1).

To be specific, let consider for $m = 7$. We find that the index $N = 4$ clearly satisfies Theorem 3.3 (i). In Theorem 3.3 (ii), we have

$$\sum_{i=0}^{20} c_i b_{\bar{l}-i+7\lfloor i/7\rfloor}(-1)^{\lfloor i/7\rfloor}\left(\frac{\bar{l}}{7}+1\right)_{\lfloor i/7\rfloor} = 0,$$

for all $\bar{l} = 0, 1, 2, \ldots, 10$ which are just

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = b_4/3, \quad b_3 = 2b_5, \tag{4.4}$$

and b_4, b_5 are arbitrary constants.

If we consider Theorem 3.3 (I), then we observe that there exists the smallest index $N = 4$ which satisfies condition (i) and there exists a nonzero solution of system (3.4) which is the same as the solution of system (4.4). Therefore, we obtain a 4/7 order of distributional solution of this fractional differential equation.

In case of $m = 2$, the differential equation is

$$12t^2u^{(5)} + (-12t^2 + 162t)u^{(4)} + (8t^2 - 126t + 468)u^{(3)} + (60t - 276)u^{(2)} + 84u^{(1)} = 0.$$

By substitution of finite fractional-order form of the solution, we have

$$(N-4)(N-5) = 0.$$

If we choose $N = 4$ and by the recurrence relation, we get a solution

$$u = \frac{b_4}{3}\left(2\delta^{(2)} + \delta^{(1)}\right). \tag{4.5}$$

If we choose $N = 5$, we get a solution

$$u = b_5\left(\delta^{(5/2)} + 2\delta^{(3/2)}\right). \tag{4.6}$$

By applying (2.1), it is easy to verify that (4.5) and (4.6) satisfy (4.1).

Example 4.2. Consider a fractional differential equation with monomial coefficients

$$t^2u^{(2)} + 2tu^{(3/2)} + 3tu^{(1)} + 5u^{(1/2)} + u = 0. \tag{4.7}$$

Suppose that $u = \sum_{k=0}^{N} b_k\delta^{(k/2)}$, $b_N \neq 0$, is a solution of the equation. Substituting the solution into the equation and using Lemma 2.1 and 2.2, we get

$$(2-N)b_N\delta^{((N+1)/2)} + \sum_{k=1}^{N}((3-k)b_{k-1} + (k^2/4)b_k)\delta^{(k/2)} = 0.$$

Since $b_N \neq 0$, we have $N = 2$. Putting $N = 2$ into the equation leads to

$$\sum_{k=1}^{2} \left[(3-k)b_{k-1} + (k^2/4)b_k\right] \delta^{(k/2)} = 0.$$

That is, the recurrence relation is equal to zero, for all $k = 1, 2$,

$$b_k = \frac{4(k-3)}{k^2} b_{k-1}.$$

That is, $b_1 = -8b_0$ and $b_2 = 8b_0$. Therefore, we obtain the solution

$$u = b_2 \left(\delta^{(1)} - \delta^{(1/2)} + \frac{1}{8}\delta\right). \tag{4.8}$$

By applying (2.1), it is easy to verify that (4.8) satisfies (4.7).

We find that index $N = 2$ clearly satisfies Theorem 3.3 (i). Now we consider Theorem 3.3 (ii) and obtain the system

$$\sum_{i=0}^{5} c_i b_{\bar{l}-i+2\lfloor i/2 \rfloor} (-1)^{\lfloor i/2 \rfloor} \left(\frac{\bar{l}}{2} + 1\right)_{\lfloor i/2 \rfloor} = 0,$$

for all $\bar{l} = 0, 1, 2, 3$. That is

$$\begin{aligned} 9/4b_3 &= 0, \\ b_2 + b_1 &= 0, \\ (1/4)b_1 + 2b_0 &= 0, \\ b_0 - 3b_0 + 2b_0 &= 0. \end{aligned} \tag{4.9}$$

Therefore, we obtain a nontrivial solution of the system which is

$$b_0 = -1/8b_2, \quad b_1 = -b_2, \quad b_3 = 0.$$

where b_2 is an arbitrary constant.

If we consider Theorem 3.3 (I), then we observe that there exists the smallest index $N = 2$ which satisfies condition (i) and there exists a nonzero solution of system (3.4) which is just the solution of system (4.9). Therefore, we obtain a 2/2 order of distributional solution of this fractional differential equation.

Example 4.3. Consider a Cauchy-Euler equation

$$m^2t^2u^{(2)} + (3m^2 + m - am)tu^{(1)} + (m^2 + m - am - n(n + a - 1))u = 0. \tag{4.10}$$

where $m, n \in \mathbb{N}$ with $m \geq 2$, and $a \in \mathbb{R} \setminus (\mathbb{Z}^- \cup \{0\})$.

Suppose $u = b_N\delta^{(N/l)}$, $b_N \neq 0$, is a solution of the equation. Substituting the solution into the equation and using Lemma 2.1 and 2.2, we get

$$\left(\frac{mN}{l} - n\right)\left(\frac{mN}{l} + (n+a-1)\right) b_N\delta^{(N/l)} = 0,$$

Therefore, we obtain $N = nl/m$ or $N = -(n+a-1)l/m$. Since N is supposed to be a non-negative integer, $m|nl$ is needed. For example, we pick $l = 3$, $m = 3$, $n = 2$, and $a = -1/2$ in (4.10), we get an equation

$$18t^2u^{(2)} + 69tu^{(1)} + 35u = 0. \tag{4.11}$$

and upon substitution, we have

$$(N-2)(N-1/2)b_N\delta^{(N/3)} = 0.$$

Therefore, $N = 1/2$ and 2. Thus, we obtain the solution of the differential equation

$$u = b_2\delta^{(2/3)}. \tag{4.12}$$

Note that $u = c\delta^{(1/6)}$ is also a solution of (4.11) for any constant c.

We find that the index $N = 2$ clearly satisfies the equation in Theorem 3.3 (i). Now we consider Theorem 3.3 (ii) and obtain the system

$$\sum_{i=0}^{6} c_i b_{\bar{l}-i+3\lfloor i/3\rfloor}(-1)^{\lfloor i/3\rfloor}\left(\frac{\bar{l}}{3}+1\right)_{\lfloor i/3\rfloor} = 0,$$

for all $\bar{l} = 0$, 1, 2, and 3. Here the system is reduced to 3 equations,

$$5b_3 = 0, \quad -b_1 = 0, \quad 2b_0 = 0, \tag{4.13}$$

Therefore, we obtain a nontrivial solution of the system, which is

$$b_0 = b_1 = b_3 = 0$$

and b_2 which is an arbitrary constant. Next, we consider Theorem 3.3 (I), we observe that $q = 2$ and then we obtain the equation,

$$\sum_{i=1}^{3} c_{3i-3}(-1)^{\lfloor (3i-3)/3\rfloor}\left(\frac{N+3}{3}\right)_{\lfloor (3i-3)/3\rfloor} = 0.$$

Hence, we get $N = 2$. After that we substitute $\bar{l} = 0$, 1, and 2 into the system (6) and find that there exists a solution of the system (6) which is just the solution of system (4.13). Therefore, we obtain a 2/3 order of distributional solution of this fractional differential equation.

Example 4.4. Consider a fractional differential equation with monomial coefficients

$$4t^2u^{(5/2)} + 18tu^{(3/2)} + (12+\epsilon)u^{(1/2)} = 0, \tag{4.14}$$

where ϵ is a nonpositive integer.

Suppose $u(t) = \sum_{k=0}^{N} b_k\delta^{(k/2)}$, $b_N \neq 0$, is a solution of the equation. Substituting the solution into the equation and using Lemma 2.1 and 2.2, we get

$$\sum_{k=0}^{N}(k^2 - k + \epsilon)b_k\delta^{((k+1)/2)} = 0.$$

Choosing $\epsilon = -6$, we have

$$\sum_{k=0}^{N}(k+2)(k-3)b_k\delta^{((k+1)/2)} = 0.$$

Since $b_N \neq 0$, we have $(N+2)(N-3) = 0$ and thus $N = 3$. Therefore, we obtain the solution of (4.15) as

$$u = b_3\delta^{(3/2)}. \tag{4.15}$$

By applying (2.1), it is easy to verify that (4.15) satisfies (4.14).

We find that the index $N = 3$ clearly satisfies the equation in Theorem 3.3 (i). Now we consider Theorem 3.3 (ii) and obtain the system

$$\sum_{i=0}^{5} c_i b_{\bar{l}-i+2\lfloor i/2\rfloor}(-1)^{\lfloor i/2\rfloor}\left(\frac{\bar{l}}{2}+1\right)_{\lfloor i/2\rfloor} = 0,$$

for all $\bar{l} = 0$, 1, 2, 3, and 4. We reduce the system into 3 equations and then we obtain a nontrivial solution of the system which is

$$-4b_2 = 0, \quad 21b_1 = 0, \quad -6b_0 = 0, \tag{4.16}$$

and b_3 which is an arbitrary constant.

If we consider Theorem 3.3 (I), then we observe that there exists the smallest index $N = 3$ which satisfies condition (i) and there exists a nonzero solution of system (3.4) which is just the solution of system (4.16). Therefore, we obtain a 3/2 order of distributional solution of this fractional differential equation.

5 Conclusions

We prove the necessary conditions for the existence of finite fractional-order solutions of the fractional differential equation (3.1) in Theorem 3.1 and the sufficient conditions for that in Theorem 3.2. We then apply both theorems for the case of monomial coefficients (3.3) and provide a simplified result in Theorem 3.3. Four examples are then provided to support our theorems.

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