

## NORMALITY CONDITIONS AND COMMUTATIVITY THEOREMS FOR RINGS

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### Abstract

Let  $R$  be a ring with center  $C$ , and let  $N$  the set of nilpotent elements. Suppose that for each  $x, y \in R \setminus N$ ,  $x^n y - xy^n \in N \cap C$ , where  $n > 1$  is a fixed integer. We shall present conditions for  $R$  to be commutative, non-commutative, normal and periodic.

Throughout,  $R$  will represent a ring with center  $C$ . Let  $N, E$  be the set of nilpotent elements of  $R$  and the set of idempotents of  $R$ , respectively; let  $N^*$  be the subset of  $N$  consisting of all elements  $x$  such that  $x^2 = 0$ . The ring  $R$  is called *normal* if  $E \subseteq C$ . For  $x, y$  in  $R$ , let  $[x, y]_1 = [x, y] = xy - yx$ , and define, recursively  $[x, y]_k = [[x, y]_{k-1}, y]$  for all integers  $k > 1$ .

Before stating and proving the main theorems of this paper, we first establish the following basic lemma.

**Lemma 1.** *Let  $n > 1$  be a fixed integer. Then*

$$\sum_{i=0}^n (-1)^i \binom{n}{i} ((n-i)^n - (n-i)) = n!.$$

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**Proof** We start with a polynomial  $f(X)$  in  $Z[X]$ , and define recursively:

$$\begin{aligned}\Delta^1 f(X) &= f(X+1) - f(X), \\ \Delta^k f(X) &= \Delta^1(\Delta^{k-1} f(X)).\end{aligned}$$

Then we can easily see that  $\Delta^k f(X) = \sum_{i=0}^k (-1)^i \binom{k}{i} (f(X+(k-i)))$ . In particular,

$$\begin{aligned}\Delta^n(X^n) &= \sum_{i=0}^n (-1)^i \binom{n}{i} (X+(n-i))^n, \\ \Delta^n(X) &= \sum_{i=0}^n (-1)^i \binom{n}{i} (X+(n-i)).\end{aligned}$$

Combining these with [8, Lemma 1], we obtain

$$\begin{aligned}\sum_{i=0}^n (-1)^i \binom{n}{i} (X+(n-i))^n &= n!, \\ \sum_{i=0}^n (-1)^i \binom{n}{i} (X+(n-i)) &= 0.\end{aligned}$$

So putting  $X = 0$  in the above, we obtain

$$\begin{aligned}\sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^n &= n!, \\ \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i) &= 0.\end{aligned}$$

Hence  $\sum_{i=0}^n (-1)^i \binom{n}{i} ((n-i)^n - (n-i)) = n!$ . □

We now proceed to prove the main theorems.

**Theorem 1.** *A ring  $R$  is normal if and only if there exists an integer  $n > 1$  for which  $R$  satisfies the following conditions:*

- (i) *For each  $x \in R \setminus N$  and  $e \in E$ ,  $[x^n - x, e] \in C$ .*
- (ii) *For each  $a \in N^*$  and  $e \in E$ ,  $n![a, e] = 0$  implies  $[a, e]_k = 0$  with some positive integer  $k$ .*

**Proof** It suffices to prove the if part only. Let  $e \in E$ , and  $x \in R$ . Obviously,  $a = ex - exe \in N^*$  and  $f = e + a \in E$ . Further, noting that  $[a, e] = -a$ , we see that  $[a, e]_k = (-1)^{k-1} [a, e]$ . Now, we shall prove that  $a = 0$ . First, suppose that there exists an integer  $i$  with  $2 \leq i \leq n$  such that  $if \in N$ , namely  $i^m f = (if)^m = 0$  for some positive integer  $m$ . Then  $i^m [a, e] = i^m [f, e] = [i^m f, e] = 0$ ,

and so  $(n!)^m[a, e] = 0$ . Hence, by (ii),  $(-1)^{k-1}(n!)^{m-1}[a, e] = (n!)^{m-1}[a, e]_k = 0$  for some positive integer  $k$ , namely  $(n!)^{m-1}[a, e] = 0$ . Therefore, we obtain eventually  $-a = [a, e] = 0$ , namely  $a = 0$ . On the other hand, if  $if \notin N$  for all  $i$  with  $2 \leq i \leq n$ , then by (i)

$$(i^n - i)[a, e] = (i^n - i)[f, e] = [(if)^n - if, e] = 0 \quad (0 \leq i \leq n).$$

Hence, by Lemma 1, we obtain  $n![a, e] = 0$ . Then, by (ii),  $(-1)^k a = [a, e]_k = 0$  for some positive integer  $k$ . We have thus seen that  $ex = exe$ . Similarly,  $xe = exe$ , and therefore  $ex = xe$ .  $\square$

**Corollary 1.** *Suppose that there exists an integer  $n > 1$  for which  $R$  satisfies the following conditions:*

- (i)' For each  $x, y \in R \setminus N$ ,  $[x^n, y] - [x, y^n] \in C$ .
- (ii) For each  $a \in N^*$  and  $e \in E$ ,  $n![a, e] = 0$  implies  $[a, e]_k = 0$  with some positive integer  $k$ .

Then  $R$  is a normal ring.

**Proof** If  $x \in R \setminus N$  and  $e \in E$ , then  $[x^n - x, e] = [x^n, e] - [x, e^n] \in C$ . Hence  $R$  is normal by Theorem 1.  $\square$

Another corollary to Theorem 1 involves periodic rings. A ring  $R$  is called *periodic* if for each  $x$  in  $R$ , there exist distinct positive integers  $n, m$  for which  $x^n = x^m$ . If  $0 < n < m$  then  $x^{n(m-n)} \in E$ . By [3, Proposition 2],  $R$  is periodic if and only if for each  $x$  in  $R$ , there exists  $f(X) \in X^2Z[X]$  such that  $x - f(x) \in N$ . We are now in a position to prove the following:

**Corollary 2.** *Suppose that there exists a fixed integer  $n > 1$ , and  $R$  is a ring which satisfies the following conditions:*

- (i)'' For each  $x, y \in R \setminus N$ ,  $x^n y - xy^n \in N \cap C$ .
- (ii) For each  $a \in N^*$  and  $e \in E$ ,  $n![a, e] = 0$  implies  $[a, e]_k = 0$  with some positive integer  $k$ .

Then  $R$  is a normal periodic ring.

**Proof** In fact, if  $x \in R \setminus N$  and  $e \in E$ , then  $[x^n - x, e] = (x^n e - x e^n) - (e x^n - e^n x) \in C$  and  $x^{n+1}(x - x^n) = x^n x^2 - x x^{2n} \in N$ . Since  $(x - x^n)^{n+3} = (x - x^n)(1 - x^{n-1})^{n+1} x^{n+1}(x - x^n)$ , it follows that  $x - x^n \in N$ . Hence,  $R$  is normal and periodic by Theorem 1 and [3, Proposition 2].  $\square$

For the conditions (i), (i)' and (i)'', we have the implications (i)''  $\Rightarrow$  (i)'  $\Rightarrow$  (i). Hence the condition (ii)'' is most strong.

Another theorem which follows at once from Theorem 1 is the following:

**Theorem 2.** *Suppose that there exists an integer  $n > 1$  for which  $R$  satisfies the following conditions:*

- (i) *For each  $x \in R \setminus N$  and  $e \in E$ ,  $[x^n - x, e] \in C$ .*
- (ii) *For each  $a \in N^*$  and  $e \in E$ ,  $n![a, e] = 0$  implies  $[a, e]_k = 0$  with some positive integer  $k$ .*

*If  $R$  is generated, as a ring, by  $E$ , then  $R$  is commutative, and isomorphic to a subdirect sum of rings isomorphic to  $Z/(p^k)$  for some prime  $p$  and some positive integer  $k$ .*

This follows by writing  $R$  as a subdirect sum of subdirectly irreducible rings, and by recalling that  $Z$  is isomorphic to a subdirect sum of prime fields  $Z/(p)'$ s.

Our next result gives a sufficient condition for a ring  $R$  to be commutative and periodic. This result makes an essential use of Corollary 2.

**Theorem 3.** *Suppose that  $n > 1$  is a fixed integer,  $R$  is a ring which satisfies the following conditions:*

- (i)'' *For each  $x, y \in R \setminus N$ ,  $x^n y - xy^n \in N \cap C$ .*
- (ii)' *For each  $a \in N$  and  $x \in R$ ,  $n![a, x] = 0$  implies  $[a, x]_k = 0$  with some positive integer  $k$ .*
- (iii) *For each  $a, b \in N$ , there exists an integer  $m = m(a, b) > 1$  such that  $[a, b] = [a, b]^m$ .*

*Then  $R$  is a commutative periodic ring.*

**Proof** By Corollary 2,  $R$  is a normal periodic ring. Then, for each  $x \in R$ , there exists a positive integer  $r$  such that  $x^r \in E \subseteq C$ . Hence, by [5, Theorem 4], the commutator ideal of  $R$  is nil, and so  $N$  forms an ideal of  $R$ . Further, in view of (iii), [6, Theorem 6] shows that  $N$  is commutative.

**Claim 1.** *If  $R$  contains 1, then it is commutative.*

**Proof** Let  $a \in N$ . Then both  $1 + a$  and  $1$  are in  $R \setminus N$ . Then, by (i)'',  $(1 + a)^n \cdot 1 - (1 + a) \cdot 1^n \in C$ . As was noted above,  $N$  is a commutative ideal, and so  $N^2 \subseteq C$ . Hence  $1 + na - (1 + a) \in C$ , namely  $(n - 1)[a, x] = 0$  for all  $x \in R$ . Then  $n![a, x] = 0$ , so that  $[a, x]_k = 0$  with some positive integer  $k$ . Now, the commutativity of  $R$  is clear by [2, Theorem].  $\square$

We now proceed to the general case ( $1 \notin R$ ). Let  $\sigma : R \rightarrow R'$  be a homomorphism of  $R$  onto a subdirectly irreducible ring  $R'$ . To complete the proof of Theorem 3, it suffices to show that  $R'$  is commutative. By [1, (c)],  $\sigma(N)$  coincides with the set  $N'$  of nilpotents in  $R'$ . Further, by [8, Lemma 1],

$R'$  is a normal periodic ring. Since  $R'$  is subdirectly irreducible, 1 and 0 are the only idempotents in  $R'$ . If  $1 \notin R'$ , then  $R' = N'$  is commutative. In what follows, we may restrict our attention to the case that  $R'$  contains 1. Then, as is easily seen, there exists a (central) idempotent  $e$  in  $R$  such that  $\sigma(e) = 1$ . Obviously,  $e$  is the unity of  $eR$  and  $eR$  satisfies all the conditions (i)'', (ii)' and (iii) in Theorem 3. Hence  $eR$  is commutative, by Claim 1; and so  $R' = \sigma(eR)$  is commutative. This completes the proof of Theorem 3.  $\square$

Related work also appears in [4].

Next, we shall present a classification theorem of rings which satisfies the conditions (i)'' and (ii) in Corollary 2.

**Theorem 4.** *For a ring  $R$  and an integer  $n > 1$ , the following conditions (1) and (2) are equivalent.*

(1)' For each  $x, y \in R \setminus N$ ,  $x^n y - x y^n \in N \cap C$ .

(ii) For each  $a \in N^*$  and  $e \in E$ ,  $n![a, e] = 0$  implies  $[a, e]_k = 0$  with some positive integer  $k$ .

(2)  $R$  is a ring which is one of the following types (a)-(d).

(a)  $R = N$ .

(b)  $R = C$  and  $x^n - x \in N$  for each  $x \in R$ .

(c) (c<sub>1</sub>)  $\{0\} \neq RE \subset C$  and  $R = RE + N$ .

(c<sub>2</sub>)  $N$  is a non-commutative ideal of  $R$ .

(c<sub>3</sub>)  $x^n - x \in N$  for each  $x \in RE$ .

(c<sub>4</sub>)  $x^n y - x y^n \in C$  for each  $x, y \in N$ .

(d) (d<sub>1</sub>)  $RE \not\subset C$ ,  $E = \{e, 0\} \subset C$  and  $R = Re + R(1 - e)$ .

(d<sub>2</sub>)  $N$  is an ideal of  $R$  containing  $R(1 - e)$ .

(d<sub>3</sub>) The factor ring  $Re/(Re \cap N)$  is a finite field  $GF(p^s)$  such that  $p^s - 1$  is a divisor of  $n - 1$ .

(d<sub>4</sub>)  $x^n y - x y^n \in C$  for each  $x, y \in R \setminus N$ .

**Proof** (1) $\Rightarrow$ (2): For each  $x \in R$ , by (i)'', we have

$$\begin{aligned} x^{n+1}(x - x^n) &= x^n x^2 - x x^{2n}, \quad \text{and} \\ (x - x^n)^{n+3} &= (x - x^n)(1 - x^{n-1})^{n+1} x^{n+1}(x - x^n). \end{aligned}$$

If  $x \notin N$  then  $x^{n+1}(x - x^n) \in N \cap C$ , whence  $x - x^n \in N$ . Hence

$$x^n - x \in N \quad \text{for all } x \in R.$$

We assume that  $R \neq N$  and  $R \neq C$ . We shall distinguish two cases:

Case 1.  $RE \subset C$ : By Corollary 2,  $R$  is normal and periodic. Hence, for each  $x \in R$ , there exists an integer  $r > 0$  such that  $x^r \in E \subset C$ . If  $RE = \{0\}$

then  $R = N$ , and this is a contradiction. Thus, we have  $\{0\} \neq RE \subset C$ . Now, we set

$$A = \{x \in R \mid x(RE) = \{0\}\}.$$

Clearly  $A$  is an ideal of  $R$ . If  $A \not\subset N$  then, for each  $x \in A \setminus N$ ,  $0 \neq x^r \in E \cap A$  for some integer  $r > 0$  and  $A(RE) \ni x^r x^r = x^r \neq 0$ , which is a contradiction. Hence we have  $A \subset N$ . Next, let  $b_1, b_2 \in N$ . Then  $b_1^{m_1} = 0$  and  $b_2^{m_2} = 0$  for some integers  $m_1 > 0$  and  $m_2 > 0$ . For each  $e \in E$ , we have  $b_1 e, b_2 e \in RE \subset C$ , and so

$$(b_1 - b_2)^{m_1+m_2} e = (b_1 e - b_2 e)^{m_1+m_2} = 0.$$

Hence  $(b_1 - b_2)^{m_1+m_2} \in A \subset N$ , and so  $b_1 - b_2 \in N$ . By a similar way, we have  $b_1 x, x b_1 \in N$  for all  $x \in R$ . Thus,  $N$  is an ideal of  $R$ . Next, we shall prove  $R = RE + N$ . Let  $x \in R \setminus N$ . Then  $0 \neq x^r \in E \subset C$  for some integer  $r > 0$ . We set  $e = x^r$  and consider  $R = Re + R(1 - e)$ . Then  $x = x_1 + x_2$  where  $x_1 \in Re$  and  $x_2 \in R(1 - e)$ . Since  $e, x_1 \in C$ , we have  $x^r = x_1^r + x_2^r$ . Since  $x^r = e$  and  $x_1^r \in Re$ , it is easily seen that  $x_2^r = 0$  and so  $x_2 \in N$ . Thus, we obtain  $x = x_1 + x_2 \in RE + N$ . Therefore, it follows that  $R = RE + N$ . Since  $R \neq C$  and  $RE \subset C$ ,  $N$  is a non-commutative ideal of  $R$ . Now, we shall prove (c<sub>4</sub>). Let  $b_1, b_2 \in N$  and  $e \neq 0 \in E$ . Then  $e + b_1, e + b_2 \notin N$ . Obviously  $RE \cap N$  is an ideal of  $R$ . Moreover

$$\begin{aligned} & (e + b_1)^n(e + b_2) - (e + b_1)(e + b_2)^n \\ &= (e + b_1^n)(e + b_2) - (e + b_1)(e + b_2^n) \quad (\text{mod } RE \cap N) \\ &= b_1^n b_2 - b_1 b_2^n \quad (\text{mod } RE \cap N). \end{aligned}$$

Hence  $C \cap N \ni (e + b_1)^n(e + b_2) - (e + b_1)(e + b_2)^n = b_1^n b_2 - b_1 b_2^n + c$  for some  $c \in RE \cap N$ . Since  $RE \cap N \subset C \cap N$ , we obtain  $b_1^n b_2 - b_1 b_2^n \in C \cap N \subset C$ . Thus, we obtain (c<sub>4</sub>) and the assertion (c).

Case 2.  $RE \not\subset C$ : In this case, we shall prove the assertion (d). Let  $a$  be an element of  $RE \setminus C$ . Then, there are elements  $e_1, \dots, e_m \in E$  and  $a_1, \dots, a_m \in R$  such that

$$a_1 e_1 + \dots + a_m e_m = a.$$

Since  $E \subset C$ , there exists an element  $f \neq 0$  in  $E$  such that  $f \geq e_i$ , that is,  $e_i f = e_i$  for  $i = 1, \dots, m$ . Then  $a \in Rf$ . We consider the Peirce decomposition

$$R = Rf + R(1 - f).$$

Since  $a \notin C$ , there exists an element  $b$  in  $R$  such that  $ab \neq ba$ . We write here

$$b = b_1 + b_2, \quad b_1 \in Rf \text{ and } b_2 \in R(1 - f).$$

Since  $a \in Rf$ , we have  $ab = ab_1$  and  $ba = b_1 a$ , whence  $ab_1 \neq b_1 a$ . Thus,  $Rf$  is a non-commutative ring, and so is  $RE$ . Now, let  $x \in RE \setminus (RE \cap N)$ . Then, there is an element  $g$  in  $E$  such that  $xg = x$ . Since  $g \in RE \setminus (RE \cap N)$ , we have

$x^n g - x g^n = x^n - x \in N \cap C$  by (i)''. Hence, it follows that  $x^n - x \in C \cap RE$  for all  $x \in RE \setminus (RE \cap N)$ . Since  $E \subset C \cap RE$ ,  $RE$  is a ring of Type (b) in [7, Theorem]. Thus, we obtain that  $E = \{e, 0\}$  and  $Re \cap N$  is an ideal of  $Re$ . We consider the Peirce decomposition

$$R = Re + R(1 - e), \quad Re = RE.$$

Since  $R$  is periodic and  $E \cap R(1 - e) = \{0\}$ , it follows that

$$R(1 - e) \subset N, \quad N = (Re \cap N) + R(1 - e)$$

and so, it is an ideal of  $R$ . Thus, we obtain (d<sub>2</sub>). Next, we shall prove (d<sub>3</sub>). By [7, Theorem, Type (b)], the factor ring  $Re/(Re \cap N)$  is a field which is algebraic over  $GF(p)$ , where  $p$  is a positive prime integer. Since  $x^n - x \in Re \cap N$  for all  $x \in Re$  and  $n$  is fixed, one will easily see that the factor ring  $Re/(Re \cap N)$  is a finite field  $GF(p^s)$  for an integer  $s > 0$ . Let  $\bar{b} = b + (Re \cap N)$  be a generating element of the multiplicative cyclic group of non-zero elements in  $Re/(Re \cap N)$ . Then

$$b^{p^s - 1} = e + c, \quad c \in Re \cap N.$$

On the other hand, since  $b \in Re \setminus (Re \cap N)$ , we have  $b^n - b \in Re \cap N$ , and so

$$b^{n-1} = e + d, \quad d \in Re \cap N.$$

Since  $p^s - 1$  is the order of  $\bar{b} = b + (Re \cap N)$ , it follows that  $p^s - 1$  is a divisor of  $n - 1$ . Thus, we obtain (d<sub>3</sub>). The assertion (d<sub>4</sub>) follows from (i)'' immediately. Therefore, for Case 2, we have the assertion (d). Next, we shall prove the converse (2) (a,b,c,d)  $\Rightarrow$  (1) in our theorem. Since the implications (a), (b), (c), (d)  $\Rightarrow$  (ii) in (1) (resp) are trivial, it suffices to prove that (a), (b), (c), (d)  $\Rightarrow$  (i)'' in (1) (resp).

(a)  $\Rightarrow$  (i)'': It is trivial.

(b)  $\Rightarrow$  (i)'': For each  $x, y \in R \setminus N$ , we have

$$x^n y - x y^n = (x^n - x)y - x(y^n - y) \in N = N \cap R = N \cap C.$$

(c)  $\Rightarrow$  (i)'': Let  $x = x_1 + x_2$ ,  $y = y_1 + y_2 \in R \setminus N$  where  $x_1, y_1 \in RE$  and  $x_2, y_2 \in N$ . Then  $x_1, y_1 \in RE \setminus N$ . Hence, by (c<sub>2</sub>), (c<sub>3</sub>) and (c<sub>4</sub>), we have

$$\begin{aligned} x_1^n y_1 - x_1 y_1^n &\in RE \cap N \subset C \cap N, \\ x_2^n y_2 - x_2 y_2^n &\in C \cap N. \end{aligned}$$

Moreover

$$\begin{aligned} x^n y - x y^n &= (x_1 + x_2)^n (y_1 + y_2) - (x_1 + x_2)(y_1 + y_2)^n \\ &= (x_1^n + x_2^n)(y_1 + y_2) - (x_1 + x_2)(y_1^n + y_2^n) \pmod{RE \cap N} \\ &= (x_1^n y_1 - x_1 y_1^n) + (x_2^n y_2 - x_2 y_2^n) \pmod{RE \cap N}. \end{aligned}$$

Therefore, it follows that  $x^n y - xy^n \in C \cap N$ .

(d)  $\Rightarrow$  (i)'': Let  $x, y \in R \setminus N$ . Then, we can write as it follows:

$$\begin{aligned} x &= x_1 + x_2, & y &= y_1 + y_2, \\ x_1, y_1 &\in Re \setminus (Re \cap N) & \text{and} & \quad x_2, y_2 \in R(1 - e). \end{aligned}$$

Since  $Re/(Re \cap N) = GF(p^s)$ , we have

$$\begin{aligned} x_1^{p^s-1} &= e + c, & c &\in Re \cap N, \\ y_1^{p^s-1} &= e + d, & d &\in Re \cap N. \end{aligned}$$

Since  $p^s - 1$  is a divisor of  $n - 1$ , we have  $n - 1 = m(p^s - 1)$  for some integer  $m > 0$ . Hence

$$\begin{aligned} x_1^n &= x_1 x_1^{n-1} = x_1 (x_1^{m(p^s-1)}) = x_1 (x_1^{p^s-1})^m = x_1 (e + c)^m = x_1 (e + c') \\ &= x_1 e + x_1 c' = x + c'', & c', c'' &= x_1 c' \in Re \cap N, \\ y_1^n &= y_1 + d'', & d'' &\in Re \cap N. \end{aligned}$$

Then, since  $x_1, y_1, e + x_2, e + y_2 \in R \setminus N$ , we have

$$\begin{aligned} x_1^n y_1 - x_1 y_1^n &= (x_1 + c'') y_1 - x_1 (y_1 + d'') \\ &= c'' y_1 - x_1 d'' \in Re \cap N \cap C \quad (\text{by } (d_2, d_4)), \quad \text{and} \\ x_2^n y_2 - x_2 y_2^n &= (e + x_2)^n (e + y_2) - (e + x_2)(e + y_2)^n \in C \cap N \quad (\text{by } (d_2, d_4)). \end{aligned}$$

Therefore, it follows that

$$x^n y - xy^n = x_1^n y_1 - x_1 y_1^n + x_2^n y_2 - x_2 y_2^n \in C \cap N.$$

Thus, we obtain the condition (i)''.  $\square$

**Lemma 2.** Let  $R$  be a ring of Type (d) in Theorem 4 for an integer  $n > 1$ , that is,  $R$  a ring which satisfies the conditions (d<sub>1</sub>)-(d<sub>4</sub>):

- (d<sub>1</sub>)  $RE \not\subset C$ ,  $E = \{e, 0\} \subset C$  and  $R = Re + R(1 - e)$ .
- (d<sub>2</sub>)  $N$  is an ideal of  $R$  containing  $R(1 - e)$ .
- (d<sub>3</sub>) The factor ring  $Re/(Re \cap N)$  is a finite field  $GF(p^s)$  such that  $p^s - 1$  is a divisor of  $n - 1$ .
- (d<sub>4</sub>)  $x^n y - xy^n \in C$  for each  $x, y \in R \setminus N$ .

Then, there hold the following (1) and (2).

- (1) If  $s = 1$  then  $N$  is non-commutative.
- (2) If  $s > 1$  and  $p$  is not a divisor of  $n - 1$  then  $N$  is non-commutative.



**Proof** (1) We assume that  $s = 1$ . The, since  $e \in Ce \setminus (Ce \cap N)$ , we have

$$Re/(Re \cap N) = GF(p) = Ce/(Ce \cap N).$$

Hence, it follows that  $Re = Ce + (Re \cap N)$ , and so,  $R = C + N$  by (d<sub>1</sub>) and (d<sub>2</sub>). Since  $Re \not\subset C$ ,  $R$  is non-commutative. This implies that  $N$  is non-commutative. (See Examples (3)).

(2) We assume that  $p$  is not a divisor of  $n - 1$  and  $N$  is commutative. Then, one will easily see that  $N^2 \subset C$ . Moreover, we have  $R(1 - e) \subset C$  by (d<sub>1</sub>) and (d<sub>2</sub>). Let  $g + (Re \cap N)$  be a generating element of the multiplicative (cyclic) group of non-zero elements of  $Re/(Re \cap N)$ . Then  $Re$  is generated by  $g$  and  $Re \cap N$ . Hence any subring of  $Re$  containing  $g$  and  $Re \cap N$  coincides with  $Re$ . Let  $C_0$  be the center of  $Re$ , and  $N_0 = Re \cap N$ . Since

$$R = Re + R(1 - e) \quad \text{and} \quad R(1 - e) \subset C$$

we have  $C_0 = Re \cap C$ . First, we shall prove that  $N_0 \cap C_0$  is an ideal of  $Re$ . We set

$$A = \{x \in Re \mid (N_0 \cap C_0)x \subset N_0 \cap C_0\}.$$

Obviously  $A$  is a subring of  $Re$ . Since  $N_0(N_0 \cap C_0) \subset N_0^2 \subset C_0$ , we have  $N_0 \subset A$ . Let  $x \in N_0 \cap C_0$ . Then  $xg \in N_0$ . We set

$$B = \{y \in Re \mid y(xg) = (xg)y\}.$$

Since  $N_0$  is commutative, we have  $N_0 \subset B$ . Moreover, since  $x \in C_0$ , we have  $g \in B$ . Hence  $B = Re$ , and  $xg \in C_0 \cap N_0$ . Therefore, it follows that  $(N_0 \cap C_0)g \subset C_0 \cap N_0$ , and so,  $g \in A$ . Since  $N_0 \subset A$ , we obtain  $A = Re$ . Thus,  $N_0 \cap C_0$  is an ideal of  $Re$ . Now, since  $Re/N_0 \cong GF(p^s)$  and  $p$  is not a divisor of  $n - 1$ , we have  $(n - 1)e \in Re \setminus N_0$ , and so,  $(n - 1)^t e \in E = \{e, 0\}$  for some integer  $t > 1$ . Hence  $(n - 1)^t e = e$ . Now, let  $v \in N_0$ . Then  $v^2 \in N_0^2 \subset N_0 \cap C_0$ , and  $e + v \in Re \setminus (Re \cap N)$ . Hence

$$Re \cap N \cap C \ni (e + v)^n e - (e + v)e^n = (n - 1)v \pmod{N_0 \cap C_0}.$$

Hence  $(n - 1)v = 0 \pmod{N_0 \cap C_0}$ , and so

$$v = (n - 1)^{t-1}(n - 1)v = 0 \pmod{N_0 \cap C_0}.$$

Therefore, it follows that  $N_0 \subset N_0 \cap C_0 \subset C_0$ . Since  $Re$  is generated by  $g$  and  $N_0$ ,  $Re$  is generated by  $g$  and  $C_0$ . Hence  $Re$  is commutative, and so,  $R$  is commutative. This is a contradiction. Thus, we obtain our assertion (2).  $\square$

Now, by virtue of Theorem 4 and Lemma 2, we shall prove a commutativity theorem in which the condition of  $R$  is weaker than that of  $R$  in Theorem 3.

**Theorem 5.** *Suppose that there exists an integer  $n > 1$  for which  $R$  satisfies the following conditions:*

- (i)'' For each  $x, y \in R \setminus N$ ,  $x^n y - xy^n \in N \cap C$ .
- (ii) For each  $a \in N^*$  and  $e \in E$ ,  $n![a, e] = 0$  implies  $[a, e]_k = 0$  with some positive integer  $k$ .
- (iii) For each  $a, b \in N$ , there exists an integer  $m = m(a, b) > 1$  such that  $[a, b] = [a, b]^m$ .

Then,  $R$  is normal periodic and  $N$  is commutative. Moreover,

(1)  $R$  is commutative, provided that  $n$  is in the following:  $\{2, 3, 4, 5, 6/8, 9, 10, 11, 12/14/16, 17, 18/20, 21, 22, 23, 24/26, 27, 28/30, \text{ and integers } t \text{ such that } t-1 \text{ is not a multiple of } p(p^s-1) \text{ for all positive prime divisors } p \text{ of } t-1 \text{ and all positive integers } s > 1\}$ .

(2)  $R$  is not always commutative, that is, there exists an example of  $R$  which is a non-commutative ring, provided that  $n$  is in the following:  $\{7, 13, 15, 19, 25, 29, \text{ and integers } t' \text{ such that } t'-1 \text{ is a multiple of } p(p^s-1) \text{ for some positive prime divisor } p \text{ of } t'-1 \text{ and some positive integer } s > 1\}$  (Examples (1) and (4)).

**Proof** By Corollary 2,  $R$  is a normal periodic ring. Moreover, by Theorem 4,  $N$  is an ideal of  $R$ . Let  $a, b \in N$ . Then, we have  $[a, b] \in N$ . Hence, it follows from (iii) that

$$[a, b] = [a, b]^m = [a, b]^{m^2} = \dots = [a, b]^{m^u} = 0$$

for some positive integers  $m > 1$  and  $u$ , and so,  $ab = ba$ . Thus,  $N$  is commutative. By Examples (1) and (4), it suffices to prove the assertion (1). Let  $n$  be an integer such that  $n-1$  is not a multiple of  $p(p^s-1)$  for all positive prime divisors  $p$  of  $n-1$  and all positive integers  $s > 1$ . Now, we assume that  $R$  is non-commutative. Then, since  $N$  is commutative, it is easily seen that  $R$  is a ring of the Type (d) in Theorem 4. Hence

$$E = \{e, 0\} \subset C, \quad Re \not\subset C, \quad R = Re + R(1-e), \quad R(1-e) \subset N,$$

and the factor ring  $Re/(Re \cap N)$  is a finite field  $GF(p^s)$  such that  $p^s-1$  is a divisor of  $n-1$ . Hence  $n-1 = q(p^s-1)$  for some integer  $q$ . If  $s=1$  then  $N$  is non-commutative by Lemma 2(1). Hence we have  $s > 1$ . By the condition on  $n$ ,  $p$  is not a divisor of  $q$ . This implies that  $p$  is not a divisor of  $n-1$ . Therefore, it follows from Lemma 2(2) that  $N$  is non-commutative. This is a contradiction. Thus,  $R$  is commutative.  $\square$

**Examples 1.** We shall present some examples of rings of Types (b), (c) and (d) in Theorem 4. In what follows,  $\{e_{ij} \mid 1 \leq i, j \leq 3\}$  means the set of matrix units in  $(GF(2))_3$ , the complete matrix ring of order 3 over  $GF(2)$ .

(1) Type (b) for any integer  $n > 1$ :  $R = GF(2)$  where  $N = \{0\}$ .

(2) Type (c) for any integer  $n > 2$ :  $R = GF(2) \oplus GF(2) \oplus N$  where  $N = \{e_{12}x + e_{13}y + e_{23}z \mid x, y, z \in GF(2)\}$ .

(3) Type (d) for  $n = 5$  (the case such that  $N$  is non-commutative):  $R = \{(e_{11} + e_{22} + e_{33})u + e_{12}x + e_{13}y + e_{23}z \mid u, x, y, z \in GF(2)\}$  where  $N = \{e_{12}x + e_{13}y + e_{23}z \mid x, y, z \in GF(2)\}$ .

(4) Type (d) for  $n = rp(p^s - 1) + 1$  with any positive prime integer  $p$  and any positive integers  $r$  and  $s > 1$  (the case such that  $N$  is commutative): We shall present an example which is a non-commutative ring of Type (d) and satisfies the condition (i)'', (ii) and (iii) in Theorem 5 for  $n = rp(p^s - 1) + 1$  with the above  $p, r$  and  $s$ . Now, we set  $h = p^s$  ( $s > 1$ ), and consider the finite field  $GF(h)$ . It is well known that  $GF(h) \setminus \{0\}$  is a multiplicative cyclic group of order  $h - 1$ . Hence  $v^{h-1} = 1$  for each  $v \in GF(h) \setminus \{0\}$ . We set

$$T = GF(h) \otimes_{GF(p)} GF(h) \quad (\text{tensor product}).$$

Then  $T$  is a right  $GF(h)$ -module by the multiplication

$$(a \otimes b)v = a \otimes (bv), \quad v \in GF(h).$$

Moreover,  $T$  is a left  $GF(h)$ -module by the multiplication

$$v(a \otimes b) = (va) \otimes b, \quad v \in GF(h).$$

Next, we consider the module

$$R = GF(h) \times T; \quad (v, t) + (v', t') = (v + v', t + t').$$

Since  $T$  is a left-right- $GF(h)$ -module, we can define a product in  $R$  by the following

$$(v, t)(v', t') = (vv', vt' + tv').$$

Then  $R$  is a non-commutative ring such that for any  $v \in GF(h) \setminus GF(p)$  ( $\neq \emptyset$ ),

$$(v, 0)(1, 1 \otimes 1) = (v, v \otimes 1) \neq (v, 1 \otimes v) = (1, 1 \otimes 1)(v, 0).$$

Obviously,  $(1, 0)$  is the identity of  $R$ . Now, we set  $N = \{(0, t) \mid t \in T\}$ . Then  $N$  is a commutative ideal of  $R$  such that  $N^2 = \{(0, 0)\}$ . If  $(v, t) \in R \setminus N$  then  $v \neq 0$  and

$$(v, t)^{p(h-1)} = ((v, t)^{h-1})^p = (v^{h-1}, t')^p = (1, t')^p = (1, pt') = (1, 0)$$

for some  $t' \in N$ . Now, let  $r$  be a positive integer, and  $d = rp(h - 1)$ . Let  $x, y \in R \setminus N$ . Then  $x^d = (1, 0)$ ,  $x^{d+1} = x$  and  $y^{d+1} = y$ . Therefore, it follows that

$$x^n y - xy^n = x^{d+1} y - xy^{d+1} = xy - xy = (0, 0) \in N \cap C.$$

Moreover, it is easily seen that  $N$  is the set of all nilpotent elements of  $R$  and  $\{(1, 0), (0, 0)\}$  is the set  $E$  of all idempotent elements of  $R$ . Further,  $N^* = N$  and it is commutative. Therefore, one will see that  $R$  satisfies the conditions (i)'', (ii) and (iii) in Theorem 5 for  $n = d + 1 = rp(h - 1) + 1$ .

**Remark.** We shall present an alternative proof of Theorem 3 in virtue of Theorem 4. Let  $R$  be a ring which satisfies the conditions (i)'', (ii)' and (iii) in Theorem 3. Obviously  $R$  satisfies the conditions (i)'' and (ii) in Corollary 2. Hence  $R$  is normal and periodic. By Theorem 4,  $N$  is an ideal of  $R$ . Hence by (iii),  $N$  is commutative and  $N^2 \subset C$ . Now, we assume that  $R$  is non-commutative. Then, it follows that  $R$  is a ring of Type (d) in Theorem 4 such that

$$E = \{e, 0\} \subset C, \quad R = Re + R(1 - e) \quad \text{and} \quad R(1 - e) \subset N \cap C.$$

Let  $a \in Re \cap N$ . Then  $e + a, e \in Re \setminus (Re \cap N)$ . Hence

$$C \ni (e + a)^n e - (e + a)e^n = (n - 1)a + c, \quad c \in C.$$

This implies that  $(n - 1)a \in C$ , and so,  $(n - 1)[a, x] = 0$  for all  $x \in Re$ . Then  $n![a, x] = 0$ , so that  $[a, x]_k = 0$  for some positive integer  $k$  (by (ii)'). Hence by [2, Theorem],  $Re$  is commutative, and so,  $R$  is commutative. This is a contradiction. Thus,  $R$  is commutative.

In Example 1(4), we set  $n = 7 = 2(2^2 - 1) + 1$  and  $GF(2^2) = \{0, g, g^2, g^3 = 1\}$ . Then, one will easily see that  $7![(0, 1 \otimes 1), (g, 0)] = 0$  and

$$0 \neq (0, 1 \otimes g - g \otimes 1) = [(0, 1 \otimes 1), (g, 0)]_{1+3n}$$

Hence, this example does not satisfy (ii)', while it satisfies (ii).

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