

AN EFFECTIVE CRITERION OF SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

New criterion of solvability of boundary value problem for system of ordinary differential equations with functional boundary conditions are constructed by method of a priori estimates.

Introduction

In this paper we will apply our result in [1] to get a new effective criterion for the existence and the uniqueness of the solutions of the following problem A:

$$x'_i(t) = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n) \quad (1)$$

$$\Phi_{0i}(x_i) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \quad (2)$$

where, for each $i \in \{1, \dots, n\}$ $f_i : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the Carathéodory conditions, Φ_{0i} - the linear nondecreasing continuous functional on $C([a, b])$ - is concentrated on $[a_i, b_i] \subseteq [a, b]$ (i.e., the value of Φ_{0i} depends only on functions restricted to $[a_i, b_i]$ and the segment can be degenerated a point) and φ_i is a continuous functional on $C_n([a, b])$. In general $\Phi_{0i}(1) = C_i$ ($i = 1, \dots, n$). Without loss of generality we can suppose that $\Phi_{0i}(1) = 1$ ($i = 1, \dots, n$) to

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simplify the notation.

We adopt the following notations:

$[a, b]$ - a segment, $-\infty < a \leq a_i \leq b_i \leq b < +\infty$, ($i = 1, \dots, n$), \mathbb{R}^n - n -dimensional real space with point $x = (x_i)_{i=1}^n$ normed by $\|x\| = \sum_{i=1}^n |x_i|$.

$C_n([a, b])$ is the spaces of continuous n -dimensional vector-valued functions on $[a, b]$ with the norm

$$\|x\| = \max \left\{ \sum_{i=1}^n |x_i(t)| : a \leq t \leq b \right\}$$

$$C^+([a, b]) = \left\{ x \in C([a, b]) : x(t) \geq 0, a \leq t \leq b \right\}.$$

$L^p([a, b])$ is the space of integrable functions on $[a, b]$ in p -power with the norm

$$\|u\|_{L^p([a, b])} = \begin{cases} \left[\int_a^b |u(t)|^p dt \right]^{\frac{1}{p}} & \text{for } 1 \leq p < +\infty \\ \text{vrai max} \{|u(t)| : a \leq t \leq b\} & \text{for } p = +\infty \end{cases}$$

$$L^p([a, b], \mathbb{R}_+) = \left\{ u \in L^p([a, b]) : u(t) \geq 0, a \leq t \leq b \right\}$$

$$l(q, q_0) = \begin{cases} 1 & \text{if } 1 \leq q \text{ and } q = q_0 \text{ or } q = +\infty \\ \left(\frac{q_0}{q} - 1 \right)^{-\frac{1}{q_0}} \left(\frac{q_0}{q\pi} \sin \frac{q\pi}{q_0} \right)^{\frac{1}{q}} & \text{if } 1 \leq q < q_0 < +\infty \end{cases}$$

Let us consider the Problem A. By a solution we mean an absolutely continuous n -dimensional vector-valued function on $[a, b]$, which satisfies the equation (1) for almost $t \in [a, b]$ and fulfills the boundary conditions (2) of Problem A.

2. Results

Definition Let $G = (g_i)_{i=1}^n : C([a, b]) \rightarrow \mathbb{R}^n$, $H = (h_{ij})_{i,j=1}^n : [a, b] \rightarrow \mathbb{R}_+^{n \times n}$ and

$\Psi = (\psi_i)_{i=1}^n : C_n(a, b) \rightarrow \mathbb{R}_+^n$ is a positively homogeneous nondecreasing operator. We say that

$$(G, H, \Psi) \in \text{Nic}_0([a, b]; a_1, \dots, a_n, b_1, \dots, b_n) \quad (3)$$

if the system of differential inequalities (we call the Problem B)

$$|x'_i(t) - g_i(t)x_i(t)| \leq \sum_{j=1}^n h_{ij}(t)|x_j(t)| \quad \text{for } a \leq t \leq b \quad (i = 1, \dots, n) \quad (4)$$

with boundary conditions

$$\min \{|x_i(t)| : a_i \leq t \leq b_i\} \leq \psi_i(|x_1(t)|, \dots, |x_n(t)|) \quad (i = 1, \dots, n) \quad (5)$$

has only one trivial solution.

The following Theorem 1 and Theorem 2 have been proved in [1] and we state here for the convenience of hte readers.

Theorem 1. *Suppose that the following inequalities hold:*

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - g_i(t)x_i] \operatorname{sign} x_i &\leq \sum_{j=1}^n h_{ij}(t)|x_j| + \omega_i\left(t, \sum_{j=1}^n |x_j|\right) \\ \text{if } t \in [a_i, b_i], \quad x \in \mathbb{R}^n, \quad (i = 1, \dots, n) \end{aligned} \quad (6_1)$$

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - g_i(t)x_i] \operatorname{sign} x_i &\geq -\sum_{j=1}^n h_{ij}(t)|x_j| - \omega_i\left(t, \sum_{j=1}^n |x_j|\right) \\ \text{if } t \in [a, b_i], \quad x \in \mathbb{R}^n, \quad (i = 1, \dots, n) \end{aligned} \quad (6_2)$$

$$\begin{aligned} |\varphi_i(x_1, \dots, x_n)| &\leq \psi_i(|x_1|, \dots, |x_n|) + r_i\left(\sum_{j=1}^n |x_j|\right) \\ \text{for all } x = (x_i)_{i=1}^n &\in C_n([a, b]) \quad (i = 1, \dots, n), \end{aligned} \quad (7)$$

where $G = (g_i)_{i=1}^n$, $H = (h_{ij})_{ij=1}^n$, and $\Psi = (\psi_i)_{i=1}^n$ satisfy the condition (3), the functions $\omega_i : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ are measurable with regard to the first and nondecreasing to the second argument, $r_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are nondecreasing and

$$\lim_{\rho \rightarrow \infty} \frac{1}{\rho} \int_a^b \omega_i(t, \rho) dt = 0 = \lim_{\rho \rightarrow \infty} \frac{1}{\rho} r_i(\rho) \quad (i = 1, \dots, n). \quad (8)$$

Then the problem (1), (2) has at least one solution.

Theorem 2. *Suppose that the following inequalities hold:*

$$\begin{aligned} \left\{ [f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] - g_i(t)(x_{1i} - x_{2i}) \right\} \operatorname{sign} (x_{1i} - x_{2i}) &\leq \\ \leq \sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}|, \text{ if } a_i \leq t \leq b, x_1 = (x_{1j})_{j=1}^n, x_2 = (x_{2j})_{j=1}^n &\in \mathbb{R}^n \\ (i = 1, \dots, n) \end{aligned} \quad (9_1)$$

$$\begin{aligned} \left\{ [f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] - g_i(t)(x_{1i} - x_{2i}) \right\} \operatorname{sign} (x_{1i} - x_{2i}) &\geq \\ \geq -\sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}|, \text{ if } a_i \leq t \leq b, x_1 = (x_{1j})_{j=1}^n, x_2 = (x_{2j})_{j=1}^n &\in \mathbb{R}^n \\ (i = 1, \dots, n) \end{aligned} \quad (9_2)$$

and

$$|\varphi_i(x_{11}, \dots, x_{1n}) - \varphi_i(x_{21}, \dots, x_{2n})| \leq \psi_i(|x_{11} - x_{12}|, \dots, |x_{1n} - x_{2n}|)$$

$$\text{for all } x_1 = (x_{1i})_{i=1}^n, x_2 = (x_{2i})_{i=1}^n \in C_n([a, b]) \quad (i = 1, \dots, n) \quad (10)$$

where $G = (g_i)_{i=1}^n$, $H = (h_{ij})_{i,j=1}^n$, and $\Psi = (\psi_i)_{i=1}^n$ satisfy the condition (3). Then the Problem A has a unique solution.

The main results in our note are Theorems 3, 4, 5 and 6. For clarity, we state our theorems first before sketching their proofs.

Theorem 3. Consider $[a, b] \times \mathbb{R}^n$ and for each $i = 1, \dots, n$, let

$$f_i(t, x_1, \dots, x_n) \text{ sign } x_i \leq \sum_{j=1}^n h_{ij}(t)|x_j| + \omega_i\left(t, \sum_{j=1}^n |x_j|\right) \text{ if } a_i \leq t \leq b$$

$$(11_1)$$

$$f_i(t, x_1, \dots, x_n) \text{ sign } x_i \geq -\sum_{j=1}^n h_{ij}(t)|x_j| - \omega_i\left(t, \sum_{j=1}^n |x_j|\right) \text{ if } a \leq t \leq b_i$$

$$(11_2)$$

and in $C_n([a, b])$,

$$|\varphi_i(x_1, \dots, x_n)| \leq \sum_{j=1}^n r_{ij} \|x_j\|_{L^{q_0}([a, b])} + r_i \left(\sum_{j=1}^n |x_j| \right) \quad (i = 1, \dots, n) \quad (12)$$

where $h_{ij} \in L^{p_{ij}}_{([a, b], \mathbb{R}_+)}$, $p_{ij} \geq 1$, $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$, $q_{ij} \leq q_0$, $r_{ij} \in \mathbb{R}_+$ ($i, j = 1, \dots, n$), $\omega_i : [a, b] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $r_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfy the conditions of Theorem 1, and the spectral radius of the matrix $S = (s_{ij})_{i,j=1}^n$,

$$s_{ij} = (b-a)^{\frac{1}{q_0}} r_{ij} + (b-a)^{\frac{1}{q_{ij}}} l(q_{ij}, q_0) \cdot h_{ij}^* \quad (i, j = 1, \dots, n)$$

$$h_{ij}^* = \max \left\{ \|h_{ij}\|_{L^{p_{ij}}_{([a_i, b])}}, \|h_{ij}\|_{L^{p_{ij}}_{([a, b_i])}} \right\} \quad (i, j = 1, \dots, n)$$

is less than 1. Then the Problem A has at least one solution.

Theorem 4. Consider $[a, b] \times \mathbb{R}^n$ and for each $i = 1, \dots, n$, let

$$[f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] \text{ sign } (x_{1i} - x_{2i}) \leq$$

$$\leq \sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}|, \text{ if } a_i \leq t \leq b \quad (13_1)$$

$$[f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] \text{ sign } (x_{1i} - x_{2i}) \geq$$

$$\geq -\sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}|, \text{ if } a \leq t \leq b_i \quad (13_2)$$

and in $C_n([a, b])$,

$$|\varphi_i(x_{11}, \dots, x_{1n}) - \varphi_i(x_{21}, \dots, x_{2n})| \leq \sum_{j=1}^n r_{ij} \|x_{1j} - x_{2j}\|_{L_{([a, b])}^{q_0}} \quad (14)$$

where $h_{ij} \in L_{([a, b], \mathbb{R}_+)}^{p_{ij}}$ and $r_{ij} \in \mathbb{R}_+$ ($i, j = 1, \dots, n$) satisfy the conditions of Theorem 3. Then the Problem A has a unique solution.

The following theorem shows the existence of our problem.

Theorem 5. Let in $[a, b] \times \mathbb{R}^n$ and for each $i = 1, \dots, n$

$$f_i(t, x_1, \dots, x_n) \text{ sign } x_i \leq g_i(t)|x_i| + \sum_{j=1}^n h_{ij}(t)|x_j| + \omega_i(t, \sum_{j=1}^n |x_j|) \quad \text{if } a_i \leq t \leq b \quad (15_1)$$

$$f_i(t, x_1, \dots, x_n) \text{ sign } x_i \geq g_i(t)|x_i| - \sum_{j=1}^n h_{ij}(t)|x_j| - \omega_i(t, \sum_{j=1}^n |x_j|) \quad \text{if } a \leq t \leq b_i \quad (15_2)$$

and in $C_n([a, b])$, $i = 1, \dots, n$

$$|\varphi_i(x_1, \dots, x_n)| \leq \sum_{j=1}^n \psi_{ij}(|x_j|) + r_i \left(\sum_{j=1}^n |x_j| \right) \quad (i = 1, \dots, n) \quad (16)$$

where $h_{ij} \in L_{([a, b], \mathbb{R}_+)}^{p_{ij}}$, $p_{ij} \geq 1$, $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$, $q_{ij} \leq q_0$, ($i, j = 1, \dots, n$), $g_i \in L([a, b])$ ($i = 1, \dots, n$), $\omega_i : [a, b] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $r_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, ($i = 1, \dots, n$) satisfy the conditions of Theorem 1, the continuous functionals $\psi_{ij} : C^+([a, b]) \rightarrow \mathbb{R}_+$ are sublinear non-decreasing and the spectral radius of the matrices

$$\Psi^* = (\psi_{ij}^*)_{i,j=1}^n = (\gamma_i \psi_{ij}(1))_{i,j=1}^n \quad (17_1)$$

and

$$S = (s_{ij})_{i,j=1}^n = (E - \Psi^*)^{-1} \left((b - a)^{\frac{1}{q_{ij}}} \gamma_i \psi_{ij}(1) h_{ij}^{**} \right)_{i,j=1}^n + \left((b - a)^{\frac{1}{q_{ij}}} l(q_{ij}, q_0) h_{ij}^{**} \right)_{i,j=1}^n \quad (17_2)$$

are less than 1 where

$$\gamma_i = \max \left\{ \exp \int_{a_i}^b |g_i(\tau)| d\tau, \exp \int_a^{b_i} |g_i(\tau)| d\tau \right\}, \quad (i = 1, \dots, n)$$

$$h_{ij}^{**} = \max \left\{ \left\| h_{ij}(t) \exp \int_{a_i}^b |g_i(s)| ds \right\|_{L_{([a_i, b])}^{p_{ij}}}, \left\| h_{ij}(t) \exp \int_a^{b_i} |g_i(s)| ds \right\|_{L_{([a, b_i])}^{p_{ij}}} \right\}$$

$$(i, j = 1, \dots, n)$$

Then Problem A has a solution.

And now for the uniqueness, we get

Theorem 6. Consider $[a, b] \times \mathbb{R}^n$ and for each $i = 1, \dots, n$, let

$$\begin{aligned} [f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] \operatorname{sign}(x_{1i} - x_{2i}) \leq g_i(t)|x_{1i} - x_{2i}| + \\ + \sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}|, \text{ if } a_i \leq t \leq b, \end{aligned} \quad (18_1)$$

$$\begin{aligned} [f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] \operatorname{sign}(x_{1i} - x_{2i}) \geq g_i(t)|x_{1i} - x_{2i}| - \\ - \sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}|, \text{ if } a \leq t \leq b_i \end{aligned} \quad (18_2)$$

and in $C_n([a, b])$, ($i = 1, \dots, n$)

$$|\varphi_i(x_{11}, \dots, x_{1n}) - \varphi_i(x_{21}, \dots, x_{2n})| \leq \sum_{j=1}^n \psi_{ij}(|x_{1j} - x_{2j}|) \quad (19)$$

where g_i , h_{ij} and ψ_{ij} ($i, j = 1, \dots, n$) satisfy the conditions in Theorem 5. Then the Problem A has a unique solution.

The proofs of our Theorems are based on the following two lemmas.

Lemma 7 Let $g_i(t) \equiv 0$, $h_{ij} \in L_{([a, b], \mathbb{R}_+)}^{p_{ij}}$, $p_{ij} \geq 1$, $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$, $q_{ij} \leq q_0$, ($i, j = 1, \dots, n$)

$$\psi_i(|x_1|, \dots, |x_n|) = \sum_{j=1}^n r_{ij} \|x_j\|_{L_{([a, b])}^{q_0}}, \quad (i = 1, \dots, n) \quad (20)$$

where each $r_{ij} \in \mathbb{R}_+$ ($i, j = 1, \dots, n$) and the spectral radius of the matrix S with elements defined in Theorem 3 is less than 1. Then (3) holds for (G, H, Ψ) .

Proof Let the vector function $x(t) = (x_i(t))_{i=1}^n$ be the solution of the problem (4), (5). We shall prove that this solution is zero. Choose $t_i \in [a_i, b_i]$ such that

$$|x_i(t_i)| = \min \{|x_i(t)| : a_i \leq t \leq b_i\}, \quad (i = 1, \dots, n) \quad (21)$$

Then by intergrating relations (4) and using relations (5), (20) and Hölder inequality, we obtain

$$\begin{aligned} |x_i(t)| &\leq |x_i(t_i)| + \sum_{j=1}^n \left| \int_{t_i}^t h_{ij}(\tau) |x_j(\tau)| d\tau \right| \leq \\ &\leq |x_i(t_i)| + \sum_{j=1}^n \left| \int_{t_i}^t h_{ij}^{p_{ij}}(\tau) d\tau \right|^{\frac{1}{p_{ij}}} \left| \int_{t_i}^t |x_j(\tau)|^{q_{ij}} d\tau \right|^{\frac{1}{q_{ij}}} \end{aligned} \quad (22)$$

and

$$\begin{aligned} \|x_i(t)\|_{L_{([a,b])}^{q_0}} &\leq (b-a)^{\frac{1}{q_0}} \sum_{j=1}^n r_{ij} \|x_j\|_{L_{([a,b])}^{q_0}} + \\ &+ \sum_{j=1}^n h_{ij}^* \left[\int_a^b \left| \int_{t_i}^t |x_j(\tau)|^{q_{ij}} d\tau \right|^{\frac{q_0}{q_{ij}}} dt \right]^{\frac{1}{q_0}} \quad (i = 1, \dots, n) \end{aligned} \quad (23)$$

By a lemma of Levin (see [2] lemma 4.7)

$$\left[\int_a^b \left| \int_{t_j}^t |x_j(\tau)|^{q_{ij}} d\tau \right|^{\frac{q_0}{q_{ij}}} dt \right]^{\frac{1}{q_0}} \leq (b-a)^{\frac{1}{q_{ij}}} l(q_{ij}, q_0) \|x_j\|_{L_{([a,b])}^{q_0}} \quad (24)$$

Consequently we obtain from (23) that

$$(E - S)(\|x_i\|_{L_{([a,b])}^{q_0}})_{i=1}^n \leq 0 \quad (25)$$

where the E -matrix unit $S = (s_{ij})_{i,j=1..n}^n$ is defined in Theorem 3. Since the spectral radius of the matrix is less than 1, it follows from (25) that

$$\|x_i\|_{L_{([a,b])}^{q_0}} = 0 \quad (i = 1, \dots, n)$$

Therefore $x_i \equiv 0 \quad (i = 1, \dots, n)$, proving our Lemma 7. □

Lemma 8 Let $g_i : [a, b] \rightarrow \mathbb{R}$, $g_i \in L([a, b]) \quad (i = 1, \dots, n)$, $h_{ij} \in L^{p_{ij}}([a, b], \mathbb{R}_+)$, $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$, $q_{ij} \leq q_0$, $(i, j = 1, \dots, n)$

$$\psi_i(|x_1|, \dots, |x_n|) = \sum_{j=1}^n \psi_{ij}(|x_j|) \quad (i = 1, \dots, n) \quad (26)$$

where each $\psi_{ij} : C^+([a, b]) \rightarrow \mathbb{R}_+ \quad (i = 1, \dots, n)$ are sublinear nondecreasing continuous functionals and the spectral radius of the matrices Ψ^* and S defined in (17) is less than 1. Then (3) holds for (G, H, Ψ) .

Proof Let the vector function $x(t) = (x_i(t))_{i=1}^n$ be the solution of the Problem B. We shall prove that this solution is zero.

Choose $t_i \in [a_i, b_i]$ such that

$$|x_i(t_i)| = \min \{ |x_i(t)| : a_i \leq t \leq b_i \}, \quad (i = 1, \dots, n) \quad (27)$$

The by integrating (4) and using (5), Hölder inequality and Lemma of Levin (see [3], Lemma 1.7) we obtain

$$\begin{aligned} |x_i(t)| &\leq \left(\exp \int_{t_i}^t g_i(\tau) \operatorname{sign}(\tau - t_i) d\tau \right) |x_i(t_i)| + \\ &+ \sum_{j=1}^n \left| \int_{t_i}^t h_{ij}(\tau) \exp \left(\int_{\tau}^t g(s) \operatorname{sign}(s - \tau) ds \right) |x_j(\tau)| d\tau \right|, \quad (i = 1, \dots, n) \end{aligned} \quad (28)$$

$$\begin{aligned} &\left\| \int_{t_i}^t h_{ij}(\tau) \exp \left(\int_{\tau}^t g_i(s) \operatorname{sign}(s - \tau) ds \right) |x_j(\tau)| d\tau \right\|_{L_{([a,b])}^{q_0}} \leq \\ &\leq \left\| \int_{t_i}^t h_{ij}^{p_{ij}}(\tau) \exp \left(p_{ij} \int_{\tau}^t g_i(s) \operatorname{sign}(s - \tau) ds \right) d\tau \right|^{\frac{1}{p_{ij}}} \left\| \int_{t_i}^t |x_j(\tau)|^{q_{ij}} d\tau \right|^{\frac{1}{q_{ij}}} \Big\|_{L_{([a,b])}^{q_0}} \\ &\leq h_{ij}^{**} \left\| \int_{t_i}^t |x(\tau)|^{q_{ij}} d\tau \right|^{\frac{1}{q_{ij}}} \Big\|_{L^{q_0}([a,b])} \leq (b-a)^{\frac{1}{q_{ij}}} l(q_{ij}, q_0) \|x_j\|_{L_{([a,b])}^{q_0}} h_{ij}^{**} \end{aligned}$$

and

$$\|x_i\|_{L_{([a,b])}^{q_0}} \leq (b-a)^{\frac{1}{q_0}} \gamma_i |x_i(t_i)| + \sum_{j=1}^n (b-a)^{\frac{1}{q_{ij}}} l(q_{ij}, q_0) h_{ij}^{**} \|x_j\|_{L_{([a,b])}^{q_0}} \quad (i = 1, \dots, n) \quad (29)$$

Substituting the inequality (28) into boundary condition (5) and using (26), (27) we have

$$\begin{aligned} |x_i(t_i)| &\leq \sum_{j=1}^n \psi_{ij}(|x_j|) \leq |x_i(t_i)| \psi_{ij}^* + \\ &+ \sum_{j=1}^n \psi_{ij} \left(\left\| \int_{t_i}^t h_{ij}^{p_{ij}}(\tau) \exp \left(p_{ij} \int_{\tau}^t g_i(s) \operatorname{sign}(s - \tau) ds \right) d\tau \right|^{\frac{1}{p_{ij}}} \left\| \int_{t_i}^t |x_j(\tau)|^{q_{ij}} d\tau \right|^{\frac{1}{q_{ij}}} \right) \leq \\ &\leq |x_i(t_i)| \psi_{ij}^* + \sum_{j=1}^n (b-a)^{\frac{1}{q_{ij}} - \frac{1}{q_0}} \psi_{ij}(1) h_{ij}^{**} \|x_j\|_{L_{([a,b])}^{q_0}}, \quad (i = 1, \dots, n). \end{aligned}$$

Since the spectral radius of the matrix Ψ^* is less than 1, we get

$$(\|x_i(t_i)\|_{i=1}^n \leq (E - \psi^*)^{-1} \left((b-a)^{\frac{1}{q_{ij}} - \frac{1}{q_0}} \psi_{ij}(1) h_{ij}^{**} \right)_{i,j=1}^n (\|x_j\|_{L_{([a,b])}^{q_0}})_{j=1}^n \quad (30)$$

Consequently, from (29), (30) we obtain

$$(\|x_i\|_{L_{([a,b])}^{q_0}})_{i=1}^n \leq S (\|x_j\|_{L_{([a,b])}^{q_0}})_{j=1}^n \quad (31)$$

Since spectral of radius of the matrix S is less than 1, we obtain

$$(\|x_i\|_{L_{([a,b])}^{q_0}})_{i=1}^n \leq 0,$$

and our Lemma has been proved. \square

Proofs of our Theorems We now can sketch the proofs of our results. By the above two Lemmas and using Theorem 1 and Theorem 2, we can get Theorem 3 and Theorem 4 easily. Applying Theorem 3 and Theorem 4, we can get Theorem 5 and Theorem 6 immediately as corollaries of Theorem 3 and Theorem 4.

References

- [1] B. Puža and Nguyen Anh Tuan, *On a boundary value problems for systems of ordinary differential equations*, East-West Journal of Mathematics, Vol 6, No 2 (2004), 139-151.
- [2] Kiguradeze.I.T, *Some singlar boundary value of problems for ordinary differential equations* (in Russian), Tbilisi Univ. Press, 1975.
- [3] Levin V.I, *On inequalities II* (in Russian) Mat. sbornik, 1938, 4 (46), No.2, 309 - 324