East-West J. of Mathematics: Vol. 7, No 1 (2005) pp. 69-77

AN EFFECTIVE CRITERION OF SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Nguyen Anh Tuan

Mathematics - Computer Science Department
University of Education of Ho Chi Minh city
280 An Duong Vuong, Ho Chi Minh City, Viet Nam

Abstract

New criterion of solvability of boundary value problem for system of ordinary differential equations with functional boundary conditions are constructed by method of a priori estimates.

Introduction

In this paper we will apply our result in [1] to get a new effective criterion for the existence and the uniqueness of the solutions of the following problem A:

$$x_i'(t) = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{1}$$

$$\Phi_{0i}(x_i) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{2}$$

where, for each $i \in \{1, \dots, n\}$ $\quad f_i : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Carathéodory conditions, Φ_{0i} - the linear nondecreasing continuous functional on $C\big([a, b]\big)$ - is concentrated on $\big[a_i, b_i\big] \subseteq \big[a, b\big]$ (i.e., the value of Φ_{0i} depends only on functions restricted to $[a_i, b_i]$ and the segment can be degenerated a point) and φ_i is a continuous functional on $C_n\big([a, b]\big)$. In general $\Phi_{0i}(1) = C_i \quad (i = 1, \dots, n)$. Without loss of generality we can suppose that $\Phi_{0i}(1) = 1 \quad (i = 1, \dots, n)$ to

Key words: boundary value problem, priory estimate, continuous functional.
2000 AMS Mathematics Subject Classification: 34K10

simplify the notation.

We adopt the following notations:

$[a,b]$– a segment, $-\infty < a \le a_i \le b_i \le b < +\infty$, $(i=1,\dots,n)$, $\mathbb{R}^n$ – n-dimensional real space with point $x = \big(x_i\big)_{i=1}^{n}$ normed by $\|x\| = \sum\limits_{i=1}^{n} |x_i|$.

$C_n\big([a,b]\big)$ is the spaces of continuous n-dimensional vector-valued functions on $[a,b]$ with the norm

$$\|x\| = \max\Big\{\sum_{i=1}^{n} |x_i(t)| : a \le t \le b\Big\}$$

$$C^+\big([a,b]\big) = \Big\{x \in C\big([a,b]\big) : x(t) \ge 0, a \le t \le b\Big\}.$$

$L^p\big([a,b]\big)$ is the space of integrable functions on $[a,b]$ in p-power with the norm

$$\|u\|_{L^p_{([a,b])}} = \begin{cases} \Big[\int\limits_a^b |u(t)|^p\,dt\Big]^{\frac{1}{p}} & \text{for} \quad 1 \le p < +\infty \\ \text{vrai}\ \max\big\{|u(t)| : a \le t \le b\big\} & \text{for} \quad p = +\infty \end{cases}$$

$$L^p\big([a,b],\mathbb{R}_+\big) = \Big\{u \in L^p\big([a,b]\big) : u(t) \ge 0, a \le t \le b\Big\}$$

$$l(q,q_0) = \begin{cases} 1 & \text{if } 1 \le q \text{ and } q = q_0 \text{ or } q = +\infty \\ \Big(\dfrac{q_0}{q} - 1\Big)^{-\frac{1}{q_0}} \Big(\dfrac{q_0}{q\pi}\sin\dfrac{q\pi}{q_0}\Big)^{\frac{1}{q}} & \text{if } 1 \le q < q_0 < +\infty \end{cases}$$

Let us consider the Problem A. By a solution we mean an absolutely continuous n-dimensional vector-valued function on $[a,b]$, which satisfies the equation (1) for almost $t \in [a,b]$ and fulfills the boundary conditions (2) of Problem A.

2. Results

Definition Let $G = (g_i)_{i=1}^n : C\big([a,b]\big) \to \mathbb{R}^n$, $H = (h_{ij})_{i,j=1}^n : [a,b] \to \mathbb{R}_+^{n\times n}$ and
$\Psi = (\psi_i)_{i=1}^n : C_n(a,b) \to \mathbb{R}_+^n$ is a positively homogeneous nondecreasing operator. We say that

$$(G,H,\Psi) \in Nic_0\big([a,b]; a_1,\dots,a_n,b_1,\dots,b_n\big) \tag{3}$$

if the system of differential inequalities (we call the Problem B)

$$|x_i'(t) - g_i(t)x_i(t)| \le \sum_{j=1}^{n} h_{ij}(t)|x_j(t)| \qquad \text{for } a \le t \le b \quad (i=1,\dots,n) \tag{4}$$

with boundary conditions

$$\min\left\{|x_i(t)| : a_i \le t \le b_i\right\} \le \psi_i\big(|x_1(t)|,\dots,|x_n(t)|\big) \quad (i=1,\dots,n) \tag{5}$$

has only one trivial solution.

The following Theorem 1 and Theorem 2 have been proved in [1] and we state here for the convenience of hte readers.

Theorem 1. *Suppose that the following inequalities hold:*

$$\begin{aligned}\big[f_i(t,x_1,\dots,x_n) \quad -g_i(t)x_i\big]\ \mathit{sign}\ x_i \le \sum_{j=1}^n h_{ij}(t)|x_j| + \omega_i\Big(t, \sum_{j=1}^n |x_j|\Big)\\ \mathit{if}\ t\in[a_i,b],\quad x\in\mathbb{R}^n,\quad (i=1,\dots,n)\end{aligned} \tag{6_1}$$

$$\begin{aligned}\big[f_i(t,x_1,\dots,x_n) \quad -g_i(t)x_i\big]\ \mathit{sign}\ x_i \ge -\sum_{j=1}^n h_{ij}(t)|x_j| - \omega_i\Big(t, \sum_{j=1}^n |x_j|\Big)\\ \mathit{if}\ t\in[a,b_i],\quad x\in\mathbb{R}^n,\quad (i=1,\dots,n)\end{aligned} \tag{6_2}$$

$$\begin{aligned}|\varphi_i(x_1,\dots,x_n)| \le \psi_i(|x_1|,\dots,|x_n|) + r_i\Big(\sum_{j=1}^n |x_j|\Big)\\ \mathit{for\ all}\ x=(x_i)_{i=1}^n \in C_n([a,b]) \quad (i=1,\dots,n),\end{aligned} \tag{7}$$

where $G=(g_i)_{i=1}^n$, $H=(h_{ij})_{ij=1}^n$, and $\Psi=(\psi_i)_{i=1}^n$ satisfy the condition (3), the functions $\omega_i : [a,b]\times\mathbb{R}^n \to \mathbb{R}_+$ are measurable with regard to the first and nondecreasing to the second argument, $r_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing and

$$\lim_{\rho\to\infty} \frac{1}{\rho}\int_a^b \omega_i(t,\rho)\,dt = 0 = \lim_{\rho\to\infty}\frac{1}{\rho} r_i(\rho) \quad (i=1,\dots,n). \tag{8}$$

Then the problem (1), (2) has at least one solution.

Theorem 2. *Suppose that the following inequalities hold:*

$$\begin{aligned}&\Big\{\big[f_i(t,x_{11},\dots,x_{1n}) - f_i(t,x_{21},\dots,x_{2n})\big] - g_i(t)(x_{1i}-x_{2i})\Big\}\ \mathit{sign}\ (x_{1i}-x_{2i}) \le\\ &\le \sum_{j=1}^n h_{ij}(t)|x_{1j}-x_{2j}|,\ \ \mathit{if}\ a_i\le t\le b, x_1=(x_{1j})_{j=1}^n, x_2=(x_{2j})_{j=1}^n \in \mathbb{R}^n\\ &\qquad (i=1,\dots,n)\end{aligned} \tag{9_1}$$

$$\begin{aligned}&\Big\{\big[f_i(t,x_{11},\dots,x_{1n}) - f_i(t,x_{21},\dots,x_{2n})\big] - g_i(t)(x_{1i}-x_{2i})\Big\}\ \mathit{sign}\ (x_{1i}-x_{2i}) \ge\\ &\ge -\sum_{j=1}^n h_{ij}(t)|x_{1j}-x_{2j}|,\ \ \mathit{if}\ a_i\le t\le b, x_1=(x_{1j})_{j=1}^n, x_2=(x_{2j})_{j=1}^n \in \mathbb{R}^n\\ &\qquad (i=1,\dots,n)\end{aligned} \tag{9_2}$$

and

$$|\varphi_i(x_{11},\dots,x_{1n})-\varphi_i(x_{21},\dots,x_{2n})|\le\psi_i(|x_{11}-x_{12}|,\dots,|x_{1n}-x_{2n}|)$$
$$\text{for all } x_1=(x_{1i})_{i=1}^n, x_2=(x_{2i})_{i=1}^n\in C_n([a,b]) \quad (i=1,\dots,n) \tag{10}$$

where $G=(g_i)_{i=1}^n$, $H=(h_{ij})_{i,j=1}^n$, *and* $\Psi=(\psi_i)_{i=1}^n$ *satisfy the condition (3). Then the Problem A has a unique solution.*

The main results in our note are Theorems 3, 4, 5 and 6. For clarity, we state our theorems first before sketching their proofs.

Theorem 3. *Consider* $[a,b]\times\mathbb{R}^n$ *and for each* $i=1,\dots,n$, *let*

$$f_i(t,x_1,\dots,x_n)\ \text{sign}\ x_i\le\sum_{j=1}^n h_{ij}(t)|x_j|+\omega_i\Big(t,\sum_{j=1}^n|x_j|\Big)\ \text{if } a_i\le t\le b \tag{11_1}$$

$$f_i(t,x_1,\dots,x_n)\ \text{sign}\ x_i\ge-\sum_{j=1}^n h_{ij}(t)|x_j|-\omega_i\Big(t,\sum_{j=1}^n|x_j|\Big)\ \text{if } a\le t\le b_i \tag{11_2}$$

and in $C_n([a,b])$,

$$|\varphi_i(x_1,\dots,x_n)|\le\sum_{j=1}^n r_{ij}\|x_j\|_{L^{q_0}([a,b])}+r_i\Big(\sum_{j=1}^n|x_j|\Big)\quad(i=1,\dots,n) \tag{12}$$

where $h_{ij}\in L^{p_{ij}}_{([a,b],\mathbb{R}_+)}$, $p_{ij}\ge1$, $\dfrac{1}{p_{ij}}+\dfrac{1}{q_{ij}}=1$, $q_{ij}\le q_0$, $r_{ij}\in\mathbb{R}_+$ $(i,j=1,\dots,n)$, $\omega_i:[a,b]\times\mathbb{R}_+\to\mathbb{R}_+$ *and* $r_i:\mathbb{R}_+\to\mathbb{R}_+$, *satisfy the conditions of Theorem 1, and the spectral radius of the matrix* $S=(s_{ij})_{i,j=1}^n$,

$$s_{ij}=(b-a)^{\frac{1}{q_0}}r_{ij}+(b-a)^{\frac{1}{q_{ij}}}l(q_{ij},q_0).h^*_{ij}\quad(i,j=1,\dots,n)$$
$$h^*_{ij}=\max\Big\{\|h_{ij}\|_{L^{p_{ij}}_{([a_i,b])}},\|h_{ij}\|_{L^{p_{ij}}_{([a,b_i])}}\Big\}\quad(i,j=1,\dots,n)$$

is less than 1. Then the Problem A has at least one solution.

Theorem 4. *Consider* $[a,b]\times\mathbb{R}^n$ *and for each* $i=1,\dots,n$, *let*

$$\big[f_i(t,x_{11},\dots,x_{1n})-f_i(t,x_{21},\dots,x_{2n})\big]\ \text{sign}\ (x_{1i}-x_{2i})\le$$
$$\le\sum_{j=1}^n h_{ij}(t)|x_{1j}-x_{2j}|,\ \text{if } a_i\le t\le b \tag{13_1}$$

$$\big[f_i(t,x_{11},\dots,x_{1n})-f_i(t,x_{21},\dots,x_{2n})\big]\ \text{sign}\ (x_{1i}-x_{2i})\ge$$
$$\ge-\sum_{j=1}^n h_{ij}(t)|x_{1j}-x_{2j}|,\ \text{if } a\le t\le b_i \tag{13_2}$$

and in $C_n([a,b])$,

$$|\varphi_i(x_{11},\dots,x_{1n}) - \varphi_i(x_{21},\dots,x_{2n})| \le \sum_{j=1}^{n} r_{ij}\|x_{1j}-x_{2j}\|_{L^{q_0}_{([a,b])}} \tag{14}$$

where $h_{ij} \in L^{p_{ij}}_{\left([a,b],\mathbb{R}_+\right)}$ *and* $r_{ij} \in \mathbb{R}_+ \quad (i,j=1,\dots,n)$ *satisfy the conditions of Theorem 3. Then the Problem A has a unique solution.*

The following theorem shows the existence of our problem.

Theorem 5. *Let in* $[a,b]\times\mathbb{R}^n$ *and for each* $i=1,\dots,n$

$$f_i(t,x_1,\dots,x_n)\ \textit{sign}\ x_i \le g_i(t)|x_i| + \sum_{j=1}^{n} h_{ij}(t)|x_j| + \omega_i\Big(t,\sum_{j=1}^{n}|x_j|\Big) \quad \textit{if } a_i \le t \le b \tag{15$_1$}$$

$$f_i(t,x_1,\dots,x_n)\ \textit{sign}\ x_i \ge g_i(t)|x_i| - \sum_{j=1}^{n} h_{ij}(t)|x_j| - \omega_i\Big(t,\sum_{j=1}^{n}|x_j|\Big) \quad \textit{if } a \le t \le b_i \tag{15$_2$}$$

and in $C_n([a,b])$, $i=1,\dots,n$

$$|\varphi_i(x_1,\dots,x_n)| \le \sum_{j=1}^{n}\psi_{ij}(|x_j|) + r_i\Big(\sum_{j=1}^{n}|x_j|\Big) \quad (i=1,\dots,n) \tag{16}$$

where $h_{ij} \in L^{p_{ij}}_{([a,b],\mathbb{R}_+)}$, $p_{ij} \ge 1, \dfrac{1}{p_{ij}} + \dfrac{1}{q_{ij}} = 1, q_{ij} \le q_0, \quad (i,j=1,\dots,n)$, $g_i \in L([a,b])$ $(i=1,\dots,n)$, $\omega_i : [a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ *and* $r_i : \mathbb{R}_+ \to \mathbb{R}_+, \quad (i=1,\dots,n)$ *satisfy the conditions of Theorem 1, the continuous functionals* $\psi_{ij} : C^+([a,b]) \to \mathbb{R}_+$ *are sublinear non-decreasing and the spectral radius of the matrices*

$$\Psi^* = \big(\psi^*_{ij}\big)_{i,j=1}^{n} = \big(\gamma_i\psi_{ij}(1)\big)_{i,j=1}^{n} \tag{17$_1$}$$

and

$$S = (s_{ij})_{i,j=1}^{n} = \big(E-\Psi^*\big)^{-1}\Big((b-a)^{\frac{1}{q_{ij}}}\gamma_i\psi_{ij}(1)h^{**}_{ij}\Big)_{i,j=1}^{n} + \Big((b-a)^{\frac{1}{q_{ij}}} l(q_{ij},q_0)h^{**}_{ij}\Big)_{i,j=1}^{n} \tag{17$_2$}$$

are less than 1 where

$$\gamma_i = \max\Big\{\exp\int_{a_i}^{b}|g_i(\tau)|d\tau, \exp\int_{a}^{b_i}|g_i(\tau)|d\tau\Big\}, \quad (i=1,\dots,n)$$

$$h_{ij}^{**} = \max\Big\{\Big\|h_{ij}(t)\exp\int\limits_{a_i}^{b}|g_i(s)|ds\Big\|_{L^{p_{ij}}_{([a_i,b])}}, \Big\|h_{ij}(t)\exp\int\limits_{a}^{b_i}|g_i(s)|ds\Big\|_{L^{p_{ij}}_{([a,b_i])}}\Big\}$$
$$(i,j=1,\dots,n)$$

Then Problem A has a solution.

And now for the uniqueness, we get

Theorem 6. *Consider $[a,b]\times\mathbb{R}^n$ and for each $i=1,\dots,n$, let*

$$\big[f_i(t,x_{11},\dots,x_{1n})-f_i(t,x_{21},\dots,x_{2n})\big]\ sign\ (x_{1i}-x_{2i}) \le g_i(t)|x_{1i}-x_{2i}|+ + \sum_{j=1}^{n} h_{ij}(t)|x_{1j}-x_{2j}|,\ \text{if } a_i \le t \le b, \tag{18_1}$$

$$\big[f_i(t,x_{11},\dots,x_{1n})-f_i(t,x_{21},\dots,x_{2n})\big]\ sign\ (x_{1i}-x_{2i}) \ge g_i(t)|x_{1i}-x_{2i}|- - \sum_{j=1}^{n} h_{ij}(t)|x_{1j}-x_{2j}|,\ \text{if } a \le t \le b_i \tag{18_2}$$

and in $C_n([a,b])$, $(i=1,\dots,n)$

$$|\varphi_i(x_{11},\dots,x_{1n})-\varphi_i(x_{21},\dots,x_{2n})| \le \sum_{j=1}^{n}\psi_{ij}(|x_{1j}-x_{2j}|) \tag{19}$$

where g_i, h_{ij} and ψ_{ij} $(i,j=1,\dots,n)$ satisfy the conditions in Theorem 5. Then the Problem A has a unique solution.

The proofs of our Theorems are based on the followoing two lemmas.

Lemma 7 *Let $g_i(t)\equiv 0, h_{ij}\in L^{p_{ij}}_{([a,b],\mathbb{R}_+)}$, $p_{ij}\ge 1$, $\dfrac{1}{p_{ij}}+\dfrac{1}{q_{ij}}=1$, $q_{ij}\le q_0$, $(i,j=1,\dots,n)$*

$$\psi_i(|x_1|,\dots,|x_n|)=\sum_{j=1}^{n} r_{ij}\|x_j\|_{L^{q_0}_{([a,b])}}, \qquad (i=1,\dots,n) \tag{20}$$

where each $r_{ij}\in\mathbb{R}_+$ $(i,j=1,\dots,n)$ and the spectral radius of the matrix S with elements defined in Theorem 3 is less than 1. Then (3) holds for (G,H,Ψ).

Proof Let the vector function $x(t)=(x_i(t))_{i=1}^n$ be the solution of the problem (4), (5). We shall prove that this solution is zero. Choose $t_i\in[a_i,b_i]$ such that

$$|x_i(t_i)| = \min\big\{|x_i(t)| : a_i\le t\le b_i\big\}, \quad (i=1,\dots,n) \tag{21}$$

Then by intergrating relations (4) and using relations (5), (20) and Hölder inequality, we obtain

$$|x_i(t)| \le |x_i(t_i)| + \sum_{j=1}^{n} \Big| \int_{t_i}^{t} h_{ij}(\tau)|x_j(\tau)|d\tau \Big| \le$$

$$\le |x_i(t_i)| + \sum_{j=1}^{n} \Big| \int_{t_i}^{t} h_{ij}^{p_{ij}}(\tau)d\tau \Big|^{\frac{1}{p_{ij}}} \Big| \int_{t_i}^{t} \big|x_j(\tau)\big|^{q_{ij}} d\tau \Big|^{\frac{1}{q_{ij}}} \quad (22)$$

and

$$\|x_i(t)\|_{L^{q_0}_{([a,b])}} \le (b-a)^{\frac{1}{q_0}} \sum_{j=1}^{n} r_{ij} \|x_j\|_{L^{q_0}_{([a,b])}} +$$

$$+ \sum_{j=1}^{n} h_{ij}^{*} \Big[\int_{a}^{b} \Big| \int_{t_i}^{t} \big|x_j(\tau)\big|^{q_{ij}} d\tau \Big|^{\frac{q_0}{q_{ij}}} \Big]^{\frac{1}{q_0}} \qquad (i = 1, \ldots, n) \quad (23)$$

By a lemma of Levin (see [2] lemma 4.7)

$$\Big[\int_{a}^{b} \Big| \int_{t_j}^{t} \big|x_j(\tau)\big|^{q_{ij}} d\tau \Big|^{\frac{q_0}{q_{ij}}} dt \Big]^{\frac{1}{q_0}} \le (b-a)^{\frac{1}{q_{ij}}} l(q_{ij}, q_0) \|x_j\|_{L^{q_0}_{([a,b])}} \quad (24)$$

Consequently we obtain from (23) that

$$(E - S)\big(\|x_i\|_{L^{q_0}_{([a,b])}}\big)_{i=1}^{n} \le 0 \quad (25)$$

where the E-matrix unit $S = (s_{ij})_{i,j=1\ldots n}^{n}$ is defined in Theorem 3. Since the spectral radius of the matrix is less than 1, it follows from (25) that

$$\|x_i\|_{L^{q_0}_{([a,b])}} = 0 \quad (i = 1, \ldots, n)$$

Therefore $x_i \equiv 0 \qquad (i = 1, \ldots, n)$, proving our Lemma 7. □

Lemma 8 *Let* $g_i : [a,b] \to \mathbb{R}$, $g_i \in L([a,b]) \quad (i = 1, \ldots, n)$, $h_{ij} \in L^{p_{ij}}([a,b], \mathbb{R}_+)$, $\dfrac{1}{p_{ij}} + \dfrac{1}{q_{ij}} = 1$, $q_{ij} \le q_0$, $(i,j = 1, \ldots, n)$

$$\psi_i(|x_1|, \ldots, |x_n|) = \sum_{j=1}^{n} \psi_{ij}(|x_j|) \quad (i = 1, \ldots, n) \quad (26)$$

where each $\psi_{ij} : C^{+}([a,b]) \to \mathbb{R}_+ \qquad (i = 1, \ldots, n)$ *are sublinear nondecreasing continuous functionals and the spectral radius of the matrices* Ψ^* *and* S *defined in (17) is less than 1. Then (3) holds for* (G, H, Ψ).

Proof Let the vector function $x(t)=(x_i(t))_{i=1}^n$ be the solution of the Problem B. We shall prove that this solution is zero.

Choose $t_i \in [a_i, b_i]$ such that

$$|x_i(t_i)| = \min\big\{|x_i(t)| : a_i \le t \le b_i\big\}, \quad (i=1,\dots,n) \tag{27}$$

The by integrating (4) and using (5), Hölder inequality and Lemma of Levin (see [3], Lemma 1.7) we obtain

$$|x_i(t)| \le \Big(\exp\int_{t_i}^{t} g_i(\tau)\ \text{sign}\ (\tau - t_i)d\tau\Big)|x_i(t_i)| +$$
$$+\sum_{j=1}^{n}\Big|\int_{t_i}^{t} h_{ij}(\tau)\exp\Big(\int_{\tau}^{t} g(s)\ \text{sign}\ (s-\tau)ds\Big)|x_j(\tau)|d\tau\Big|, \quad (i=1,\dots,n) \tag{28}$$

$$\Big\|\int_{t_i}^{t} h_{ij}(\tau)\exp\Big(\int_{\tau}^{t} g_i(s)\ \text{sign}\ (s-\tau)ds\Big)|x_j(\tau)|d\tau\Big\|_{L^{q_0}_{([a,b])}} \le$$
$$\le \Big\|\Big|\int_{t_i}^{t} h_{ij}^{p_{ij}}(\tau)\exp\big(p_{ij}\int_{\tau}^{t} g_i(s)\ \text{sign}\ (s-\tau)ds\big)d\tau\Big|^{\frac{1}{p_{ij}}}\Big|\int_{t_i}^{t}|x_j(\tau)^{q_{ij}}d\tau\Big|^{\frac{1}{q_{ij}}}\Big\|_{L^{q_0}_{([a,b])}}$$
$$\le h_{ij}^{**}\Big\|\Big|\int_{t_i}^{t}|x(\tau)|^{q_{ij}}d\tau\Big|^{\frac{1}{q_{ij}}}\Big\|_{L^{q_o}([a,b])} \le (b-a)^{\frac{1}{q_{ij}}}\, l(q_{ij},q_0)\|x_j\|_{L^{q_0}_{([a,b])}} h_{ij}^{**}$$

and

$$\|x_i\|_{L^{q_0}_{([a,b])}} \le (b-a)^{\frac{1}{q_0}}\gamma_i|x_i(t_i)| + \sum_{j=1}^{n}(b-a)^{\frac{1}{q_{ij}}}\, l(q_{ij},q_0)h_{ij}^{**}\|x_j\|_{L^{q_0}_{([a,b])}}\ (i=1,\dots,n) \tag{29}$$

Substituting the inequality (28) into boundary condition (5) and using (26), (27) we have

$$|x_i(t_i)| \le \sum_{j=1}^{n}\psi_{ij}(|x_j|) \le |x_i(t_i)|\psi_{ij}^{*} \quad +$$
$$+\sum_{j=1}^{n}\psi_{ij}\Big(\Big|\int_{t_i}^{t} h_{ij}^{p_{ij}}(\tau)\exp\big(p_{ij}\int_{\tau}^{t} g_i(s)\ \text{sign}(s-\tau)ds\big)d\tau\Big|^{\frac{1}{p_{ij}}}\Big|\int_{t_i}^{t}|x_j(\tau)|^{q_{ij}}d\tau\Big|^{\frac{1}{q_{ij}}}\Big) \le$$
$$\le |x_i(t_i)|\psi_{ij}^{*} + \sum_{j=1}^{n}(b-a)^{\frac{1}{q_{ij}}-\frac{1}{q_0}}\psi_{ij}(1)h_{ij}^{**}\|x_j\|_{L^{q_0}_{([a,b])}}\quad, \quad (i=1,\dots,n).$$

Since the spectral radius of the matrix Ψ^* is less than 1, we get

$$(|x_i(t_i)|)_{i=1}^n \leq (E-\psi^*)^{-1}\Big((b-a)^{\frac{1}{q_{ij}}-\frac{1}{q_0}}\psi_{ij}(1)h_{ij}^{**}\Big)_{i,j=1}^n \big(\|x_j\|_{L^{q_o}_{([a,b])}}\big)_{j=1}^n \quad (30)$$

Consequently, from (29), (30) we obtain

$$\big(\|x_i\|_{L^{q_0}_{([a,b])}}\big)_{i=1}^n \leq S\big(\|x_j\|_{L^{q_0}_{([a,b])}}\big)_{j=1}^n \quad (31)$$

Since spectral of radius of the matrix S is less than 1, we obtain

$$\big(\|x_i\|_{L^{q_0}_{([a,b])}}\big)_{i=1}^n \leq 0,$$

and our Lemma has been proved. □

Proofs of our Theorems We now can sketch the proofs of our results. By the above two Lemmas and using Theorem 1 and Theorem 2, we can get Theorem 3 and Theorem 4 easily. Applying Theorem 3 and Theorem 4, we can get Theorem 5 and Theorem 6 immediately as corollaries of Theorem 3 and Theorem 4.

References

[1] B.Půža and Nguyen Anh Tuan, *On a boundary value problems for systems of ordinary differential equations*, East-West Journal of Mathematics, Vol 6, No 2 (2004), 139-151.

[2] Kiguradeze.I.T, *Some singluar boundary value of problems for ordinary differential equations* (in Russian), Tbilisi Univ. Press, 1975.

[3] Levin V.I, *On inequalities II* (in Russian) Mat. sbornik, 1938, 4 (46), No.2, 309 - 324