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BOUNDED p-VARIATION IN THE MEAN

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Abstract

We introduce the notion of bounded p-variation in the sense of L_p-norm. We obtain a Riesz type result for functions of bounded p-variation, and study conditions under which the Nemytskii operator maps the space of bounded p-variation into itself.

1 Introduction

The circle group T is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$, where, as indicated by the notation, $2\pi\mathbb{Z}$ is the group of integral multiples of 2π. There is a natural identification between functions on T and 2π-periodic functions on $\mathbb{R}$, which allows an implicit introduction of notions such as continuity, differentiability, etc for functions on T. The Lebesgue measure on T also can be defined by means of the preceding identification: a function f is integrable on T if the corresponding 2π-periodic function, which we denote again by f, is integrable on $[0, 2\pi]$, and we set

$$\int_T f(t)dt = \int_0^{2\pi} f(x)dx.$$

Let f be a real-value function in L_1 on the circle group T. We define the corresponding interval function by $f(I) = f(b) - f(a)$, where I denotes the interval $[a, b]$. Let $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$ and $I_{kx} = [x + t_{k-1}, x + t_k]$, if

$$V_m(f, T) = \sup \left\{ \int_T \sum_{k=1}^{n} |f(I_{kx})|\, dx \right\} < \infty$$

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where the supremum is taken over all partition of $[0, 2\pi]$, then f is said to be of variation in the mean (or bounded variation in L_1-norm).

We denote the class of all functions which are of bounded variation in the mean by BVM. This concept was introduce by Móricz and Siddiqi [1], who investigated the convergence in the mean of the partial sums of $S[f]$, the Fourier series of f.

If f is of bounded variation ($f \in BV$) with variation $V(f,T)$, then

$$\int_T \sum_{k=1}^{n} |f(I_{kx})|\, dx \le 2\pi V(f,T),$$

and so it is clear that $BV \subset BVM$. A straightforward calculation shows that BVM is a Banach space with norm

$$\|f\|_{BVM} = \|f\|_1 + V_m(f,T).$$

In the present paper we introduce the concept of bounded p-variation in the mean in the sense of $L_p[0,2\pi]$ norm (see Definition 2.1) and prove a characterization of the class BV_pM in terms of this concept.

In 1910 in [2], F. Riesz defined the concept of bounded p-variation $(1 \le p < \infty)$ and proved that for $1 < p < \infty$ this class coincides with the class of functions f, absolutely continuous with derivative $f' \in L_p[a,b]$. Moreover, the p-variation of a function f on $[a,b]$ is given by $\|f'\|_{Lp[a,b]}$, that is

$$V_p(f;[a,b]) = \|f'\|_{Lp[a,b]} \tag{1}$$

In this paper we obtain an analogous result for the class BV_pM. More precisely we show that if $f \in BV_pM$ is such that f' is continuous on $[0,2\pi]$, then $f' \in Lp[0,2\pi]$ and

$$V_p^m(f) = 2\pi \|f'\|_{Lp}^p$$

(See theorem 2.5).

2 Bounded p-variation in the mean

Definition 2.1. let $f \in L_p[0,2\pi]$ with $1 < p < \infty$. Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0,2\pi]$ if

$$V_p^m(f,T) = sup \left\{ \sum_{k=1}^{n} \int_T \frac{|f(x+t_k) - f(x+t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx \right\} < \infty, \tag{2}$$

where the supremum is taken over all partitions P of $[0,2\pi]$, then f is said to be of bounded p-variation in the mean.

We denote the class of all functions which are of bounded p-variation in the mean by BV_pM, that is

$$BV_pM = \left\{ f \in L_p[0,2\pi] : V_p^m(f,T) < \infty \right\} \tag{3}$$

Remark 2.1. For $1 < p < \infty$, it is not hard to prove that

$$\|f\|_{BV_pM} = \|f\|_{L_p} + \left\{ V_p^m(f,T) \right\}^{1/p} \tag{4}$$

defines a norm on BV_pM.

Proposition 2.1. Let f and g be two functions in BV_pM, then

$$\begin{array}{ll} i) & f+g \in BV_pM, \\ ii) & kf \in BV_pM, \quad \text{for any} \quad k \in \mathbb{R}. \end{array}$$

In order words, BV_pM is a vector space.

Moreover

$$V_p^m(f+g,T) \le 2^{p-1}\left[V_p^m(f,T) + V_p^m(g,T) \right],$$

and

$$V_p^m(kf,T) = |k|^p V_p^m(f,T).$$

Theorem 2.1. For $1 < p < \infty$, $BV_pM \subset BVM$ and

$$V_m(f,T) \le (2\pi)^{2-\frac{2}{p}} \left[V_p^m(f,T) \right]^{\frac{1}{p}}. \tag{5}$$

Proof Let $P: 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0,2\pi]$ and consider $f \in BV_pM$, then by Hölder's inequality we obtain

$$\begin{aligned} &\sum_{k=1}^{n} \int_0^{2\pi} |f(x+t_k) - f(x+t_{k-1})|\, dx \\ &\le \left(\sum_{k=1}^{n} \int_0^{2\pi} |t_k - t_{k-1}|\, dx \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} \int_0^{2\pi} \frac{|f(x+t_k) - f(x+t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx \right)^{\frac{1}{p}} \\ &\le (2\pi)^{2-\frac{2}{p}} \left[V_p^m(f,T) \right]^{\frac{1}{p}}. \end{aligned} \tag{6}$$

Thus $f \in BVM$, therefore $BV_pM \subset BVM$. By (6) we obtain (5). This completes the proof of Theorem 2.1. □

Theorem 2.2. $Lip\,[0,2\pi] \subset BV_pM$, where $Lip\,[0,2\pi]$ denotes the class of all functions which are Lipschitz on $[0,2\pi]$.

Proof Let $f \in Lip\,[0, 2\pi]$, then there exists a positive constant $M > 0$ such that

$$|f(x) - f(y)| \leq M\,|x - y|\,,$$

for all $x, y \in [0, 2\pi]$.

Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$, thus

$$|f\,(x + t_k) - f\,(x + t_{k-1})| \leq M\,|t_k - t_{k-1}|\,, \tag{7}$$

from (7) we have

$$\sum_{k=1}^{n} \int_0^{2\pi} \frac{|f\,(x + t_k) - f\,(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx \leq 4\pi M^p \tag{8}$$

by (8) we get $f \in BV_pM$. This completes the proof of the Theorem 2.2. □

Remark 2.2 By Theorem 2.1 and 2.2, we can observe the following embedding:

$$Lip\,[0, 2\pi] \subset BV_pM \subset BVM.$$

Theorem 2.3. *Let $f \in Lip\,[0, 2\pi]$ and $g \in BV_pM$. Then $f \circ g \in BV_pM$.*

Proof Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$ then

$$\begin{aligned}
&\sum_{k=1}^{n} \int_0^{2\pi} \frac{|f\,(g\,(x + t_k)) - f\,(g\,(x + t_{k-1}))|^p}{|t_k - t_{k-1}|^{p-1}} dx \\
&\leq M^p \sum_{k=1}^{n} \int_0^{2\pi} \frac{|g\,(x + t_k) - g\,(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx.
\end{aligned}$$

Thus

$$\sum_{k=1}^{n} \int_0^{2\pi} \frac{|f\,(g\,(x + t_k)) - f\,(g\,(x + t_{k-1}))|^p}{|t_k - t_{k-1}|^{p-1}} dx \leq M^p V_p^m\,(g, T) \tag{9}$$

for all partitions of $[0, 2\pi]$. By (9) we obtain

$$V_p^m\,(f \circ g, T) \leq M^p V_p^m\,(g, T)\,.$$

Hence $f \circ g \in BV_pM$. □

Theorem 2.4. BV_pM, equipped with the norm defined in Remark 2.1. is a Banach space.

Proof Let $\{f_n\}_n$ be a Cauchy sequence in BV_pM. Then for any $\epsilon > 0$ there exists a positive integer no such that

$$\|f_n - f_m\|_{BV_pM} < \epsilon \quad \text{whenever} \quad n, m \geq n_0, \tag{10}$$

From (3) and (9) we have

$$\|f_n - f_m\|_{L_p} \quad \leq \quad \|f_n - f_m\|_{BV_pM} < \epsilon$$

Whenever $n, m \geq n_0$, this implies that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in Lp since this space is complete, thus $\lim_{n\to\infty} f_n$ exits, call it f. By Fatou's lemma and (3) we obtain

$$\|f_n - f_m\|_{BV_pM} \leq \liminf_{m\to\infty} \|f_n - f_m\|_{L_p} + \liminf_{m\to\infty} \left\{V_p^m(f_n - f_m, T)\right\}^{1/p} < \epsilon$$

whenever $n \geq n_0$.

Finally we need to prove that $f \in BV_pM$. In other to do that we invoke Fatou's lemma again.

$$\|f\|_{BV_pM} \leq \liminf_{n\to\infty} \|f_n\|_{L_p} + \liminf \left\{V_p^m(f_n, T)\right\}^{1/p} < \infty.$$

Thus $f \in BV_pM$.

This completes the proof of Theorem 2.4 □

Theorem 2.5. Let $f \in BV_pM$ such that f' is continuous on $[0, 2\pi]$, then $f' \in L_p[0, 2\pi]$ and

$$V_m^p(f) = 2\pi\|f'\|_{L_p}^p \tag{11}$$

Proof Let $P : 0 = t_0 < t_1 < \ldots < t_n = 2\pi$ be a partition of $[0, 2\pi]$. By the Mean value theorem there exists $\epsilon_k \in (x + t_{k-1}, x + t_k)$ for any $x \in [0, 2\pi]$ such that for $1 < p < \infty$

$$\frac{|f(x + t_k) - f(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} = |f'(\epsilon_k)|^p(t_k - t_{k-1}) \tag{12}$$

by (12) we obtain

$$2\pi \lim_{\|p\|\to 0} \sum_{k=1}^{n} |f'(\epsilon_k)|^p(t_k - t_{k-1}) \leq \sum_{k=1}^{n} \int_0^{2\pi} \frac{|f(x + t_k) - f(x + t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} dx \tag{13}$$

from (13) we have

$$2\pi \int_0^{2\pi} |f'(x)|^p dx \leq V_p^m(f). \tag{14}$$

Thus (14) implies that $f' \in L_p[0, 2\pi]$ and also we have

$$2\pi\|f'\|_{L_p}^p \leq V_p^m(f) \tag{15}$$

on the other hand

$$f(x + t_k) - f(x + t_{k-1}) = \int_{x+t_{k-1}}^{x+t_{k-1}} f'(t)dt \tag{16}$$

by Hölder's inequality we obtain

$$\left|\int_{x+t_{k-1}}^{x+t_{k-1}} f'(t)dt\right|^p \leq \left(\int_{x+t_{k-1}}^{x+t_{k-1}} |f'(t)|^p dt\right) |t_k - t_{k-1}|^{p-1},$$

hence by (16) we get

$$\frac{|f(x+t_k) - f(x+t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} \leq \int_{x+t_{k-1}}^{x+t_{k-1}} |f'(t)|, dt$$

then

$$\sum_{k=1}^{n} \int_0^{2\pi} \frac{|f(x+t_k) - f(x+t_{k-1})|^p}{|t_k - t_{k-1}|^{p-1}} \leq 2\pi \int_0^{2\pi} |f'(x)|^p dx. \tag{17}$$

From (17) we finally have

$$V_p^m(f) \leq 2\pi \|f'\|_{L_p}^p \tag{18}$$

Combining (15) and (16) we obtain (11)

□

3 Substitution Operators

Let $\Omega \subset \mathbb{R}$ be a bounded open set. A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the Carathéodory conditions if

i) For every $t \in \mathbb{R}$, the function $f(\cdot, t) : \Omega \to \mathbb{R}$ is Lebesgue measurable

ii) For a.e. $x \in \Omega$, the fuction $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.

Set

$$\mathcal{M} = \{\varphi : \Omega \to \mathbb{R} : \varphi \quad \text{is Lebesgue measurable}\}$$

for each $\varphi \in \mathcal{M}$ define the operator

$$(N_f\varphi)(t) = f(t, \varphi(t))$$

The operator N_f is said to be the substitution or Nemytskii operator generated by the function f.
The purpose of this section is to present one condition under which the operator N_f maps BV_pM into itself.

Lemma 3.1. $N_f : BV_pM \to BV_pM$ if there exits a constant $L > 0$ such that $|f(s, \varphi(s) - f(t, \varphi(t))| \leq L|\varphi(s) - \varphi(t)|$ for every $\varphi \in \mathcal{M}$.

Proof Let $\varphi \in BV_pM$, then

$$\begin{aligned}
&\sup\left\{\sum_{k=1}^{n}\int_{\Gamma}\frac{|(N_f\varphi)(x+t_k)-(N_f\varphi)(x+t_{k-1})|^p}{|t_k-t_{k-1}|^{p-1}}dx\right\}\\
=&\sup\left\{\sum_{k=1}^{n}\int_{\Gamma}\frac{|f(x+t_k,\varphi(x+t_k))-f(x+t_{k-1},\varphi(x+t_{k-1}))|^p}{|t_k-t_{k-1}|^{p-1}}dx\right\}\\
\leq& L\sup\left\{\sum_{k=1}^{n}\int_{\Gamma}\frac{|\varphi(x+t_k)-\varphi(x+t_{k-1})|^p}{|t_k-t_{k-1}|^{p-1}}dx\right\}<\infty.
\end{aligned}$$

Thus $N_f \in BV_pM$. $\square$

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