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Annihilator of Tensor Product of S-acts

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Abstract

For S-acts ${}_SM$ and U_S, let $Ann_{_M}(U) = \{(m, m') \in M \times M | u \otimes m = u \otimes m'$ for any $u \in U\}$. Then U_S is called ${}_SM$-faithful if $Ann_{_M}(U)$ is the identity relation on M. If U_S is ${}_SM$-faithful for any S-act ${}_SM$, then we call U_S completely faithful. The present paper discusses proerties of ${}_SM$-faithful(completely faithful) S-acts. The structures of ${}_SM$-faithful(completely faithful) right S-acts are characterized. Some related results are also obtained.

1 Preliminaries

In this paper, we shall always let semigroup S mean a monoid and all S-acts be unitary. We denote the category of all right (left) S-acts by $Act - S$ ($S - Act$). Let A_S be a right S-act. An equivalence relation ρ on A is called an S-congruence or a congruence on A_S if for any $a, a' \in A$, $(a, a') \in \rho$ implies $(as, a's) \in \rho$ for any $s \in S$.

If ${}_SM$ is a left S-act, then the cartesian product $M \times M$ with the operation $s \cdot (m, m') = (sm, sm')$ for all $s \in S$, $m, m' \in M$ is a left S-act. Let $f : {}_SM \longrightarrow {}_SN$ be an S-homomorphism. We denote by $Imf = \{f(m) | m \in M\}$ and $kerf = \{(m, m') \in M \times M | f(m) = f(m')\}$. It is clear that $(f, f) : {}_S(M \times M) \longrightarrow {}_S(N \times N)$ with $(f, f)((m, m')) := (f(m), f(m'))$, $m, m' \in M$, is an S-homomorphism, and $kerf$ is a congruence on ${}_SM$.

Let X be a set. Denote by $\triangle_X = \{(x, x) | x \in X\}$ and $\nabla_X = X \times X$. For a subact ${}_SN$ of ${}_SM$, $\rho_{_N} = (N \times N) \cap \triangle_M$ is clearly a congruence on ${}_SM$ which is called the Rees congruence and we denote the quotient act $M/\rho_{_N}$ by M/N.

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Let U_S, M_S be right S-acts. As in module theory, the trace and the reject of U in M, respectively, are defined by

$$Tr_{_M}(U) = \cup\{Imf | f \in Hom_{_S}(U, M)\}$$

and

$$Rej_{_M}(U) = \cap\{\ker f | f \in Hom_{_S}(M, U)\}.$$

We say that U_S generates (cogenerates) M_S in case $Tr_{_M}(U) = M$ ($Rej_{_M}(U) = \triangle_M$). U_S is called a generator (cogenerator) of $Act - S$ in case $Tr_{_M}(U) = M$ ($Rej_{_M}(U) = \triangle_M$) for all $M_S \in Act - S$. Denoted by $\mathbf{r_S}(M) := \{(s, s') \in S \times S \mid ms = ms',\ \forall m \in M\}$ the annihilator of right S-act M_S. It is clear that $\mathbf{r_S}(M)$ is a congruence on M_S.

Let $(A_\alpha)_{\alpha\in I}$ be a family of right S-acts. Then, the coproduct $\coprod_{\alpha\in I} A_\alpha$ of $(A_\alpha)_{\alpha\in I}$ is the disjoint union of $(A_\alpha)_{\alpha\in I}$.

We call A_S a faithful right S-act if for any $s,\ t \in S$ the equality $as = at$ for all $a \in A$ implies $s = t$. Obviously, A_S is faithful if and only if $\mathbf{r_S}(A) = \triangle_S$. A_S is called a strongly faithful right S-act if for any $s,\ t \in S$ the equality $as = at$ for some $a \in A$ implies $s = t$.

For other definitions and terminologies not mentioned in this paper, the reader is refered to [3].

2 Faithfulness

Definition 2.1. *Let U_S and ${}_SM$ be S-acts, $U \otimes M$ the tensor product of U and M. Then*

$$Ann_{_M}(U) = \{(m, m') \in M \times M \mid u \otimes m = u \otimes m',\ \forall u \in U\}$$

is called the annihilator in M of U. Call U_S to be ${}_SM$-faithful in case $Ann_{_M}(U) = \triangle_M$.

It is obvious that $Ann_{_S}(U) = \mathbf{r_S}(U)$ for any right S-act U_S.

Proposition 2.2. *Let U_S and ${}_SM$ be S-acts. Then $Ann_{_M}(U)$ is the unique smallest congruence λ on ${}_SM$ such that U is M/λ-faithful.*

Proof Suppose that $\lambda = Ann_{_M}(U) = \{(m_1, m_2) \in M \times M \mid u \otimes m_1 = u \otimes m_2,\ \forall u \in U\}$. Clearly, λ is a congruence on ${}_SM$.

Assume that $(\bar{m}_1, \bar{m}_2) \in Ann_{_{M/\lambda}}(U)$. Then, we have $u \otimes \bar{m}_1 = u \otimes \bar{m}_2$ for all $u \in U$. Thus, there exist $x_1, x_2, \cdots, x_n \in U$, $\bar{y}_2, \cdots, \bar{y}_n \in M/\lambda$, $s_1, t_1, \cdots, s_n, t_n \in S$ such that

$$\begin{aligned} u &= x_1 s_1, \\ x_1 t_1 &= x_2 s_2, \quad s_1\bar{m}_1 = t_1\bar{y}_2, \\ &\cdots\cdots \\ x_n t_n &= u, \qquad s_n\bar{y}_n = t_n\bar{m}_2. \end{aligned}$$

This implies that $(s_1m_1, t_1y_2), \cdots, (s_ny_n, t_nm_2) \in \lambda$, and then, for any $u \in U$,

$$\begin{aligned} u \otimes m_1 &= x_1s_1 \otimes m_1 = x_1 \otimes s_1m_1 = x_1 \otimes t_1y_2 = x_1t_1 \otimes y_2 \\ &= x_2s_2 \otimes y_2 = \cdots = x_ns_n \otimes y_n = x_n \otimes s_ny_n \\ &= x_n \otimes t_nm_2 = x_nt_n \otimes m_2 = u \otimes m_2 \end{aligned}$$

which shows that $(m_1, m_2) \in \lambda$ and $\bar{m}_1 = \bar{m}_2$. Therefore $Ann_{M/\lambda}(U) = \triangle_{M/\lambda}$.

Let now σ be a congruence on ${}_SM$ with $Ann_{M/\sigma}(U) = \triangle_{M/\sigma}$. Assume that $(m, m') \in \lambda$. Then $u \otimes m = u \otimes m'$ for all $u \in U$. Let $n: M \longrightarrow M/\sigma$ be the canonical epimorphism. Then $1_U \otimes n: U \otimes M \longrightarrow U \otimes M/\sigma$ is surjective and $u \otimes (m\sigma) = (1_U \otimes n)(u \otimes m) = (1_U \otimes n)(u \otimes m') = u \otimes (m'\sigma)$ for all $u \in U$. Thus, $(m\sigma, m'\sigma) \in Ann_{M/\sigma}(U) = \triangle_{M/\sigma}$ and $m\sigma = m'\sigma$, i.e., $(m, m') \in \sigma$. Hence $\lambda \subseteq \sigma$. $\square$

Proposition 2.3. *Let U_S, ${}_SM$ and ${}_SN$ be S-acts and let $f \in Hom_S(M, N)$. Then*

(a) $(f, f)(Ann_M(U)) \subseteq Ann_N(U)$. In particular, $Ann_M(U)$ is stable under endomorphisms of ${}_SM$.

(b) If f is epic and $Kerf \subseteq Ann_M(U)$, then $(f, f)(Ann_M(U)) = Ann_N(U)$.

Proof (a) Assume that $(m, m') \in Ann_M(U)$ and $u \in U$. Since $u \otimes m = u \otimes m'$ we have

$$u \otimes f(m) = (1_U \otimes f)(u \otimes m) = (1_U \otimes f)(u \otimes m') = u \otimes f(m').$$

Thus $(f(m), f(m')) \in Ann_N(U)$ and therefore, $(f, f)(Ann_M(U)) \subseteq Ann_N(U)$.

(b) It will suffice to prove that $Ann_N(U) \subseteq (f, f)(Ann_M(U))$. Let $\phi: M \longrightarrow M/Kerf$ be the canonical epimorphism. Because f is epic there exists a unique isomorphism $\bar{f}: M/Kerf \longrightarrow N$ such that $f = \bar{f}\phi$.

Assume that $(\bar{m}, \bar{m}') \in Ann_{M/Kerf}(U)$ and $u \in U$. Since $u \otimes \bar{m} = u \otimes \bar{m}'$, there exist $x_1, x_2, \cdots, x_n \in U$, $\bar{y}_2, \cdots, \bar{y}_n \in M/Kerf$, $s_1, t_1, \cdots, s_n, t_n \in S$ such that

$$\begin{aligned} u &= x_1s_1, \\ x_1t_1 &= x_2s_2, \quad s_1\bar{m} = t_1\bar{y}_2, \\ &\cdots\cdots \\ x_nt_n &= u, \quad\quad s_n\bar{y}_n = t_n\bar{m}'. \end{aligned}$$

Thus $(s_1m, t_1y_2), \cdots, (s_ny_n, t_nm') \in Kerf \subseteq Ann_M(U)$ and so

$$\begin{aligned} u \otimes m &= x_1s_1 \otimes m = x_1 \otimes s_1m = x_1 \otimes t_1y_2 = x_1t_1 \otimes y_2 \\ &= x_2s_2 \otimes y_2 = \cdots = x_ns_n \otimes y_n = x_n \otimes s_ny_n \\ &= x_n \otimes t_nm_2 = x_nt_n \otimes m' = u \otimes m'. \end{aligned}$$

Therefore, $(m, m') \in Ann_M(U)$. Hence $(\bar{m}, \bar{m}') = (\phi, \phi)((m, m')) \in (\phi, \phi)(Ann_M(U))$, i.e., $Ann_{M/Kerf}(U) \subseteq (\phi, \phi)(Ann_M(U))$.

Now, for any $(n, n') \in Ann_N(U)$, there exist unique $\bar{m}, \bar{m}' \in M/Kerf$ such that $n = \bar{f}(\bar{m})$ and $n' = \bar{f}(\bar{m}')$. Noting that $\bar{f}$ is an isomorphism, we know that $1_U \otimes \bar{f}$ is a bijection. Since $(1_U \otimes \bar{f})(u \otimes \bar{m}) = u \otimes \bar{f}(\bar{m}) = u \otimes n = u \otimes n' = u \otimes \bar{f}(\bar{m}') = (1_U \otimes \bar{f})(u \otimes \bar{m}')$, we have $u \otimes \bar{m} = u \otimes \bar{m}'$ for all $u \in U$ which shows that $(\bar{m}, \bar{m}') \in Ann_{M/Kerf}(U) \subseteq (\phi, \phi)(Ann_M(U))$. Hence

$$\begin{aligned} (n, n') &= (\bar{f}, \bar{f})((\bar{m}, \bar{m}')) \in (\bar{f}, \bar{f})(Ann_{M/Kerf}(U)) \subseteq (\bar{f}, \bar{f})((\phi, \phi)(Ann_M(U))) \\ &= (\bar{f}\phi, \bar{f}\phi)(Ann_M(U)) = (f, f)(Ann_M(U)). \end{aligned}$$

We complete the proof. □

Lemma 2.4. *Let $(A_\alpha)_{\alpha\in I}$ be a family of right S-acts, $(B_\beta)_{\beta\in J}$ a family of left S-acts and $a \otimes b, c \otimes d$ in $(\coprod_{\alpha\in I} A_\alpha) \otimes_S (\coprod_{\beta\in J} B_\beta)$. Then $a \otimes b = c \otimes d$ in $(\coprod_{\alpha\in I} A_\alpha) \otimes_S (\coprod_{\beta\in J} B_\beta)$ if and only if $a \otimes b = c \otimes d$ in $A_\alpha \otimes_S B_\beta$ for some $\alpha \in I, \beta \in J$.*

Proof *sufficiency* is obvious.

Necessity. Suppose $a \otimes b = c \otimes d$ in $(\coprod_{\alpha\in I} A_\alpha) \otimes_S (\coprod_{\beta\in J} B_\beta)$. Then there exist $a_1, a_2, \cdots, a_n \in \coprod_{\alpha\in I} A_\alpha$, $b_2, \cdots, b_n \in \coprod_{\beta\in J} B_\beta$, $u_1, v_1, \cdots, u_n, v_n \in S$, such that

$$\begin{aligned} a &= a_1u_1, \\ a_1v_1 &= a_2u_2, \quad u_1b = v_1b_2, \\ &\cdots\cdots \\ a_nv_n &= c, \quad\ \ u_nb_n = v_nd. \end{aligned}$$

Since $a \in \coprod_{\alpha\in I} A_\alpha$ and $b \in \coprod_{\beta\in J} B_\beta$, there uniquely exist $\alpha \in I, \beta \in J$ such that $a \in A_\alpha$ and $b \in B_\beta$. Now, $a_1u_1 = a \in A_\alpha$ implies that $a_1 \in A_\alpha$. Otherwise, if $a_1 \in A_{\alpha'}$ with $\alpha \neq \alpha'$, then $a_1u_1 \in A_\alpha \cap A_{\alpha'}$ which contradicts that $A_\alpha \cap A_{\alpha'} = \emptyset$. So $a_2u_2 = a_1v_1 \in A_\alpha$ and $a_2 \in A_\alpha$. Repeating this process, we conclude $a_3, \cdots, a_n, c \in A_\alpha$. Similarly, we have $b, b_2, \cdots, b_n, d \in B_\beta$. This shows that $a \otimes b = c \otimes d$ in $A_\alpha \otimes_S B_\beta$. □

Proposition 2.5. *Let I, J be index sets, U, $U_j \in Act - S$, $j \in J$ and M, $M_i \in S - Act$, $i \in I$. Then*

(a) $Ann_{\coprod_{i\in I} M_i}(U) = \coprod_{i\in I} Ann_{M_i}(U)$.

(b) $Ann_M(\coprod_{j\in J} U_j) = \bigcap_{j\in J} Ann_M(U_j)$.

Proof (a) It is obvious that $\coprod_{i\in I} Ann_{M_i}(U) \subseteq Ann_{\coprod_{i\in I} M_i}(U)$. Also, $\forall (m, m') \in Ann_{\coprod_{i\in I} M_i}(U)$, $\forall u \in U$, we have $u \otimes m = u \otimes m'$ in $U \otimes (\coprod_{i\in I} M_i)$. From Lemma 2.4 it follows that $u \otimes m = u \otimes m'$ in $U \otimes M_i$ for some $i \in I$, and so $(m, m') \in Ann_{M_i}(U) \subseteq \coprod_{i\in I} Ann_{M_i}(U)$. This shows (a).

(b) Clearly, $\bigcap_{j\in J} Ann_M(U_j) \subseteq Ann_M(\coprod_{j\in J} U_j)$. Conversely, if $(m, m') \in Ann_M(\coprod_{j\in J} U_j)$ and $u \in U_j \subseteq \coprod_{j\in J} U_j$, $j \in J$, then $u \otimes m = u \otimes m'$ in $(\coprod_{j\in J} U_j) \otimes M$. By Lemma 2.4, we get $u \otimes m = u \otimes m'$ in $U_j \otimes M$. Thus, $Ann_M(\coprod_{j\in J} U_j) \subseteq \bigcap_{j\in J} Ann_M(U_j)$. This shows (b). □

It is well known that each S-act has a unique indecomposable decomposition (see [4] or [2]). Now, by our Lemma 2.4, we have the following lemma.

Lemma 2.6. *Let A_S and $_SB$ be S-acts and $a\otimes b = a'\otimes b'$ in $A\otimes_S B$. Then a, a' and b, b' are in the same indecomposable subacts of A_S and $_SB$, respectively.*

Theorem 2.7. *If I is an ideal of S and $_SM \in S-Act$, then*

$$Ann_M(S/I) \subseteq (IM \times IM) \cup \triangle_M.$$

Moreover, $Ann_M(S/I) = (IM\times IM)\cup\triangle_M$ if and only if M is indecomposable.

Proof If we define

$$S/I \times M/IM \longrightarrow M/IM, \quad (\bar{s}, \tilde{m}) \longmapsto \widetilde{sm},$$

then M/IM is an S/I-act and $_S(M/IM) =_{S/I}(M/IM)$. Let

$$\phi: \ S/I \otimes_S M \longrightarrow M/IM, \quad \bar{s} \otimes m \longmapsto \widetilde{sm}.$$

Then ϕ is well-defined. In fact, suppose that $\bar{s} \otimes m = \bar{s}' \otimes m'$ for some $\bar{s}$, $\bar{s}' \in S/I$, m, $m' \in M$. Then there exist $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n \in S/I$, $y_2, \cdots, y_n \in M$, $r_1, t_1, \cdots, r_n, t_n \in S$ such that

$$\begin{array}{rcl} \bar{s} & = & \bar{x}_1 r_1, \\ \bar{x}_1 t_1 & = & \bar{x}_2 r_2, \quad r_1 m = t_1 y_2, \\ & & \cdots\cdots \\ \bar{x}_n t_n & = & \bar{s}', \qquad r_n y_n = t_n m'. \end{array}$$

Thus

$$\begin{array}{rcl} \widetilde{sm} & = & \bar{s}\tilde{m} = \bar{x}_1 r_1 \tilde{m} = \bar{x}_1 \widetilde{r_1 m} = \bar{x}_1 \widetilde{t_1 y_2} \\ & = & \bar{x}_1 t_1 \tilde{y}_2 = \cdots = \bar{x}_n t_n \tilde{m}' = \bar{s}' \tilde{m}' = \widetilde{s'm'}, \end{array}$$

i.e., ϕ is well-defined.

If $(m_1, m_2) \in Ann_M(S/I)$, then $\bar{s} \otimes m_1 = \bar{s} \otimes m_2$ and $\widetilde{sm_1} = \widetilde{sm_2}$ for all $s \in S$, in particular, $(m_1, m_2) \in (IM \times IM) \cup \triangle_M$. Thus, $Ann_M(S/I) \subseteq (IM \times IM) \cup \triangle_M$.

Suppose that $Ann_M(S/I) = (IM \times IM) \cup \triangle_M$. Then, for any $(m_1, m_2) \in M \times M$ and $a \in I$, we have $(am_1, am_2) \in Ann_M(S/I)$, in particular, $\bar{1} \otimes am_1 = \bar{1} \otimes am_2$. By Lemma 2.6, am_1, am_2 is in the same indecomposable subact of M. This implies that m_1, m_2 is in the same indecomposable subact. Hence, M is indecomposable.

Conversely, suppose M is indecomposable. It will suffice to prove that $(IM \times IM) \subseteq Ann_{M}(S/I)$. For any $(a_1m_1, a_2m_2) \in IM \times IM$, where $a_1, a_2 \in I$, $m_1, m_2 \in M$, and for any $\bar{s} \in S/I$, we have

$$\bar{s} \otimes a_1m_1 = \bar{s}a_1 \otimes m_1 = \overline{sa_1} \otimes m_1 = 0 \otimes m_1,$$

$$\bar{s} \otimes a_2m_2 = \bar{s}a_2 \otimes m_2 = \overline{sa_2} \otimes m_2 = 0 \otimes m_2.$$

Since M is indecomposable, there exist $y_2, \cdots, y_n \in M$, $r_1, t_1, \cdots, r_n, t_n \in S$ such that

$$\begin{array}{rcl} r_1m_1 & = & t_1y_2, \\ r_2y_2 & = & t_2y_3, \\ & \cdots & \\ r_ny_n & = & t_nm_2. \end{array}$$

It follows from this that $0 \otimes m_1 = 0 \otimes m_2$, i.e., $\bar{s} \otimes a_1m_1 = \bar{s} \otimes a_2m_2$. Hence, $(a_1m_1, a_2m_2) \in Ann_{M}(S/I)$. We complete the proof. $\square$

Theorem 2.8. *Let U_S and ${}_SM$ be S-acts and $M = \coprod_{\alpha \in I} M_\alpha$ the indecomposable decomposition of M. Then the following statements are equivalent:*

(a) U_S is ${}_SM$-faithful.

(b) $\forall \alpha \in I$, U is M_α-faithful.

(c) For any ${}_SN \in S$-Act and every homomorphism $f : {}_SM \longrightarrow_S N$, if $1_U \otimes f$ is monic then f is monic.

(d) For any ${}_SN \in S$-Act and every homomorphism $f : {}_SN \longrightarrow_S M$, $Ann_{N}(U) \subseteq Kerf$.

Proof (a)$\Leftrightarrow$(b). By Proposition 2.5, we have $Ann_{M}(U) = \coprod_{\alpha \in I} Ann_{M_\alpha}(U)$. Thus, $Ann_{M}(U) = \triangle_M = \coprod_{\alpha \in I} \triangle_{M_\alpha} \iff Ann_{M_\alpha}(U) = \triangle_{M_\alpha} (\forall \alpha \in I) \iff \forall \alpha \in I$, U is M_α-faithful.

(a)$\Rightarrow$(c). Suppose that $Ann_{M}(U) = \triangle_M$, $f \in Hom_{S}(M, N)$ and $1_U \otimes f$ is monic. If $(m_1, m_2) \in Kerf$, then $f(m_1) = f(m_2) \in N$ and we have $u \otimes f(m_1) = u \otimes f(m_2)$ for all $u \in U$, i.e., $(1_U \otimes f)(u \otimes m_1) = (1_U \otimes f)(u \otimes m_2)$. This implies $u \otimes m_1 = u \otimes m_2 (\forall u \in U)$. Thus $(m_1, m_2) \in Ann_{M}(U) = \triangle_M$ and hence $m_1 = m_2$. So, $Kerf = \triangle_M$, i.e., f is monic.

(c)$\Rightarrow$(a). Assume (c). If $(m_1, m_2) \in Ann_{M}(U)$, then $u \otimes m_1 = u \otimes m_2 (\forall u \in U)$. Let $f : M \longrightarrow M/\lambda(m_1, m_2)$ be canonical epimorphism where $\lambda(m_1, m_2)$ is a congruence on ${}_SM$ generated by (m_1, m_2). Then

$$1_U \otimes f : U \otimes M \longrightarrow U \otimes M/\lambda(m_1, m_2), \quad u \otimes m \longmapsto u \otimes f(m) = u \otimes \bar{m}$$

is monic. In fact, for any $u\otimes m,\ u'\otimes m'\in U\otimes M$, if $u\otimes\bar{m}=u'\otimes\bar{m'}$, then there exist $x_1,x_2,\cdots,x_n\in U,\ \bar{y}_2,\cdots,\bar{y}_n\in M/\lambda(m_1,m_2)\ s_1,t_1,\cdots,s_n,t_n\in S$ such that

$$\begin{array}{rcl} u &=& x_1s_1, \\ x_1t_1 &=& x_2s_2, \quad s_1\bar{m}=t_1\bar{y}_2, \\ && \cdots\cdots \\ x_nt_n &=& u', \qquad s_n\bar{y}_n=t_n\bar{m'}. \end{array}$$

Thus, we get $(s_1m,t_1y_2),\cdots,(s_ny_n,t_nm')\in\lambda(m_1,m_2)$. If $s_1m=t_1y_2$, then

$$u\otimes m=x_1s_1\otimes m=x_1\otimes s_1m=x_1\otimes t_1y_2.$$

If $s_1m\neq t_1y_2$, then there exist $p_1,\cdots,p_k\in S$, such that

$$s_1m=p_1c_1,\quad p_2d_2=p_3c_3,\cdots,\quad p_{k-1}d_{k-1}=p_kc_k,$$

$$p_1d_1=p_2c_2,\ \cdots,\ p_{k-1}d_{k-1}=p_kc_k,\ p_kd_k=t_1y_2,$$

where $(c_j,d_j)\in\{(m_1,m_2),(m_2,m_1)\}$, $j=1,\cdots,k$. So

$$\begin{array}{rcl} u\otimes m &=& x_1s_1\otimes m=x_1\otimes s_1m=x_1\otimes p_1c_1 \\ &=& x_1p_1\otimes c_1=x_1p_1\otimes d_1=x_1\otimes p_1d_1 \\ &=& \cdots=x_1\otimes p_kd_k=x_1\otimes t_1y_2. \end{array}$$

By repeating the above arguments, we have

$$\begin{array}{rcl} u\otimes m &=& x_1\otimes t_1y_2=x_1t_1\otimes y_2=x_2s_2\otimes y_2 \\ &=& x_2\otimes s_2y_2=x_2\otimes t_2y_3=\cdots \\ &=& x_n\otimes t_nm'=x_nt_n\otimes m'=u'\otimes m'. \end{array}$$

Therefore $1_U\otimes f$ is monic. Now, by (c), f is monic and so $\lambda(m_1,m_2)=Kerf=\triangle_M$, i.e., $m_1=m_2$, whence $Ann_{_M}(U)=\triangle_M$.

(a)$\Rightarrow$(d). Suppose that $Ann_{_M}(U)=\triangle_M$. For any $f\in Hom_{_S}(N,M)$, $(n_1,n_2)\in Ann_{_N}(U)$, we have $u\otimes n_1=u\otimes n_2$ for all $u\in U$. Thus

$$u\otimes f(n_1)=(1_U\otimes f)(u\otimes n_1)=(1_U\otimes f)(u\otimes n_2)=u\otimes f(n_2)$$

for all $u\in U$. This means $(f(n_1),f(n_2))\in Ann_{_M}(U)=\triangle_M$ and $f(n_1)=f(n_2)$. Hence $(n_1,n_2)\in Kerf$. This shows that $Ann_{_N}(U)\subseteq Kerf$.

(d)$\Rightarrow$(a). Assume (d). If we take $f=id_M:\ M\longrightarrow M$, then $Ann_{_M}(U)\leq Kerf=\triangle_M$ and the result follows. $\square$

3 Completely faithfulness

Definition 3.1. *An S-act U_S is said to be completely faithful in case $Ann_{_M}(U) = \triangle_M$ for every left S-act M.*

For example, since S_S is a generator in Act-S, S_S is completely faithful (see Proposition 3.7).

Theorem 3.2. *For an S-act U_S, the following statements are equivalent:*

(a) U_S *is completely faithful.*

(b) *For every indecomposable left S-act T, U is ${}_ST$-faithful.*

(c) *For any ${}_SN$, ${}_SM \in S$-Act and every homomorphism $f : {}_SM \longrightarrow {}_SN$, if $1_U \otimes f$ is monic, then f is monic.*

(d) *For any ${}_SN$, ${}_SM \in S$-Act and every homomorphism $f : {}_SM \longrightarrow {}_SN$, $Ann_{_M}(U) \subseteq \ker f$.*

Proof The proof is similar to the one of Theorem 2.8 □

Let $Z = \{z\}$ be a set of one-element. Then Z is an S-act with only one way. Such an S-act is called the zero S-act.

Proposition 3.3. *Let Z be the zero right S-act and M a left S-act. Then M is indecomposable S-act if and only if $Ann_{_M}(Z) = \nabla_M$.*

Proof It is obvious that M is indecomposable $\Longleftrightarrow |Z \otimes M| = 1 \Longleftrightarrow Ann_{_M}(Z) = M \times M = \nabla_M$. □

Theorem 3.4. *The following statements are equivalent:*

(a) *Each right S-act is completely faithful.*

(b) *The zero right S-act is completely faithful.*

(c) $S = \{1\}$.

Proof (a)$\Rightarrow$ (b) is clear.

(b)$\Rightarrow$(c). Let Z be the zero right S-act. Since ${}_SS = S1$ is indecomposable, we have, by Proposition 3.3, $Ann_{_S}(Z) = \nabla_S$. Now, $Ann_{_S}(Z) = \triangle_S$ implies $S = \{1\}$.

(c)$\Rightarrow$(a). Suppose that $S = \{1\}$. Then, for any ${}_SM \in S{-}Act$, $U_S \in Act{-}S$, we have $U \otimes M = U \times M$. Hence, $Ann_{_M}(U) = \triangle_M$, i.e., U is ${}_SM$-faithful. □

The proof of the following proposition is straightforward.

Proposition 3.5. *Let S and T be monoids, and let A_S, ${}_SB_T$ be acts. Then*

(a) *If A_S and B_T are completely faithful, then $(A \otimes B)_T$ is completely faithful.*

(b) *If $(A \otimes B)_T$ is completely faithful, then B_T is completely faithful.*

Proposition 3.6. *Let U_S, V_S, and ${}_SM$ be S-acts. If U_S generates V_S, then $Ann_M(U) \subseteq Ann_M(V)$.*

Proof For any $(m_1, m_2) \in Ann_M(U)$ and $x \in V$, there exist $f \in Hom_S(U, V)$ and $u \in U$ such that $x = f(u)$ since $Tr_V(U) = \cup\{Imf | f \in Hom_S(U, V)\} = V$. So $x \otimes m_1 = f(u) \otimes m_1 = (f \otimes 1_M)(u \otimes m_1) = (f \otimes 1_M)(u \otimes m_2) = f(u) \otimes m_2 = x \otimes m_2$, and thus $(m_1, m_2) \in Ann_M(V)$. Hence $Ann_M(U) \subseteq Ann_M(V)$. □

Proposition 3.7. *Every generator in $Act - S$ is completely faithful.*

Proof Suppose that G_S is a generator in $Act - S$. Since $Tr_S(G) = S$, there exist $f \in Hom_S(G, S)$ and $x \in G$ such that $f(x) = 1$. Let M be an arbitrary left S-act and $(m_1, m_2) \in Ann_M(G)$. Then $x \otimes m_1 = x \otimes m_2$. So

$$1 \otimes m_1 = f(x) \otimes m_1 = (f \otimes 1_M)(x \otimes m_1) = (f \otimes 1_M)(x \otimes m_2) = f(x) \otimes m_2 = 1 \otimes m_2$$

which shows that $m_1 = m_2$. Hence $Ann_M(G) = \triangle_M$. □

Theorem 3.8. *Let T and S be monoids, ${}_TU_S$ the $S-T$-biact, ${}_SM \in S$-Act and ${}_TC \in T$-Act. Let $U^* = Hom_T(U, C) \in S - Act$. Then*

(a) $Ann_M(U) \subseteq Rej_M(U^*)$.

(b) *If ${}_TC$ cogenerates $U \otimes M$, then $Ann_M(U) = Rej_M(U^*)$.*

(c) *If ${}_TC$ is a cogenerator, then U_S is completely faithful if and only if ${}_SU^*$ is a cogenerator in $S - Act$.*

Proof By [3] Proposition 2.5.19,

$$\phi : \; Hom_S(M, Hom_T(U, C)) \longrightarrow Hom_T(U \otimes_S M, C)$$

defined by

$$\phi(\gamma)(x \otimes m) = (\gamma(m))(x)$$

for any $x \in U$, $m \in M$ and $\gamma \in Hom_S(M, Hom_T(U, C))$, is a bijection.

(a) For any $\gamma \in Hom_S(M, U^*)$, $(m_1, m_2) \in Ann_M(U)$ and $x \in U$, we have $x \otimes m_1 = x \otimes m_2$, and then $\phi(\gamma)(x \otimes m_1) = \phi(\gamma)(x \otimes m_2)$. Thus, $(\gamma(m_1))(x) = (\gamma(m_2))(x)$ for all $x \in U$ which shows that $\gamma(m_1) = \gamma(m_2)$, that is, $(m_1, m_2) \in Ker\gamma$. Therefore, $Ann_M(U) \subseteq Rej_M(U^*)$.

(b) It will suffice to prove that $Rej_M(U^*) \subseteq Ann_M(U)$. For any $h \in Hom_T(U \otimes_S M, C)$, there exists a unique $\gamma \in Hom_S(M, U^*)$ such that $\phi(\gamma) = h$. Also, for any $(m, m') \in Rej_M(U^*)$ and $u \in U$, we have

$$\begin{aligned} h(u \otimes m) &= \phi(\gamma)(u \otimes m) = (\gamma(m))(u) = (\gamma(m'))(u) \\ &= \phi(\gamma)(u \otimes m') = h(u \otimes m') \end{aligned}$$

since $\gamma(m) = \gamma(m')$. This implies that $(u \otimes m, u \otimes m') \in Rej_{U \otimes M}(C)$. By noting that C cogenerates $U \otimes M$, $Rej_{U \otimes M}(C) = \triangle_{U \otimes M}$. So, $u \otimes m = u \otimes m'$ for all $u \in U$. Hence $(m, m') \in Ann_M(U)$.

(c) This part follows (b). $\square$

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