

Annihilator of Tensor Product of S -acts

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Abstract

For S -acts ${}_S M$ and U_S , let $Ann_M(U) = \{(m, m') \in M \times M | u \otimes m = u \otimes m' \text{ for any } u \in U\}$. Then U_S is called ${}_S M$ -faithful if $Ann_M(U)$ is the identity relation on M . If U_S is ${}_S M$ -faithful for any S -act ${}_S M$, then we call U_S completely faithful. The present paper discusses properties of ${}_S M$ -faithful (completely faithful) S -acts. The structures of ${}_S M$ -faithful (completely faithful) right S -acts are characterized. Some related results are also obtained.

1 Preliminaries

In this paper, we shall always let semigroup S mean a monoid and all S -acts be unitary. We denote the category of all right (left) S -acts by $Act - S$ ($S - Act$). Let A_S be a right S -act. An equivalence relation ρ on A is called an S -congruence or a congruence on A_S if for any $a, a' \in A$, $(a, a') \in \rho$ implies $(as, a's) \in \rho$ for any $s \in S$.

If ${}_S M$ is a left S -act, then the cartesian product $M \times M$ with the operation $s \cdot (m, m') = (sm, sm')$ for all $s \in S$, $m, m' \in M$ is a left S -act. Let $f : {}_S M \rightarrow {}_S N$ be an S -homomorphism. We denote by $Im f = \{f(m) | m \in M\}$ and $ker f = \{(m, m') \in M \times M | f(m) = f(m')\}$. It is clear that $(f, f) : {}_S(M \times M) \rightarrow {}_S(N \times N)$ with $(f, f)((m, m')) := (f(m), f(m'))$, $m, m' \in M$, is an S -homomorphism, and $ker f$ is a congruence on ${}_S M$.

Let X be a set. Denote by $\Delta_X = \{(x, x) | x \in X\}$ and $\nabla_X = X \times X$. For a subact ${}_S N$ of ${}_S M$, $\rho_N = (N \times N) \cap \Delta_M$ is clearly a congruence on ${}_S M$ which is called the Rees congruence and we denote the quotient act M/ρ_N by M/N .

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Let U_S, M_S be right S -acts. As in module theory, the trace and the reject of U in M , respectively, are defined by

$$Tr_M(U) = \cup\{Imf | f \in Hom_S(U, M)\}$$

and

$$Rej_M(U) = \cap\{\ker f | f \in Hom_S(M, U)\}.$$

We say that U_S generates (cogenerates) M_S in case $Tr_M(U) = M$ ($Rej_M(U) = \Delta_M$). U_S is called a generator (cogenerator) of $Act - S$ in case $Tr_M(U) = M$ ($Rej_M(U) = \Delta_M$) for all $M_S \in Act - S$. Denoted by $\mathbf{r}_S(M) := \{(s, s') \in S \times S \mid ms = ms', \forall m \in M\}$ the annihilator of right S -act M_S . It is clear that $\mathbf{r}_S(M)$ is a congruence on M_S .

Let $(A_\alpha)_{\alpha \in I}$ be a family of right S -acts. Then, the coproduct $\coprod_{\alpha \in I} A_\alpha$ of $(A_\alpha)_{\alpha \in I}$ is the disjoint union of $(A_\alpha)_{\alpha \in I}$.

We call A_S a faithful right S -act if for any $s, t \in S$ the equality $as = at$ for all $a \in A$ implies $s = t$. Obviously, A_S is faithful if and only if $\mathbf{r}_S(A) = \Delta_S$. A_S is called a strongly faithful right S -act if for any $s, t \in S$ the equality $as = at$ for some $a \in A$ implies $s = t$.

For other definitions and terminologies not mentioned in this paper, the reader is referred to [3].

2 Faithfulness

Definition 2.1. Let U_S and ${}_S M$ be S -acts, $U \otimes M$ the tensor product of U and M . Then

$$Ann_M(U) = \{(m, m') \in M \times M \mid u \otimes m = u \otimes m', \forall u \in U\}$$

is called the annihilator in M of U . Call U_S to be ${}_S M$ -faithful in case $Ann_M(U) = \Delta_M$.

It is obvious that $Ann_S(U) = \mathbf{r}_S(U)$ for any right S -act U_S .

Proposition 2.2. Let U_S and ${}_S M$ be S -acts. Then $Ann_M(U)$ is the unique smallest congruence λ on ${}_S M$ such that U is M/λ -faithful.

Proof Suppose that $\lambda = Ann_M(U) = \{(m_1, m_2) \in M \times M \mid u \otimes m_1 = u \otimes m_2, \forall u \in U\}$. Clearly, λ is a congruence on ${}_S M$.

Assume that $(\bar{m}_1, \bar{m}_2) \in Ann_{M/\lambda}(U)$. Then, we have $u \otimes \bar{m}_1 = u \otimes \bar{m}_2$ for all $u \in U$. Thus, there exist $x_1, x_2, \dots, x_n \in U, \bar{y}_2, \dots, \bar{y}_n \in M/\lambda, s_1, t_1, \dots, s_n, t_n \in S$ such that

$$\begin{aligned} u &= x_1 s_1, \\ x_1 t_1 &= x_2 s_2, \quad s_1 \bar{m}_1 = t_1 \bar{y}_2, \\ &\dots\dots \\ x_n t_n &= u, \quad s_n \bar{y}_n = t_n \bar{m}_2. \end{aligned}$$

This implies that $(s_1 m_1, t_1 y_2), \dots, (s_n y_n, t_n m_2) \in \lambda$, and then, for any $u \in U$,

$$\begin{aligned} u \otimes m_1 &= x_1 s_1 \otimes m_1 = x_1 \otimes s_1 m_1 = x_1 \otimes t_1 y_2 = x_1 t_1 \otimes y_2 \\ &= x_2 s_2 \otimes y_2 = \dots = x_n s_n \otimes y_n = x_n \otimes s_n y_n \\ &= x_n \otimes t_n m_2 = x_n t_n \otimes m_2 = u \otimes m_2 \end{aligned}$$

which shows that $(m_1, m_2) \in \lambda$ and $\bar{m}_1 = \bar{m}_2$. Therefore $\text{Ann}_{M/\lambda}(U) = \Delta_{M/\lambda}$.

Let now σ be a congruence on ${}_S M$ with $\text{Ann}_{M/\sigma}(U) = \Delta_{M/\sigma}$. Assume that $(m, m') \in \lambda$. Then $u \otimes m = u \otimes m'$ for all $u \in U$. Let $n : M \rightarrow M/\sigma$ be the canonical epimorphism. Then $1_U \otimes n : U \otimes M \rightarrow U \otimes M/\sigma$ is surjective and $u \otimes (m\sigma) = (1_U \otimes n)(u \otimes m) = (1_U \otimes n)(u \otimes m') = u \otimes (m'\sigma)$ for all $u \in U$. Thus, $(m\sigma, m'\sigma) \in \text{Ann}_{M/\sigma}(U) = \Delta_{M/\sigma}$ and $m\sigma = m'\sigma$, i.e., $(m, m') \in \sigma$. Hence $\lambda \subseteq \sigma$. \square

Proposition 2.3. *Let U_S , ${}_S M$ and ${}_S N$ be S -acts and let $f \in \text{Hom}_S(M, N)$. Then*

(a) $(f, f)(\text{Ann}_M(U)) \subseteq \text{Ann}_N(U)$. In particular, $\text{Ann}_M(U)$ is stable under endomorphisms of ${}_S M$.

(b) If f is epic and $\text{Ker} f \subseteq \text{Ann}_M(U)$, then $(f, f)(\text{Ann}_M(U)) = \text{Ann}_N(U)$.

Proof (a) Assume that $(m, m') \in \text{Ann}_M(U)$ and $u \in U$. Since $u \otimes m = u \otimes m'$ we have

$$u \otimes f(m) = (1_U \otimes f)(u \otimes m) = (1_U \otimes f)(u \otimes m') = u \otimes f(m').$$

Thus $(f(m), f(m')) \in \text{Ann}_N(U)$ and therefore, $(f, f)(\text{Ann}_M(U)) \subseteq \text{Ann}_N(U)$.

(b) It will suffice to prove that $\text{Ann}_N(U) \subseteq (f, f)(\text{Ann}_M(U))$. Let $\phi : M \rightarrow M/\text{Ker} f$ be the canonical epimorphism. Because f is epic there exists a unique isomorphism $\bar{f} : M/\text{Ker} f \rightarrow N$ such that $f = \bar{f}\phi$.

Assume that $(\bar{m}, \bar{m}') \in \text{Ann}_{M/\text{Ker} f}(U)$ and $u \in U$. Since $u \otimes \bar{m} = u \otimes \bar{m}'$, there exist $x_1, x_2, \dots, x_n \in U$, $\bar{y}_2, \dots, \bar{y}_n \in M/\text{Ker} f$, $s_1, t_1, \dots, s_n, t_n \in S$ such that

$$\begin{aligned} u &= x_1 s_1, \\ x_1 t_1 &= x_2 s_2, \quad s_1 \bar{m} = t_1 \bar{y}_2, \\ &\dots\dots \\ x_n t_n &= u, \quad s_n \bar{y}_n = t_n \bar{m}'. \end{aligned}$$

Thus $(s_1 m, t_1 y_2), \dots, (s_n y_n, t_n m') \in \text{Ker} f \subseteq \text{Ann}_M(U)$ and so

$$\begin{aligned} u \otimes m &= x_1 s_1 \otimes m = x_1 \otimes s_1 m = x_1 \otimes t_1 y_2 = x_1 t_1 \otimes y_2 \\ &= x_2 s_2 \otimes y_2 = \dots = x_n s_n \otimes y_n = x_n \otimes s_n y_n \\ &= x_n \otimes t_n m_2 = x_n t_n \otimes m' = u \otimes m'. \end{aligned}$$

Therefore, $(m, m') \in \text{Ann}_M(U)$. Hence $(\bar{m}, \bar{m}') = (\phi, \phi)((m, m')) \in (\phi, \phi)(\text{Ann}_M(U))$, i.e., $\text{Ann}_{M/\text{Ker}f}(U) \subseteq (\phi, \phi)(\text{Ann}_M(U))$.

Now, for any $(n, n') \in \text{Ann}_N(U)$, there exist unique $\bar{m}, \bar{m}' \in M/\text{Ker}f$ such that $n = \bar{f}(\bar{m})$ and $n' = \bar{f}(\bar{m}')$. Noting that \bar{f} is an isomorphism, we know that $1_U \otimes \bar{f}$ is a bijection. Since $(1_U \otimes \bar{f})(u \otimes \bar{m}) = u \otimes \bar{f}(\bar{m}) = u \otimes n = u \otimes n' = u \otimes \bar{f}(\bar{m}') = (1_U \otimes \bar{f})(u \otimes \bar{m}')$, we have $u \otimes \bar{m} = u \otimes \bar{m}'$ for all $u \in U$ which shows that $(\bar{m}, \bar{m}') \in \text{Ann}_{M/\text{Ker}f}(U) \subseteq (\phi, \phi)(\text{Ann}_M(U))$. Hence

$$\begin{aligned} (n, n') &= (\bar{f}, \bar{f})((\bar{m}, \bar{m}')) \in (\bar{f}, \bar{f})(\text{Ann}_{M/\text{Ker}f}(U)) \subseteq (\bar{f}, \bar{f})((\phi, \phi)(\text{Ann}_M(U))) \\ &= (\bar{f}\phi, \bar{f}\phi)(\text{Ann}_M(U)) = (f, f)(\text{Ann}_M(U)). \end{aligned}$$

We complete the proof. \square

Lemma 2.4. *Let $(A_\alpha)_{\alpha \in I}$ be a family of right S -acts, $(B_\beta)_{\beta \in J}$ a family of left S -acts and $a \otimes b, c \otimes d$ in $(\prod_{\alpha \in I} A_\alpha) \otimes_S (\prod_{\beta \in J} B_\beta)$. Then $a \otimes b = c \otimes d$ in $(\prod_{\alpha \in I} A_\alpha) \otimes_S (\prod_{\beta \in J} B_\beta)$ if and only if $a \otimes b = c \otimes d$ in $A_\alpha \otimes_S B_\beta$ for some $\alpha \in I, \beta \in J$.*

Proof *sufficiency* is obvious.

Necessity. Suppose $a \otimes b = c \otimes d$ in $(\prod_{\alpha \in I} A_\alpha) \otimes_S (\prod_{\beta \in J} B_\beta)$. Then there exist $a_1, a_2, \dots, a_n \in \prod_{\alpha \in I} A_\alpha$, $b_2, \dots, b_n \in \prod_{\beta \in J} B_\beta$, $u_1, v_1, \dots, u_n, v_n \in S$, such that

$$\begin{aligned} a &= a_1 u_1, \\ a_1 v_1 &= a_2 u_2, \quad u_1 b = v_1 b_2, \\ &\dots\dots \\ a_n v_n &= c, \quad u_n b_n = v_n d. \end{aligned}$$

Since $a \in \prod_{\alpha \in I} A_\alpha$ and $b \in \prod_{\beta \in J} B_\beta$, there uniquely exist $\alpha \in I, \beta \in J$ such that $a \in A_\alpha$ and $b \in B_\beta$. Now, $a_1 u_1 = a \in A_\alpha$ implies that $a_1 \in A_\alpha$. Otherwise, if $a_1 \in A_{\alpha'}$ with $\alpha \neq \alpha'$, then $a_1 u_1 \in A_\alpha \cap A_{\alpha'}$ which contradicts that $A_\alpha \cap A_{\alpha'} = \emptyset$. So $a_2 u_2 = a_1 v_1 \in A_\alpha$ and $a_2 \in A_\alpha$. Repeating this process, we conclude $a_3, \dots, a_n, c \in A_\alpha$. Similarly, we have $b, b_2, \dots, b_n, d \in B_\beta$. This shows that $a \otimes b = c \otimes d$ in $A_\alpha \otimes_S B_\beta$. \square

Proposition 2.5. *Let I, J be index sets, $U, U_j \in \text{Act} - S$, $j \in J$ and $M, M_i \in S - \text{Act}$, $i \in I$. Then*

- (a) $\text{Ann}_{\prod_{i \in I} M_i}(U) = \prod_{i \in I} \text{Ann}_{M_i}(U)$.
- (b) $\text{Ann}_M(\prod_{j \in J} U_j) = \bigcap_{j \in J} \text{Ann}_M(U_j)$.

Proof (a) It is obvious that $\prod_{i \in I} \text{Ann}_{M_i}(U) \subseteq \text{Ann}_{\prod_{i \in I} M_i}(U)$. Also, $\forall (m, m') \in \text{Ann}_{\prod_{i \in I} M_i}(U)$, $\forall u \in U$, we have $u \otimes m = u \otimes m'$ in $U \otimes (\prod_{i \in I} M_i)$. From Lemma 2.4 it follows that $u \otimes m = u \otimes m'$ in $U \otimes M_i$ for some $i \in I$, and so $(m, m') \in \text{Ann}_{M_i}(U) \subseteq \prod_{i \in I} \text{Ann}_{M_i}(U)$. This shows (a).

(b) Clearly, $\bigcap_{j \in J} \text{Ann}_M(U_j) \subseteq \text{Ann}_M(\prod_{j \in J} U_j)$. Conversely, if $(m, m') \in \text{Ann}_M(\prod_{j \in J} U_j)$ and $u \in U_j \subseteq \prod_{j \in J} U_j$, $j \in J$, then $u \otimes m = u \otimes m'$ in $(\prod_{j \in J} U_j) \otimes M$. By Lemma 2.4, we get $u \otimes m = u \otimes m'$ in $U_j \otimes M$. Thus, $\text{Ann}_M(\prod_{j \in J} U_j) \subseteq \bigcap_{j \in J} \text{Ann}_M(U_j)$. This shows (b). \square

It is well known that each S -act has a unique indecomposable decomposition (see [4] or [2]). Now, by our Lemma 2.4, we have the following lemma.

Lemma 2.6. *Let A_S and ${}_S B$ be S -acts and $a \otimes b = a' \otimes b'$ in $A \otimes_S B$. Then a, a' and b, b' are in the same indecomposable subacts of A_S and ${}_S B$, respectively.*

Theorem 2.7. *If I is an ideal of S and ${}_S M \in S\text{-Act}$, then*

$$\text{Ann}_M(S/I) \subseteq (IM \times IM) \cup \Delta_M.$$

Moreover, $\text{Ann}_M(S/I) = (IM \times IM) \cup \Delta_M$ if and only if M is indecomposable.

Proof If we define

$$S/I \times M/IM \longrightarrow M/IM, \quad (\bar{s}, \tilde{m}) \longmapsto \widetilde{sm},$$

then M/IM is an S/I -act and ${}_S(M/IM) = {}_{S/I}(M/IM)$. Let

$$\phi: S/I \otimes_S M \longrightarrow M/IM, \quad \bar{s} \otimes m \longmapsto \widetilde{sm}.$$

Then ϕ is well-defined. In fact, suppose that $\bar{s} \otimes m = \bar{s}' \otimes m'$ for some $\bar{s}, \bar{s}' \in S/I$, $m, m' \in M$. Then there exist $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n \in S/I$, $y_2, \dots, y_n \in M$, $r_1, t_1, \dots, r_n, t_n \in S$ such that

$$\begin{aligned} \bar{s} &= \bar{x}_1 r_1, \\ \bar{x}_1 t_1 &= \bar{x}_2 r_2, \quad r_1 m = t_1 y_2, \\ &\dots\dots\dots \\ \bar{x}_n t_n &= \bar{s}', \quad r_n y_n = t_n m'. \end{aligned}$$

Thus

$$\begin{aligned} \widetilde{sm} &= \bar{s} \tilde{m} = \bar{x}_1 r_1 \tilde{m} = \bar{x}_1 r_1 \widetilde{m} = \bar{x}_1 t_1 \widetilde{y_2} \\ &= \bar{x}_1 t_1 \tilde{y}_2 = \dots = \bar{x}_n t_n \tilde{m}' = \bar{s}' \tilde{m}' = \widetilde{s'm'}, \end{aligned}$$

i.e., ϕ is well-defined.

If $(m_1, m_2) \in \text{Ann}_M(S/I)$, then $\bar{s} \otimes m_1 = \bar{s} \otimes m_2$ and $\widetilde{sm}_1 = \widetilde{sm}_2$ for all $s \in S$, in particular, $(m_1, m_2) \in (IM \times IM) \cup \Delta_M$. Thus, $\text{Ann}_M(S/I) \subseteq (IM \times IM) \cup \Delta_M$.

Suppose that $\text{Ann}_M(S/I) = (IM \times IM) \cup \Delta_M$. Then, for any $(m_1, m_2) \in M \times M$ and $a \in I$, we have $(am_1, am_2) \in \text{Ann}_M(S/I)$, in particular, $\bar{1} \otimes am_1 = \bar{1} \otimes am_2$. By Lemma 2.6, am_1, am_2 is in the same indecomposable subact of M . This implies that m_1, m_2 is in the same indecomposable subact. Hence, M is indecomposable.

Conversely, suppose M is indecomposable. It will suffice to prove that $(IM \times IM) \subseteq \text{Ann}_M(S/I)$. For any $(a_1m_1, a_2m_2) \in IM \times IM$, where $a_1, a_2 \in I$, $m_1, m_2 \in M$, and for any $\bar{s} \in S/I$, we have

$$\bar{s} \otimes a_1m_1 = \bar{s}a_1 \otimes m_1 = \overline{\bar{s}a_1} \otimes m_1 = 0 \otimes m_1,$$

$$\bar{s} \otimes a_2m_2 = \bar{s}a_2 \otimes m_2 = \overline{\bar{s}a_2} \otimes m_2 = 0 \otimes m_2.$$

Since M is indecomposable, there exist $y_2, \dots, y_n \in M$, $r_1, t_1, \dots, r_n, t_n \in S$ such that

$$\begin{aligned} r_1m_1 &= t_1y_2, \\ r_2y_2 &= t_2y_3, \\ &\dots \\ r_ny_n &= t_nm_2. \end{aligned}$$

It follows from this that $0 \otimes m_1 = 0 \otimes m_2$, i.e., $\bar{s} \otimes a_1m_1 = \bar{s} \otimes a_2m_2$. Hence, $(a_1m_1, a_2m_2) \in \text{Ann}_M(S/I)$. We complete the proof. \square

Theorem 2.8. *Let U_S and ${}_S M$ be S -acts and $M = \coprod_{\alpha \in I} M_\alpha$ the indecomposable decomposition of M . Then the following statements are equivalent:*

- (a) U_S is ${}_S M$ -faithful.
- (b) $\forall \alpha \in I$, U is M_α -faithful.
- (c) For any ${}_S N \in S\text{-Act}$ and every homomorphism $f : {}_S M \rightarrow {}_S N$, if $1_U \otimes f$ is monic then f is monic.
- (d) For any ${}_S N \in S\text{-Act}$ and every homomorphism $f : {}_S N \rightarrow {}_S M$, $\text{Ann}_N(U) \subseteq \text{Ker } f$.

Proof (a) \Leftrightarrow (b). By Proposition 2.5, we have $\text{Ann}_M(U) = \coprod_{\alpha \in I} \text{Ann}_{M_\alpha}(U)$. Thus, $\text{Ann}_M(U) = \Delta_M = \coprod_{\alpha \in I} \Delta_{M_\alpha} \iff \text{Ann}_{M_\alpha}(U) = \Delta_{M_\alpha} (\forall \alpha \in I) \iff \forall \alpha \in I$, U is M_α -faithful.

(a) \Rightarrow (c). Suppose that $\text{Ann}_M(U) = \Delta_M$, $f \in \text{Hom}_S(M, N)$ and $1_U \otimes f$ is monic. If $(m_1, m_2) \in \text{Ker } f$, then $f(m_1) = f(m_2) \in N$ and we have $u \otimes f(m_1) = u \otimes f(m_2)$ for all $u \in U$, i.e., $(1_U \otimes f)(u \otimes m_1) = (1_U \otimes f)(u \otimes m_2)$. This implies $u \otimes m_1 = u \otimes m_2 (\forall u \in U)$. Thus $(m_1, m_2) \in \text{Ann}_M(U) = \Delta_M$ and hence $m_1 = m_2$. So, $\text{Ker } f = \Delta_M$, i.e., f is monic.

(c) \Rightarrow (a). Assume (c). If $(m_1, m_2) \in \text{Ann}_M(U)$, then $u \otimes m_1 = u \otimes m_2 (\forall u \in U)$. Let $f : M \rightarrow M/\lambda(m_1, m_2)$ be canonical epimorphism where $\lambda(m_1, m_2)$ is a congruence on ${}_S M$ generated by (m_1, m_2) . Then

$$1_U \otimes f : U \otimes M \rightarrow U \otimes M/\lambda(m_1, m_2), \quad u \otimes m \mapsto u \otimes f(m) = u \otimes \bar{m}$$

is monic. In fact, for any $u \otimes m, u' \otimes m' \in U \otimes M$, if $u \otimes \bar{m} = u' \otimes \bar{m}'$, then there exist $x_1, x_2, \dots, x_n \in U, \bar{y}_2, \dots, \bar{y}_n \in M/\lambda(m_1, m_2)$ $s_1, t_1, \dots, s_n, t_n \in S$ such that

$$\begin{aligned} u &= x_1 s_1, \\ x_1 t_1 &= x_2 s_2, \quad s_1 \bar{m} = t_1 \bar{y}_2, \\ &\dots\dots\dots \\ x_n t_n &= u', \quad s_n \bar{y}_n = t_n \bar{m}'. \end{aligned}$$

Thus, we get $(s_1 m, t_1 y_2), \dots, (s_n y_n, t_n m') \in \lambda(m_1, m_2)$. If $s_1 m = t_1 y_2$, then

$$u \otimes m = x_1 s_1 \otimes m = x_1 \otimes s_1 m = x_1 \otimes t_1 y_2.$$

If $s_1 m \neq t_1 y_2$, then there exist $p_1, \dots, p_k \in S$, such that

$$s_1 m = p_1 c_1, \quad p_2 d_2 = p_3 c_3, \dots, \quad p_{k-1} d_{k-1} = p_k c_k,$$

$$p_1 d_1 = p_2 c_2, \quad \dots, \quad p_{k-1} d_{k-1} = p_k c_k, \quad p_k d_k = t_1 y_2,$$

where $(c_j, d_j) \in \{(m_1, m_2), (m_2, m_1)\}$, $j = 1, \dots, k$. So

$$\begin{aligned} u \otimes m &= x_1 s_1 \otimes m = x_1 \otimes s_1 m = x_1 \otimes p_1 c_1 \\ &= x_1 p_1 \otimes c_1 = x_1 p_1 \otimes d_1 = x_1 \otimes p_1 d_1 \\ &= \dots = x_1 \otimes p_k d_k = x_1 \otimes t_1 y_2. \end{aligned}$$

By repeating the above arguments, we have

$$\begin{aligned} u \otimes m &= x_1 \otimes t_1 y_2 = x_1 t_1 \otimes y_2 = x_2 s_2 \otimes y_2 \\ &= x_2 \otimes s_2 y_2 = x_2 \otimes t_2 y_3 = \dots \\ &= x_n \otimes t_n m' = x_n t_n \otimes m' = u' \otimes m'. \end{aligned}$$

Therefore $1_U \otimes f$ is monic. Now, by (c), f is monic and so $\lambda(m_1, m_2) = \text{Ker} f = \Delta_M$, i.e., $m_1 = m_2$, whence $\text{Ann}_M(U) = \Delta_M$.

(a) \Rightarrow (d). Suppose that $\text{Ann}_M(U) = \Delta_M$. For any $f \in \text{Hom}_S(N, M)$, $(n_1, n_2) \in \text{Ann}_N(U)$, we have $u \otimes n_1 = u \otimes n_2$ for all $u \in U$. Thus

$$u \otimes f(n_1) = (1_U \otimes f)(u \otimes n_1) = (1_U \otimes f)(u \otimes n_2) = u \otimes f(n_2)$$

for all $u \in U$. This means $(f(n_1), f(n_2)) \in \text{Ann}_M(U) = \Delta_M$ and $f(n_1) = f(n_2)$. Hence $(n_1, n_2) \in \text{Ker} f$. This shows that $\text{Ann}_N(U) \subseteq \text{Ker} f$.

(d) \Rightarrow (a). Assume (d). If we take $f = \text{id}_M : M \longrightarrow M$, then $\text{Ann}_M(U) \subseteq \text{Ker} f = \Delta_M$ and the result follows. \square

3 Completely faithfulness

Definition 3.1. An S -act U_S is said to be completely faithful in case $Ann_M(U) = \Delta_M$ for every left S -act M .

For example, since S_S is a generator in $Act-S$, S_S is completely faithful (see Proposition 3.7).

Theorem 3.2. For an S -act U_S , the following statements are equivalent:

- (a) U_S is completely faithful.
- (b) For every indecomposable left S -act T , U is ${}_S T$ -faithful.
- (c) For any ${}_S N$, ${}_S M \in S-Act$ and every homomorphism $f : {}_S M \rightarrow {}_S N$, if ${}_1 U \otimes f$ is monic, then f is monic.
- (d) For any ${}_S N$, ${}_S M \in S-Act$ and every homomorphism $f : {}_S M \rightarrow {}_S N$, $Ann_M(U) \subseteq \ker f$.

Proof The proof is similar to the one of Theorem 2.8 \square

Let $Z = \{z\}$ be a set of one-element. Then Z is an S -act with only one way. Such an S -act is called the zero S -act.

Proposition 3.3. Let Z be the zero right S -act and M a left S -act. Then M is indecomposable S -act if and only if $Ann_M(Z) = \nabla_M$.

Proof It is obvious that M is indecomposable $\iff |Z \otimes M| = 1 \iff Ann_M(Z) = M \times M = \nabla_M$. \square

Theorem 3.4. The following statements are equivalent:

- (a) Each right S -act is completely faithful.
- (b) The zero right S -act is completely faithful.
- (c) $S = \{1\}$.

Proof (a) \implies (b) is clear.

(b) \implies (c). Let Z be the zero right S -act. Since ${}_S S = S1$ is indecomposable, we have, by Proposition 3.3, $Ann_S(Z) = \nabla_S$. Now, $Ann_S(Z) = \Delta_S$ implies $S = \{1\}$.

(c) \implies (a). Suppose that $S = \{1\}$. Then, for any ${}_S M \in S-Act$, $U_S \in Act-S$, we have $U \otimes M = U \times M$. Hence, $Ann_M(U) = \Delta_M$, i.e., U is ${}_S M$ -faithful. \square

The proof of the following proposition is straightforward.

Proposition 3.5. Let S and T be monoids, and let $A_S, {}_S B_T$ be acts. Then

- (a) If A_S and B_T are completely faithful, then $(A \otimes B)_T$ is completely faithful.
 (b) If $(A \otimes B)_T$ is completely faithful, then B_T is completely faithful.

Proposition 3.6. *Let U_S , V_S , and ${}_S M$ be S -acts. If U_S generates V_S , then $Ann_M(U) \subseteq Ann_M(V)$.*

Proof For any $(m_1, m_2) \in Ann_M(U)$ and $x \in V$, there exist $f \in Hom_S(U, V)$ and $u \in U$ such that $x = f(u)$ since $Tr_V(U) = \cup\{Imf | f \in Hom_S(U, V)\} = V$. So $x \otimes m_1 = f(u) \otimes m_1 = (f \otimes 1_M)(u \otimes m_1) = (f \otimes 1_M)(u \otimes m_2) = f(u) \otimes m_2 = x \otimes m_2$, and thus $(m_1, m_2) \in Ann_M(V)$. Hence $Ann_M(U) \subseteq Ann_M(V)$. \square

Proposition 3.7. *Every generator in $Act - S$ is completely faithful.*

Proof Suppose that G_S is a generator in $Act - S$. Since $Tr_S(G) = S$, there exist $f \in Hom_S(G, S)$ and $x \in G$ such that $f(x) = 1$. Let M be an arbitrary left S -act and $(m_1, m_2) \in Ann_M(G)$. Then $x \otimes m_1 = x \otimes m_2$. So

$$1 \otimes m_1 = f(x) \otimes m_1 = (f \otimes 1_M)(x \otimes m_1) = (f \otimes 1_M)(x \otimes m_2) = f(x) \otimes m_2 = 1 \otimes m_2$$

which shows that $m_1 = m_2$. Hence $Ann_M(G) = \Delta_M$. \square

Theorem 3.8. *Let T and S be monoids, ${}_T U_S$ the $S - T$ -biact, ${}_S M \in S\text{-Act}$ and ${}_T C \in T\text{-Act}$. Let $U^* = Hom_T(U, C) \in S - Act$. Then*

- (a) $Ann_M(U) \subseteq Rej_M(U^*)$.
 (b) If ${}_T C$ cogenerates $U \otimes M$, then $Ann_M(U) = Rej_M(U^*)$.
 (c) If ${}_T C$ is a cogenerator, then U_S is completely faithful if and only if ${}_S U^*$ is a cogenerator in $S - Act$.

Proof By [3] Proposition 2.5.19,

$$\phi : Hom_S(M, Hom_T(U, C)) \longrightarrow Hom_T(U \otimes_S M, C)$$

defined by

$$\phi(\gamma)(x \otimes m) = (\gamma(m))(x)$$

for any $x \in U$, $m \in M$ and $\gamma \in Hom_S(M, Hom_T(U, C))$, is a bijection.

(a) For any $\gamma \in Hom_S(M, U^*)$, $(m_1, m_2) \in Ann_M(U)$ and $x \in U$, we have $x \otimes m_1 = x \otimes m_2$, and then $\phi(\gamma)(x \otimes m_1) = \phi(\gamma)(x \otimes m_2)$. Thus, $(\gamma(m_1))(x) = (\gamma(m_2))(x)$ for all $x \in U$ which shows that $\gamma(m_1) = \gamma(m_2)$, that is, $(m_1, m_2) \in Ker\gamma$. Therefore, $Ann_M(U) \subseteq Rej_M(U^*)$.

(b) It will suffice to prove that $Rej_M(U^*) \subseteq Ann_M(U)$. For any $h \in Hom_T(U \otimes_S M, C)$, there exists a unique $\gamma \in Hom_S(M, U^*)$ such that $\phi(\gamma) = h$. Also, for any $(m, m') \in Rej_M(U^*)$ and $u \in U$, we have

$$\begin{aligned} h(u \otimes m) &= \phi(\gamma)(u \otimes m) = (\gamma(m))(u) = (\gamma(m'))(u) \\ &= \phi(\gamma)(u \otimes m') = h(u \otimes m') \end{aligned}$$

since $\gamma(m) = \gamma(m')$. This implies that $(u \otimes m, u \otimes m') \in \text{Rej}_{U \otimes M}(C)$. By noting that C cogenerates $U \otimes M$, $\text{Rej}_{U \otimes M}(C) = \Delta_{U \otimes M}$. So, $u \otimes m = u \otimes m'$ for all $u \in U$. Hence $(m, m') \in \text{Ann}_M(U)$.

(c) This part follows (b). \square

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