***East-West J. of Mathematics: Vol. 7, No 1*** (2005) pp.45-49

# Inclusions among Multipliers from $L_r^p$ to $l_q$

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**Abstract**

Let $G$ be a compact abelian group with dual group $\Gamma$. We study multipliers from the space of $p$-integrable functions on $G$ with Fourier transform in the sequence space $l_r(\Gamma)$ into the sequence space $l_q(\Gamma)$ and prove some new results. We suggest some open problems.

## Multipliers from $L_r^p$ to $l_q$:

For all unexplained notation see the articles [4],[5]. Let $G$ be a compact abelian group with dual $\Gamma$. For $1 \leq p < \infty$, define

$$L_r^p(G) = \{f : f \in L^p(G), \hat{f} \in l_r(\Gamma)\}.$$

We will write $L_r^p(G)$ as $L_r^p$. Note that $L_r^p = L^p$ if $r \geq p' \geq 2$. For $p = 1$, set $L_r^p = A_r$. A function $\phi$ on $\Gamma$ is said to be a multiplier from $L_r^p$ to $l_q(\Gamma)$ if $\phi\widehat{f} \in l_q$ for every $f \in L_r^p$. The set of all multipliers from $L_r^p$ to $l_q$ is denoted by $(L_r^p, l_q)$. It is easily seen that $\phi$ induces a bounded linear operator from $L_r^p$ to $l_q$.

We note that $(L^p, l_q)$ and $(A_p, A_q)$-multipliers have been studied in [1], [3], [5], [8], [9], [10]. Let $\phi$ be a Young's function (for definition, see [4]or [7]) and $L^\phi(G)$ be the corresponding Orlicz space. Let

$$L_r^\phi(G) = \{f : f \in L^\phi(G), \hat{f} \in l_r(\Gamma)\}.$$

A simple use of Hölder's inequality shows that

$$l_{(rq/(r-q))} \subseteq (A_r, l_q) \subseteq (L_r^\phi, l_q) \subseteq (L_r^p, l_q), r > 2, 1 \leq q \leq r < \infty, p > 1.$$

**Key words:**
2000 AMS Mathematics Subject Classification: 43A22, 42A45.

Note that if $1 \leq r \leq 2$, then $\hat{L_r^\phi} = \hat{A}_r = l_r$ by a simple use of Plancherel Theorem and problems posed in this article are not interesting then.

Let $A \subset B$ denote that $A$ is a proper subset of $B$. It was shown in [4] that $l_{(rq/(r-q))} \subset (L_r^p, l_q), r > 2, 1 \leq q \leq r < \infty, p > 1$.(In fact it was proved that $l_{(rq/(r-q))} \subset (L_r^\phi, l_q), r > 2, 1 \leq q \leq r < \infty, p > 1$ for $\phi_\alpha(t) = t(\ln^+ t)^\alpha$, $\alpha > 1/2$).

Naturally, one may ask whether $(A_r, l_q)$ is properly contained in $(L_r^\phi, l_q)$ for every Young's function $\phi$. This remains open. Therefore the following weaker question that $(A_r, l_q)$ is properly contained in $(L_r^p, l_q)$ for every $p > 1$ is worth exploring. We show that this is so if $p = r'$ (then $L_r^{r'} = L^{r'}$). In fact, we prove a stronger result:

## Main Theorem

Let G be a compact abelian, not totally disconnected group, $r > 2$ and $1 \leq p < r'$. Then

$$(L_r^p, l_q) \subset (L^{r'}, l_q), 1 \leq q < r < \infty.$$

The proof consists of three steps: In Step 1, we prove Theorem for the case $G = \mathbb{T}$, the circle group. Then the proof of the general case is reduced to the case $G = \mathbb{T}$, by using the fact that there exists a closed subgroup $H$ of $G$ such that $G/H$ is isomorphic with $\mathbb{T}$. To prove Step 1, we need the following Theorem from (Theorem 8, [5]):

**Theorem** Let $1 < p \leq 2$, and $\phi$ be a complex-valued function defined on $\mathbf{Z}$, then $\phi \in (L^p, l_q)$ if

(a)

$$M = \sum_{n \in \mathbf{Z} \setminus \{0\}} \frac{|\phi(n)|^{(pq/(p-q))}}{(|n|)^{((p-2)q/(p-q))}} < \infty \text{ when } q < p;$$

(b)

$$M = sup_{n \in \mathbf{Z}} |n|^{1/s} |\phi(n)| < \infty \text{ when } p \leq q \leq p' \text{ and } 1/s = (1/q - 1/p').$$

**Proof of Main Theorem**

**Step 1**($G = \mathbb{T}$) We shall use the following fact: if $\psi \in l_r (r > 2)$, then

$$\sum_{n \in \mathbb{Z}} |n|^{p-2} |\psi(n)|^p < \infty \text{ when } 1 \leq p < r'.$$

Indeed, by Hölder's inequality, we get

$$\sum_{n \in \mathbb{Z}} |n|^{p-2} |\psi(n)|^p \leq \left( \sum_{n \in \mathbb{Z}} |\psi(n)|^r \right)^{p/r} \left( \sum_{n \in \mathbb{Z}} |n|^{\frac{(p-2)r}{r-p}} \right)^{(r-p)/r} < \infty$$

as $\frac{r(2-p)}{r-p} > 1$.

Case 1. $q < r'$. We shall construct a $\phi$ satisfying the condition (a) of Lemma for $p = r'$, and a $\psi \in l_r$ such that $\psi$ is non-negative, even, decreasing and $\phi\psi \notin l_q$. Then $\phi \in (L^{r'}, l_q)$ by Lemma (a) and $\psi \in \hat{L}_r^p$ for $1 \leq p < r'$ (see [2]). Hence $\phi \notin (L_r^p, l_q)$.

Define

$$\psi(n) = \begin{cases} \frac{1}{|n|^{1/r}(\ln|n|)^{1/2}}, & |n| \geq 2; \\ 1, & \text{otherwise.} \end{cases}$$

$$\phi(n) = \begin{cases} \frac{|n|^{((q-r)/qr)}}{(\ln|n|)^{((2-q)/2q)}}, & |n| \geq 2; \\ 1, & \text{otherwise.} \end{cases}$$

Since $r > 2$, $\psi$ satisfies the desired conditions. We show below that $\phi$ satisfies condition (a) of Lemma for $p = r'$.

$$\begin{aligned} \sum_{|n|\geq 2} \frac{|\phi(n)|^{\frac{qr'}{r'-q}}}{|n|^{\left(\frac{r'-2}{r'-q}\right)q}} &= 2\sum_{n=2}^{\infty} \frac{n^{(\frac{q-r}{qr})(\frac{qr'}{r'-q})}}{(\ln n)^{(\frac{2-q}{2q})(\frac{qr'}{r'-q})} n^{\frac{q(r'-2)}{r'-q}}} \\ &= 2\sum_{n=2}^{\infty} \frac{1}{n \ln n^{\frac{(2-q)r'}{2(r'-q)}}} \end{aligned}$$

as $(\frac{q-r}{qr})(\frac{qr'}{r'-q}) = \left(\frac{(r'-2)q}{r'-q} - 1\right)$. Since $\frac{(2-q)r'}{2(r'-q)} > 1$, $\phi$ satisfies condition (a) of Theorem. Next we show that $\phi\psi \notin l_q$,

$$\begin{aligned} \sum_{|n|\geq 2} |\phi(n)|^q |\psi(n)|^q &= 2\sum_{n=2}^{\infty} \frac{n^{\frac{q-r}{r}}}{(\ln n)^{\frac{(2-q)}{2}} n^{q/r} (\ln n)^{q/2}} \\ &= 2\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty \end{aligned}$$

Hence the proof of case 1 is complete.

Case 2. $r' \leq q$. Define

$$\phi(n) = |n|^{((1/r)-(1/q))} \quad \forall n \in \mathbb{Z}$$

and

$$\psi(n) = \begin{cases} \frac{1}{|n|^{1/r}(\ln|n|)^{1/q}}, & |n| \geq 2; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\phi$ satisfies condition (b) of Theorem for $p = r'$. Hence $\phi \in (L^r, l_q)$. Since $q < r, \psi \in l_r$. Therefore,$\psi \in \hat{L}_r^p$ for $1 \leq p < r'$ (see, [2]). We show that $\phi\psi \notin l_q$.

$$\sum_{|n|\geq 2} |\phi(n)|^q |\psi(n)|^q = 2\sum_{n=2}^{\infty} \frac{n^{(q-r)/r}}{n^{q/r}(\ln n)} = 2\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty.$$

Hence $\phi \notin (L^p_r, l_q)$.

This completes the proof of step 1.

**Step 2** Let $G$ be a compact abelian group such that Theorem holds for $G/H$, for some closed subgroup $H$ of $G$. Then it holds for $G$.

**Proof** Let $\phi \in (L^p_r(G/H), l_q(H^\perp))$ such that $\phi \notin (L^{r'}(G/H), l_q(H^\perp))$. Let $f \in L^{r'}(G/H)$ be such that $\phi\widehat{f} \notin l_q(H^\perp)$. Define $\phi = 0$ on $\Gamma\backslash H^\perp$. We show that $\phi \in (L^p_r, l_q)$ and $\phi \notin (L^{r'}, l_q)$. Let $g \in L^p_r$, then $\pi_H(g) \in L^p_r(G/H)$ and $\widehat{(\pi_H(g))} = \widehat{g}$ on $H^\perp$ $\left(\pi_H(f) = \int_H f(x+y)dm_H(y)\right.$, where $m_H$ denotes the Haar measure on $H$). Therefore

$$\phi\widehat{g} = \begin{cases} \phi\widehat{(\pi_H(g))} & \text{on } H^\perp; \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\phi\widehat{g} \in l_q$ as $\phi\widehat{(\pi_H(g))} \in l_q(H^\perp)$. Therefore, $\phi \in (L^p_r, l_q)$. Also, $fo\pi_H \in L^{r'}(G)$ and $\widehat{(fo\pi_H)} = \widehat{f}\chi_{H^\perp}$. Hence $\phi\widehat{(fo\pi_H)} \notin l_q$ and so $\phi \notin (L^{r'}, l_q)$. This completes the proof of step 2.

**Step 3** Since $G$ is not totally disconnected, $\Gamma$ contains an element of infinite order (see,[6]) say, $\gamma_0$. Let $S$ denote the subgroup generated by $\gamma_0$ and $H = S^\perp$. Then $G/H$ is isomorphic with the circle group $\mathbb{T}$. Now the proof of the theorem follows from step 1 and step 2.

**Corollary** *Let $r > 2$ and $1 \le q < r < \infty$, then*

$$(A_r, l_q) \subset (L^{r'}, l_q)$$

**Note** Let $r > 2$, $1 \le q < r < \infty$ and $1 \le p < r'$. It remains open whether

$$(A_r, l_q) \subset (L^p_{r'}, l_q)$$

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