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PRINCIPALLY QUASI-BAER RINGS AND GENERALIZED PRINCIPALLY QUASI-BAER RINGS

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Abstract

In this paper, we investigate the question whether the p.q.-Baer center of a ring R can be extended to R. We give several counterexamples this question and consider some conditions under which the answers of this may be affirmative. The concept of a generalized p.q.-Baer property which is a generalization of Baer property of a ring is also introduced.

1. Introduction

In [15], Kaplansky introduced *Baer* rings as rings in which every right (left) annihilator ideal is generated by an idempotent. According to Clark [9], a ring R is called *quasi-Baer* if the right annihilator of every right ideal is generated (as a right ideal) by an idempotent. Further works on quasi-Baer rings appear in [4], [6], and [17]. Recently, Birkenmeier, Kim and Park [8] called a ring R to be a *right* (resp. *left*) *principally quasi-Baer* (or simply *right* (resp. *left*) *p.q.-Baer*) ring if the right (resp. left) annihilator of a principal right (resp. left) ideal is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer. The class of right or left p.q.-Baer rings is a nontrivial generalization of the class of quasi-Baer rings. For example, if R is a

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commutative von Neumann regular ring which is not complete, then R is p.q.-Baer but not quasi-Baer. Observe that every biregular ring is also a p.q.-Baer ring.

A ring satisfying a generalization of Rickart's condition (i.e., every right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right PP-ring which is also another generalization of a Baer ring. A ring R is called a *right* (resp. *left*) *PP*-ring if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of R is generated (as a right (resp. left) ideal) by an idempotent of R). R is called a *PP*-ring (also called a *Rickart* ring [3, p. 18]) if it is both right and left PP. Baer rings are clearly right (left) PP-rings, and von Neumann regular rings are also right (left) PP-rings by Goodearl [10, Theorem 1.1]. Note that the conditions right PP and right p.q.-Baer are distinct [8, Example 1.3 and 1.5], but R is an abelian PP-ring if and only if R is a reduced p.q.-Baer ring [8, Corollary 1.15].

Throughout this paper R denotes an associative ring with identity. For a nonempty subset X of R, we write $r_R(X) = \{a \in R \mid Xa = 0\}$ and $\ell_R(X) = \{a \in R \mid aX = 0\}$, which are called the right annihilator of X in R and the left annihilator of X in R, respectively.

2. Principally quasi-Baer centers

As a motivation for this section, we recall the following results:

(1) [15, Theorem 7] The center of a Baer ring is Baer.

(2) [7, Proposition 1.8] The center of a quasi-Baer ring is quasi-Baer.

(3) [8, Proposition 1.12] The center of a right p.q.-Baer ring is PP (hence p.q.-Baer).

(4) [1, Theorem D] Every reduced PI-ring with the Baer center is a Baer ring.

It is natural to ask if the p.q.-Baer center of a ring R can be extended to R. In this section, we show that this question has a negative answer, and so we investigate the class of rings with some conditions under which the answer to this question is affirmative.

Let $C(R)$ denote the center of a ring R.

Example 1. (1) Let K be a field. We consider the ring $R = K[X, Y, Z]$ with $XY = XZ = ZX = YX = 0$ and $YZ \neq ZY$. Then R is reduced and $C(R) = K[X]$ is Baer and so p.q-Baer. But $r_R(Y)$ has no idempotents. Thus R is not right p.q.-Baer. Note that

$$I = \{f(Y, Z) \in K[Y, Z] \mid f(0, 0) = 0\}$$

is a two-sided ideal of R and $I \cap C(R) = 0$.

(2) Let

$$R = \left\{ \begin{pmatrix} x & y & z \\ 0 & x & u \\ 0 & 0 & v \end{pmatrix} \mid x, y, z, u, v \in \mathbb{R} \right\} \subseteq \operatorname{Mat}_3(\mathbb{R}),$$

where $\mathbb{R}$ denotes the set of real numbers. Then R is a PI-ring which is not semiprime. Then we see that

$$r_R\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \right) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b, c \in \mathbb{R} \right\}.$$

But this cannot be generated by an idempotent. Hence R is not right p.q.-Baer. On the other hand,

$$C(R) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \mid x \in \mathbb{R} \right\} \cong \mathbb{R}.$$

Therefore $C(R)$ is Baer.

Observe that Example 1(2) also shows that there exists a PI-ring R with the Baer center, but R is not right p.q.-Baer.

However, we have the following results:

Lemma 2. [8, Proposition 1.7] *R is a right p.q.-Baer ring if and only if the right annihilator of any finitely generated right ideal is generated (as a right ideal) by an idempotent.*

Proposition 3. *Let R be a ring with the p.q.-Baer center $C(R)$. If R satisfies any of the following conditions for any nonzero two-sided ideal I of R, then R is quasi-Baer (and hence right p.q.-Baer).*

(1) *$I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$.*

(2) *$I \cap C(R) \neq 0$ and every central idempotent of R is orthogonal.*

(3) *$I \cap C(R) \neq 0$ and every right ideal of R generated by a central element contains $C(R)$.*

Proof. Let I be a nonzero two-sided ideal of R. If $r_R(I) = 0$, then we are done. Thus we assume $r_R(I) \neq 0$.

(1) By hypothesis and Lemma 2, $I \cap C(R) \neq 0$ and $r_{C(R)}(I \cap C(R)) = eC(R)$ for some $e^2 = e \in C(R)$. We claim that $r_R(I) = eR$. If $Ie \neq 0$, then Ie is a nonzero two-sided ideal of R. Thus, by hypothesis, $0 \neq Ie \cap C(R) \subseteq I \cap C(R)$. Let $0 \neq x \in Ie \cap C(R)$. Then $x = ye \in I \cap C(R)$ for some $y \in I$, and so $x = xe = 0$; which is a contradiction. Hence $eR \subseteq r_R(I)$, and then $r_R(I) = R \cap r_R(I) = (eR \oplus (1-e)R) \cap r_R(I) = eR \oplus ((1-e)R \cap r_R(I))$. We show that $(1-e)R \cap r_R(I) = 0$. Suppose that $0 \neq (1-e)R \cap r_R(I)$.

Then $(1-e)R \cap C(R)$ is a nonzero two-sided ideal of R. Thus, by hypothesis, $0 \neq (1-e)R \cap r_R(I) \cap C(R) = (1-e)R \cap r_{C(R)}(I) \subseteq (1-e)R \cap r_{C(R)}(I \cap C(R)) \subseteq (1-e)R \cap eC(R) \subseteq (1-e)R \cap eR = 0$; which is also a contradiction. Therefore $r_R(I) = eR$ and thus R is quasi-Baer.

(2) There exists $0 \neq a \in C(R)$ such that $a \in I$, and so $r_{C(R)}(aC(R)) = eC(R)$ for some $e^2 = e \in C(R)$ by hypothesis. Then $r_R(aR) = eR$. For, since $r_R(aR) \cap C(R) = r_{C(R)}(aC(R)) = eC(R)$, $e \in r_R(aR)$ and so $eR \subseteq r_R(aR)$, and thus $r_R(aR) = eR$ by the similar method to (1). Hence $r_R(I) \subseteq eR$. Now, we claim $eR \subseteq r_R(I)$. If not, there exists $0 \neq x \in R$ such that $x \in I \cap C(R)$ by the same arguments as in (1). Then $r_{C(R)}(xC(R)) = fC(R)$ for some $f^2 = f \in C(R)$ and so $r_R(xR) = fR$. Hence $r_R(I) \subseteq fR \cap eR = 0$; which is a contradiction. Thus $r_R(I) = eR$ for some $e^2 = e \in R$ and therefore R is a quasi-Baer ring.

(3) By hypothesis, there exists $0 \neq a \in I \cap C(R)$ and so $r_{C(R)}(aC(R)) = eC(R)$ for some $e^2 = e \in C(R)$. Then $r_R(aR) = eR$, and this implies $r_R(I) \subseteq eR$ by the same method as in (2). Now, we claim that $eR \subseteq r_R(I)$. If not, there exists $0 \neq x \in Ie \cap C(R) \subseteq I \cap aR \subseteq aR$, by hypothesis. We put $x = ye \in C(R)$ for some $y \in I$. Since $r_R(x) \supseteq r_R(aR) = eR$, we obtain $x = xe = 0$; which is a contradiction. Thus $eR \subseteq r_R(I)$, and consequently $r_R(I) = eR$. Therefore R is a quasi-Baer ring. □ □

Corollary 4. *Let R be a semiprime PI-ring with the p.q.-Baer center $C(R)$. If either every central idempotent of R is orthogonal or every right ideal of R generated by a central element contains $C(R)$, then R is quasi-Baer.*

Proof. It follows from [18, Theorem 6.1.28] and Proposition 3. □ □

Part (1) of the following example shows that the condition "$I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$" and the condition "every central idempotent of R is orthogonal" in Proposition 3 (1) and (2) are not superfluous, respectively, and parts (2) and (3) show that in Proposition 3, the condition (1) is not equivalent to the condition (2).

Example 5. (1) Let $R = \{\langle a_i \rangle \in \prod_{i=1}^{\infty} T_i \mid a_i \text{ is eventually constant}\}$, where $T_i = \mathrm{Mat}_2(F)$ for all i and F is a field. For a two-sided ideal $I = \{\langle a_i \rangle \in R \mid a_i = 0 \text{ if } i \text{ is even}\}$, $r_R(I) = \{\langle b_j \rangle \in R \mid b_j = 0 \text{ if } i \text{ is odd}\}$. Since

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots \right\rangle \notin R,$$

$r_R(I)$ cannot be generated by an idempotent of R. Thus R is not quasi-Baer. Note that

$$C(R) = \left\{ \langle a_i \rangle \in R \mid a_i = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \text{ for some } k \in F \right\}$$

is p.q.-Baer. Now,

$$I \cap C(R) = \left\{ \langle a_i \rangle \in R \mid a_i = 0 \text{ if } i \text{ is even}, a_i = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \mathrm{Mat}_2(F) \text{ if } i \text{ is odd} \right\}$$

is not finitely generated. Moreover,

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots \right\rangle$$

and

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots \right\rangle$$

are idempotents, but they are not orthogonal.

(2) Let $R = F[x_1, x_2, \ldots]$, where F is a field. Then R is a commutative quasi-Baer ring whose the only idempotents 0 and 1 are orthogonal. But the two-sided ideal $\langle x_1^2, x_2^2, \ldots \rangle$ of R is not finitely generated.

(3) Let $R = \mathbb{Z} \oplus \mathbb{Z}$. Then R is a commutative quasi-Baer ring. Since R is Noetherian, every two-sided ideal of R is finitely generated. But the central idempotents $(1, 0)$ and $(1, 1)$ are not orthogonal.

Related to the result of [1, Theorem D], we have the next example.

Example 6. (1) Let $R = \mathcal{C}[0, 1]$ be the ring of all real-valued continuous functions on $[0, 1]$. Then R is commutative (and so PI) and reduced. But R is not p.q.-Baer. For, let

$$f : [0, 1] \to \mathbb{R}$$

defined by

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2}, & \frac{1}{2} < x \le 1 \end{cases}.$$

Then $f \in R$, and so

$$r_R(f) = \{g \in R \mid g((1/2, 1]) = 0\} \neq 0.$$

Suppose that $r_R(f) = eR$ for some nonzero idempotent $e \in R$. Then $e(x)^2 = e(x)$, for each $x \in [0, 1]$. Thus $e(x) = 0$ or $e(x) = 1$. Since $e \in r_R(f)$, $e((\frac{1}{2}, 1]) = \{0\}$. But e is continuous, and so $e(x) = 0$ for each $x \in [0, 1]$. Hence $r_R(f) = 0$; which is a contradiction. Thus R is a reduced PI-ring which is not right p.q.-Baer.

(2) We take the ring in [12, Example 2(1)]. Let $\mathbb{Z}$ be the ring of integers and $\mathrm{Mat}_2(\mathbb{Z})$ the 2×2 full matrix ring over $\mathbb{Z}$. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \;\middle|\; a - d \equiv b \equiv c \equiv 0 \,(\mathrm{mod} 2) \right\}.$$

Then R is right p.q.-Baer, but R is neither right PP nor left PP by [12, Example 2(1)]. Moreover, it can be easily checked that R is an abelian PI-ring with the PP center.

3. Generalized p.q.-Baer rings

Regarding to a generalization of Baer rings as well as a PP-ring, recall that a ring R is called a *generalized right PP*-ring if for any $x \in R$ the right ideal $x^n R$ is projective for some positive integer n, depending on n, equivalently, if for any $x \in R$ the right annihilator of x^n is generated by an idempotent for some positive integer n, depending on n. Left cases may be defined analogously. A ring R is called a *generalized PP*-ring if it is both generalized right and left PP-ring. Right PP-rings are generalized right PP obviously. A number of papers have been written on generalized PP-rings. For basic and other results on generalized PP-rings, see e.g. [11, 14, 16].

As a parallel definition to the generalized PP-property related to the p.q.-Baer property, we define the following.

Definition 7. A ring R is called a *generalized right p.q.-Baer* ring if for any $x \in R$ the right annihilator of $x^n R$ is generated by an idempotent for some positive integer n, depending on n. Left cases is defined analogously. A ring R is called a *generalized p.q.-Baer* ring if it is both generalized right and left p.q.-Baer ring.

We have the following connections.

Lemma 8. [12, Lemma 1] *Let R be a reduced ring. The following are equivalent:*

(1) *R is right PP.*
(2) *R is PP.*
(3) *R is generalized right PP.*
(4) *R is generalized PP.*
(5) *R is right p.q.-Baer.*
(6) *R is p.q.-Baer.*
(7) *R is generalized right p.q.-Baer.*
(8) *R is generalized p.q.-Baer.*

Shin [19] defined that a ring R *satisfies* (S I) if for each $a \in R$, $r_R(a)$ is a two-sided ideal of R, and proved that R satisfies (S I) if and only if $ab = 0$ implies $aRb = 0$ for $a, b \in R$ [19, Lemma 1.2]. The (S I) property was studied in the context of near rings by Bell, in [2], where it is called the insertion of factors principle (simply, IFP). It is well known that every reduced ring has the IFP, and if R has the IFP then it is abelian, but the converses do not hold, respectively.

Recall from [8, Corollary 1.15], R is an abelian PP-ring if and only if R is a reduced p.q.-Baer ring. Similarly, we have the following.

Proposition 9. *Let a ring R have the IFP. Then R is a generalized right PP-ring if and only if R is a generalized right p.q.-Baer ring.*

Proof. For any $x \in R$ and positive integer n, $r_R(x^n) = r_R(x^n R)$ since R has the IFP. □ □

Every right p.q.-Baer rings is generalized right p.q.-Baer, but the converse does not hold, by the next example.

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}$, where $a \in R$ and $m \in M$ and the usual matrix operations are used.

Example 10. [14, Example 2] Let D be a domain and $R = T(D, D)$ be the trivial extension of D. Then R has the IFP and R is a generalized right PP-ring, but it is not a right PP-ring. Thus R is a generalized right p.q.-Baer ring by Proposition 9, but it is not right p.q.-Baer by [8, Proposition 1.14].

Recall from [5], an idempotent $e \in R$ is called *left* (resp. *right*) *semicentral* if $xe = exe$ (resp. $ex = exe$) for all $x \in R$. The set of left (resp. right) semicentral idempotents of R is denoted by $S_\ell(R)$ (resp. $S_r(R)$). Note that $S_\ell(R) \cap S_r(R) = \mathbf{B}(R)$, where $\mathbf{B}(R)$ is the set of all central idempotents of R, and if R is semiprime then $S_\ell(R) = S_r(R) = \mathbf{B}(R)$. Some of the basic properties of these idempotents are indicated in the following.

Lemma 11. [7, Lemma 1.1] *For an idempotent $e \in R$, the following are equivalent:*

(1) $e \in S_\ell(R)$.
(2) $1 - e \in S_r(R)$.
(3) $(1 - e)Re = 0$.
(4) eR *is a two-sided ideal of* R.
(5) $R(1 - e)$ *is a two-sided ideal of* R.

The following example shows that the condition "R has the IFP" in Proposition 9 cannot be dropped.

Example 12. [8, Example 1.6] For a field F, take $F_n = F$ for $n = 1, 2, \ldots$, and let

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \oplus_{n=1}^{\infty} F_n \\ \oplus_{n=1}^{\infty} F_n & < \oplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix},$$

which is a subring of the 2×2 matrix ring over the ring $\prod_{n=1}^{\infty} F_n$, where $< \oplus_{n=1}^{\infty} F_n, 1 >$ is the F-algebra generated by $\oplus_{n=1}^{\infty} F_n$ and 1. Then R is a regular ring by [10, Lemma 1.6], and so R is a generalized PP-ring.

Let $a \in (a_n) \in \prod_{n=1}^{\infty} F_n$ such that $a_n = 1$ if n is odd and $a_n = 0$ if n is even, and let $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$. Now we assume that there exists an idempotent $e \in R$ such that $r_R(\alpha^k R) = eR$ for a positive integer k. Then e is left semicentral, and so e is central since R is semiprime. But this is impossible. Thus R is not generalized right p.q.-Baer. Similarly R is not generalized left p.q.-Baer.

Proposition 13. *Let R be a ring. The following are equivalent:*

(1) *R is generalized right p.q.-Baer.*

(2) *For any principal ideal I of the form Ra^nR of R, where n is a positive integer, there exists $e \in S_r(R)$ such that $I \subseteq Re$ and $r_R(I) \cap Re = (1-e)Re$.*

Proof. The proof is an adaptation from [8, Proposition 1.9]. (1)$\Rightarrow$(2): Assume (1). Then $r_R(I) = r_R(Ra^nR) = r_R(a^nR) = fR$ with $f \in S_\ell(R)$. So $I \subseteq \ell_R(r_R(I)) = R(1-f)$. Let $e = 1-f$, then $e \in S_r(R)$. Hence $r_R(I) \cap Re = (1-e)R \cap Re = (1-e)Re$.

(2)$\Rightarrow$(1): Assume (2). Clearly $(1-e)R \subseteq r_R(I)$ for any ideal I of the form Ra^nR. Let $\alpha \in r_R(I)$, then $\alpha e = e\alpha e + (1-e)\alpha e \in r_R(I) \cap Re = (1-e)Re$. So $e\alpha = e\alpha e = 0$. Hence $\alpha = (1-e)\alpha \in (1-e)R$. Thus $r_R(I) = (1-e)R$, and therefore R is generalized right p.q.-Baer.□ □

Corollary 14. *Let R be a generalized right p.q.-Baer ring. If I is a principal ideal of the form Ra^nR of R, then there exists $e \in S_r(R)$ such that $I \subseteq Re$, $(1-e)Re$ is an ideal of R, and $I + (1-e)Re$ is left essential in Re.*

As a parallel result to [8, Proposition 1.12], we have the following whose proof is also an adaptation from [8].

Proposition 15. *If R is a generalized right p.q.-Baer ring, then the center $C(R)$ of R is a generalized PP-ring.*

Proof. Let $a \in C(R)$. For any positive integer n, there exists $e \in S_\ell(R)$ such that $\ell_R(a^n) = \ell_R(Ra^n) = r_R(a^n) = r_R(a^nR) = eR$. Observe that $\ell_R(Ra^n) = \ell_R r_R \ell_R(Ra^n) = \ell_R r_R(eR)$. Let $r_R(eR) = r_R(e^nR) = fR$ with $f \in S_\ell(R)$, then $1-f \in S_r(R)$. Hence $eR = \ell_R(Ra^n) = \ell_R r_R(eR) = \ell_R(fR) = R(1-f)$. So there exists $x \in R$ such that $e = x(1-f)$ and hence $ef = x(1-f)f = 0$. Now $fe = efe = 0$ because $e \in S_\ell(R)$, and so $ef = fe = 0$. Since $eR = R(1-f)$, there is $y \in R$ such that $1-f = ey$ and so $e = e(1-f) = ey = 1-f$. Thus

$e \in S_\ell(R) \cap S_r(R) = \mathbf{B}(R)$. Consequently, $r_{C}(R)(a^n) = r_R(a^n) \cap C(R) = eR \cap C(R) = eC(R)$. Therefore the center $C(R)$ of R is a generalized PP-ring. □ □

The following example shows that there exists a semiprime ring $\mathcal{R}$ whose center is generalized PP, but $\mathcal{R}$ is not generalized right p.q.-Baer.

Example 16. Let $\mathcal{R} = R \oplus \mathrm{Mat}_2(\mathbb{Z}[x])$, where

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \oplus_{n=1}^{\infty} F_n \\ \oplus_{n=1}^{\infty} F_n & < \oplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix},$$

in Example 12. Then the center of $\mathcal{R}$ is generalized PP. Since R is not generalized right p.q.-Baer by Example 12, $\mathcal{R}$ is not generalized right p.q.-Baer either. Furthermore, due to [14, Example 4], $\mathrm{Mat}_2(\mathbb{Z}[x])$ is not generalized right PP. Thus $\mathcal{R}$ is not generalized right PP.

Note that given a reduced ring R the trivial extension of R (by R) has the IFP by simple computations. However, the trivial extension of a ring R which has the IFP does not have the IFP by [13, Example 11]. We give examples of generalized right p.q.-Baer rings, which are extensions of the trivial extension, as in the following.

Lemma 17. *Let S be a ring and for $n \geq 2$*

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S \right\}.$$

If S has the IFP, then for any $A \in R_n$ and any $E^2 = E \in R_n$, $AE = \mathbf{0}$ implies $AR_nE = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix in R_n.

Proof. Note that every idempotent E in R_n is of the form

$$\begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix}$$

with $e^2 = e \in S$ by [14, Lemma 2]. Suppose that $AE = \mathbf{0}$ for any

$$A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n.$$

Then we have the following: $ae = 0$ and $a_{ij}e = 0$ for $i < j$, $1 \leq i$ and $2 \leq j$. Since S has the IFP, $aSe = 0$ and $a_{ij}Se = 0$ for $i < j$, $1 \leq i$ and $2 \leq j$. These imply $AR_nE = \mathbf{0}$. □ □

Proposition 18. *Let a ring S have the IFP and let R_n for $n \geq 2$ be the ring in Lemma 17. Then the following are equivalent:*

(1) *S is generalized right p.q.-Baer.*

(2) *R_n is generalized right PP.*

(2) *R_n is generalized right p.q.-Baer.*

Proof. (1)⇒(2): Suppose that S is generalized right p.q.-Baer. By Proposition 9, S is generalized right PP. Hence R_n is also generalized right PP by [14, Proposition 3].

(2)⇒(3): Suppose that R_n is generalized right PP. Then for any

$$A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n$$

and a positive integer k, there exists an idempotent

$$E = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \in R_n$$

with $e^2 = e \in S$ such that $r_{R_n}(A^k) = ER_n$. Note that $r_{R_n}(A^kR_n) \subseteq ER_n$. From $r_{R_n}(A^k) = ER_n$, $A^kE = \mathbf{0}$, and so $A^kR_nE = \mathbf{0}$ by Lemma 17. Thus we have $E \in r_{R_n}(A^kR_n)$, and so $ER_n \subseteq r_{R_n}(A^kR_n)$. Consequently, $r_{R_n}(A^kR_n) = ER_n$, and therefore R_n is generalized right p.q.-Baer.

(3)⇒(1): Suppose that R_n is generalized right p.q.-Baer. Let $a \in S$ and consider

$$A = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n.$$

Since R_n is generalized right p.q.-Baer, $r_{R_n}(A^kR_n) = ER_n$ for some $E^2 = E \in R_n$ and a positive integer k. Then by [14, Lemma 2], there is $e^2 = e \in S$ such that

$$E = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \in R_n.$$

Hence $eS \subseteq r_S(a^k S)$. Let $b \in r_S(a^k S)$, then

$$\begin{pmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \in R_n$$

is contained in $r_{R_n}(A^k R_n) = ER_n$, so $b \in eS$. Thus S is also a generalized right p.q.-Baer ring. $\square$ $\blacksquare$

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