

ON \overline{Z}_M -SEMIPERFECT MODULES

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Abstract

Let τ_M be any preradical for $\sigma[M]$ and N any module in $\sigma[M]$. N is called a τ_M -semiperfect module if for every submodule K of N , there is a decomposition $K = A \oplus B$ such that A is a projective direct summand of N in $\sigma[M]$ and $B \subseteq \tau_M(N)$. In this paper we prove that any finite direct sum of τ_M -semiperfect modules is τ_M -semiperfect. It is also shown that if M is a local projective module in $\sigma[M]$, then for every index set Λ , the sum $M^{(\Lambda)}$ is \overline{Z}_M -semiperfect in $\sigma[M]$ if and only if every factor module of $M^{(\Lambda)}$ has a projective \overline{Z}_M -cover.

1 Introduction

Throughout this paper all rings are associative with identity and modules are unitary right modules. For any module M , τ_M will denote a preradical in $\sigma[M]$. Like in [2], a module $N \in \sigma[M]$ is called τ_M -lifting if for every submodule K of N , there exists a decomposition $K = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$. According to [9], any module N in $\sigma[M]$ is called *semiperfect* in $\sigma[M]$ if every factor module of N has a projective cover. By [9, 41.14 and 42.1], if $P \in \sigma[M]$ is projective in $\sigma[M]$, then P is semiperfect if and only if for every submodule K of P , there exists a decomposition $K = A \oplus B$ such that A is a direct summand of P and $B \ll P$. Recently, Özcan and Alkan [7] have defined the τ_M -semiperfect modules in $\sigma[M]$ for any preradical τ_M on $\sigma[M]$. Inspired by this work, we mainly study \overline{Z}_M -semiperfect modules in

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$\sigma[M]$ in this paper. A module $N \in \sigma[M]$ is called \overline{Z}_M -semiperfect in $\sigma[M]$ if for every submodule K of N , there is a decomposition $K = A \oplus B$ such that A is a projective direct summand of N in $\sigma[M]$ and $B \subseteq \overline{Z}_M(N)$.

In Section 2, we will be concerned with the structure of $\overline{Z}_R(R_R)$. It is shown that if R is a right good ring, then $\overline{Z}_R(R_R) = \text{ann}_r(\text{Rad}(R))$. Section 3 is devoted to the study of τ_M -lifting modules. Some results deal with the case $\tau_M = \overline{Z}_M$. In Section 4, we prove that any finite direct sum of τ_M -semiperfect modules is τ_M -semiperfect. In [2], the authors called a module $L \in \sigma[M]$ τ_M -semiperfect, if every factor module of N has a projective τ_M -cover in $\sigma[M]$. Section 5 establishes the relation between this definition of τ_M -semiperfect modules and the one given in [7] in some special cases. In particular, we prove the following Proposition:

Let M be a local projective module in $\sigma[M]$. The following are equivalent for a module N in $\sigma[M]$ which is isomorphic to $M^{(\Lambda)}$ for some index set Λ :

- (1) Every factor module of N has a projective \overline{Z}_M -cover.
- (2) N is \overline{Z}_M -semiperfect.

2 Some Properties of $\overline{Z}_R(R_R)$

Let M be an R -module. A module $N \in \sigma[M]$ is said to be M -small if there exists a module $L \in \sigma[M]$ such that $N \ll L$.

Let $N \in \sigma[M]$. In [8], Talebi and Vanaja define $\overline{Z}_M(N)$ as follows: $\overline{Z}_M(N) = \cap \{ \text{Ker}(g) \mid g \in \text{Hom}(N, L), L \text{ is } M\text{-small} \}$. The module N is called M -cosingular (non- M -cosingular) if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$).

The following proposition maybe well-known. We give here its proof for the sake of completeness.

Proposition 2.1. *Let M be any R -module. Then we have:*

- (1) $\text{Rad}(M)\overline{Z}_R(R_R) = 0$.
- (2) $M\overline{Z}_R(R_R) \leq \overline{Z}_R(M)$.

Proof (1) Let $x \in \text{Rad}(M)$ and let $a \in \overline{Z}_R(R_R)$. Consider the homomorphism $f : R \rightarrow xR$ defined by $f(r) = xr$. Since $xR \ll M$, $\overline{Z}_R(R_R) \leq \text{Ker}(f)$. Thus $f(a) = 0$. That is $xa = 0$.

(2) Let $x \in M$ and consider the homomorphism $f : R \rightarrow M$ defined by $f(r) = xr$. Since $f(\overline{Z}_R(R_R)) \leq \overline{Z}_R(M)$ (see [8, Proposition 2.1(2)]), it follows that $x\overline{Z}_R(R_R) \leq \overline{Z}_R(M)$. Therefore $M\overline{Z}_R(R_R) \leq \overline{Z}_R(M)$. \square

Corollary 2.2. *If R is a ring having a radical R -module M with $\text{ann}_r(M) = 0$, then $\overline{Z}_R(R_R) = 0$.*

Proof Since $\text{Rad}(M)\overline{Z}_R(R_R) = 0$ and $\text{Rad}(M) = M$, we have $M\overline{Z}_R(R_R) = 0$. But $\text{ann}_r(M) = 0$. So $\overline{Z}_R(R_R) = 0$. \square

Example 2.3. We consider the ring \mathbb{Z} . We know that \mathbb{Q} is a radical faithful \mathbb{Z} -module. Then $\overline{\mathbb{Z}}_{\mathbb{Z}}(\mathbb{Z}) = 0$ by Corollary 2.2.

The proof of the next Proposition is similar to the proof of [1, Proposition 17.10].

Proposition 2.4. Let R be any ring with $I = \overline{\mathbb{Z}}_R(R_R)$. If P is a projective right R -module, then $\overline{\mathbb{Z}}_R(P) = PI$.

Proof Since P is projective, P is a direct summand of a free module $R^{(A)}$. So there exists a submodule Q of $R^{(A)}$ such that $R^{(A)} = P \oplus Q$. By [8, Proposition 2.1(4)], $\overline{\mathbb{Z}}_R(P) \oplus \overline{\mathbb{Z}}_R(Q) = (\overline{\mathbb{Z}}_R(R_R))^{(A)} = I^{(A)} = R^{(A)}.I = PI \oplus QI$. But $PI \leq \overline{\mathbb{Z}}_R(P)$ by Proposition 2.1. Then $\overline{\mathbb{Z}}_R(P) = PI$. \square

Definition 2.5. Following [3, p. 236], a ring R is called a right good ring if for every right R -module M we have $M\text{Rad}(R) = \text{Rad}(M)$.

Clearly, any semilocal ring is a good ring (see [3, Theorem 9.7.1]). Especially, every artinian ring is a good ring.

Proposition 2.6. Let R be a right good ring. We have $\overline{\mathbb{Z}}_R(R_R) = \text{ann}_r(\text{Rad}(R))$.

Proof By Proposition 2.1(1), $\overline{\mathbb{Z}}_R(R_R) \subseteq \text{ann}_r(\text{Rad}(R))$.

Now let $r \in \text{ann}_r(\text{Rad}(R))$ i.e. $\text{Rad}(R)r = 0$ and let $f : R \rightarrow L$ be a homomorphism where L is a small submodule of an R -module X . Since $\text{Rad}(X) = X\text{Rad}(R)$, we have $\text{Rad}(X)r = 0$. Thus $Lr = 0$ and hence $f(1)r = 0$. That is $r \in \text{Ker}(f)$. Therefore $r \in \overline{\mathbb{Z}}_R(R_R)$ and so $\text{ann}_r(\text{Rad}(R)) \subseteq \overline{\mathbb{Z}}_R(R_R)$. Consequently, $\overline{\mathbb{Z}}_R(R_R) = \text{ann}_r(\text{Rad}(R))$. \square

Corollary 2.7. Let R be a semilocal ring. Then:

- (1) $\overline{\mathbb{Z}}_R(R_R) = \text{Soc}({}_R R)$ is an ideal of R .
- (2) If P is a projective R -module, then $\overline{\mathbb{Z}}_R(P) = P\text{Soc}({}_R R)$.
- (3) If $\text{Soc}({}_R R) = \text{Soc}(R_R)$, then for every projective right R -module P we have $\overline{\mathbb{Z}}_R(P) = \text{Soc}(P)$.

Proof (1) By Proposition 2.6, $\overline{\mathbb{Z}}_R(R_R) = \text{ann}_r(\text{Rad}(R))$. By [1, Proposition 15.17], $\text{Soc}({}_R R) = \text{ann}_r(\text{Rad}(R))$. The result follows.

(2) By (1) and Proposition 2.4.

(3) It is known that $\text{Soc}(P) = P\text{Soc}({}_R R)$ (See [1, Exercise 17.12]). But by hypothesis, we have $\text{Soc}({}_R R) = \text{Soc}(R_R)$. So $\text{Soc}(P) = P\text{Soc}({}_R R)$. By (2), we get $\overline{\mathbb{Z}}_R(P) = \text{Soc}(P)$. \square

Corollary 2.8. Let R be a QF ring. Then for every projective right R -module P we have $\overline{\mathbb{Z}}_R(P) = \text{Soc}(P)$.

Proof By Corollary 2.7 and [5, Corollary 15.7]. \square

Examples 2.9. (1) We consider the ring \mathbb{Z} . We have $\text{Rad}(\mathbb{Z}) = 0$. Then $\text{ann}_r(\text{Rad}(\mathbb{Z})) = \mathbb{Z}$. But $\overline{Z}_{\mathbb{Z}}(\mathbb{Z}) = 0$. Therefore $\overline{Z}_{\mathbb{Z}}(\mathbb{Z}) \neq \text{ann}_r(\text{Rad}(\mathbb{Z}))$.

(2) Let R be a Dedekind domain and let M be any radical R -module. Then $\overline{Z}_R(M) = M$. In fact, if $f : M \rightarrow L$ is a homomorphism where L is a small R -module, then $\frac{M}{\text{Ker}(f)} \cong \text{Im}(f)$. Hence $\text{Im}(f)$ is radical. Since R is a Dedekind domain, $\text{Im}(f)$ is injective. But $\text{Im}(f)$ is a small R -module. Thus $\text{Im}(f) = 0$ and $\text{Ker}(f) = M$. In particular, we have $\overline{Z}_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}$.

(3) If R is semisimple, then $\overline{Z}_R(R_R) = R$ (because the only R -small module is 0).

Recall that an R -module M is called V -module if every simple module in $\sigma[M]$ is M -injective. By [8, Proposition 2.5], if M is an R -module, then M is a V -module if and only if every module in $\sigma[M]$ is non- M -cosingular. Moreover, by [8, Corollary 2.6], R is a right V -ring if and only if the module R is non- R -cosingular. We can also give the following Proposition:

Proposition 2.10. A ring R is a right V -ring if and only if for every right R -modules M and N with $N \in \sigma[M]$, we have $\overline{Z}_M(N) = N$.

Proof Assume that R is a right V -ring. Then for every right R -module L we have $\text{Rad}(L) = 0$. So 0 is the only M -small R -module. Then $\overline{Z}_M(N) = N$. The converse is clear by [8, Corollary 2.6]. \square

3 τ_M -lifting modules

Lemma 3.1. Let $N \in \sigma[M]$. The following are equivalent:

(i) For every submodule K of N , there is a decomposition $K = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$.

(ii) For every submodule K of N , there is a direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$.

(iii) For every submodule K of N , there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$.

Proof This is clear. \square

A module $N \in \sigma[M]$ is called τ_M -lifting if it satisfies one of the equivalent conditions of Lemma 3.1. τ_M -lifting modules are studied in [2] for any radical τ on $\sigma[M]$. Note that by [2, Proof of 2.10(2)], any direct summand of a τ_M -lifting module is again τ_M -lifting for every preradical τ_M .

Theorem 3.2. *Let $N \in \sigma[M]$ with $N = N_1 \oplus N_2$ be a direct sum of relatively projective modules N_1 and N_2 such that N_1 is semisimple and N_2 is τ_M -lifting. Then N is τ_M -lifting.*

Proof We follow the proof in [4, Theorem 6]. Let L be any submodule of N . Note that $K = N_1 \cap (L + N_2)$ is a submodule of N_1 . Then $N_1 = K \oplus K'$ for some submodule K' of N_1 and hence $N = K \oplus K' \oplus N_2 = L + (N_2 \oplus K')$. It is easy to see that K is $(N_2 \oplus K')$ -projective. Then there exists a submodule L' of L such that $N = L' \oplus (N_2 \oplus K')$ by [9, 41.14]. Since $K' \cap (L + N_2) = K' \cap K = 0$, it is easy to check that $L \cap (X + K') = X \cap (L + K')$ for every $X \leq N_2$. Note that N_2 is τ_M -lifting and $N_2 \cap (L + K')$ is a submodule of N_2 . Therefore there exists a decomposition $N_2 = A_1 \oplus A_2$ such that $A_1 \subseteq N_2 \cap (L + K')$ and $A_2 \cap (L + K') \subseteq \tau_M(N_2)$. Thus $N = (L' \oplus A_1) \oplus (A_2 \oplus K')$, $L' \oplus A_1 \subseteq L$ and $L \cap (A_2 \oplus K') = A_2 \cap (L + K') \subseteq \tau_M(N)$. Consequently, N is τ_M -lifting. \square

The following result can be found also in [2] or in [7].

Lemma 3.3. *Let N be τ_M -lifting in $\sigma[M]$. Then $\text{Rad}(N) \subseteq \tau_M(N)$.*

Proof Since N is τ_M -lifting in $\sigma[M]$, $N/\tau_M(N)$ is semisimple. The result follows. \square

Proposition 3.4. *Let $N \in \sigma[M]$ be an indecomposable module such that N is τ_M -lifting in $\sigma[M]$. Then either $\tau_M(N) = N$ or N is a local module with $\tau_M(N) = \text{Rad}(N)$.*

Proof Suppose that $\tau_M(N) \neq N$. Let X be any submodule of N . Then $X = A \oplus B$ with A is a direct summand of N and $B \leq \tau_M(N)$. But N is indecomposable. Thus $A = 0$ or $A = N$. So $X \leq \tau_M(N)$ or $X = N$. Therefore N is a local module and $\tau_M(N)$ is the maximal submodule of N . \square

Corollary 3.5. *Let $N \in \sigma[M]$ be a local module. The following are equivalent:*

- (1) N is τ_M -lifting in $\sigma[M]$.
- (2) either $\tau_M(N) = N$ or $\tau_M(N) = \text{Rad}(N)$.

Proof (1) \Rightarrow (2) By Proposition 3.4.

(2) \Rightarrow (1) Immediate. \square

The following result gives some examples of modules M such that M is $\overline{\tau}_M$ -lifting and shows that, in general, a local module over a ring R need not be τ_M -lifting.

Proposition 3.6. *Let R be a local ring with maximal ideal m . Then the following are equivalent:*

- (i) R_R is $\overline{\tau}_R$ -lifting.
- (ii) $m^2 = 0$.

Proof From Proposition 2.6 we have $\overline{Z}_R(R_R) = \text{ann}_r(\text{Rad}(R)) = \text{ann}_r(m)$. The result is a consequence of Corollary 3.5. \square

Proposition 3.7. *Let N be a τ_M -lifting module in $\sigma[M]$ such that $N = \bigoplus_{j \in J} N_j$ is a finite direct sum of indecomposable submodules N_j ($j \in J$). Then $N = K \oplus L$ such that $\tau_M(K) = K$ and L is a direct sum of local submodules with $\tau_M(L) = \text{Rad}(L)$.*

Proof By [2, Proof of 2.10(2)], every N_j ($j \in J$) is τ_M -lifting. It follows that for every $j \in J$, we have $\tau_M(N_j) = N_j$ or $\tau_M(N_j) = \text{Rad}(N_j)$ by Proposition 3.4. The result follows. \square

Corollary 3.8. *Let $N \in \sigma[M]$ be a module with finite hollow dimension such that N is projective in $\sigma[M]$. If N is τ_M -lifting, then $N = K \oplus L$ with K is semiperfect in $\sigma[M]$ and $L = \tau_M(L)$.*

Proof It is well-known that N is a finite direct sum of indecomposable submodules. The result follows from Proposition 3.7 and [9, 42.3]. \square

Corollary 3.9. *Let N be a \overline{Z}_M -lifting module in $\sigma[M]$ such that $N = \bigoplus_{i \in I} N_i$ is a direct sum of indecomposable submodules N_i ($i \in I$). Then $N = K \oplus L$ such that $\overline{Z}_M(K) = K$, L is a direct sum of local submodules and $\overline{Z}_M(L) = \text{Rad}(L)$.*

Proof By Proposition 3.4 and [8, Proposition 2.1(4)]. \square

Proposition 3.10. *Let R be a commutative ring and M an R -module. Let L be a local \overline{Z}_M -lifting module in $\sigma[M]$. Then L is simple or $\overline{Z}_M(L) = \text{Rad}(L)$.*

Proof Suppose that L is not simple. Let $x \in L$ such that $L = xR$ and let N be the maximal submodule of L . By Lemma 3.3, we have $N \subseteq \overline{Z}_M(L)$. Let $a \in N$ such that $a \neq 0$. It is clear that aR is an M -small module. Consider the homomorphism $f : xR \rightarrow aR$ defined by $f(xr) = ar$ for all $r \in R$. It is well defined, because if $xr = 0$ for some $r \in R$, then $r \in \text{ann}_r(L)$ (R is commutative), and hence $ar = 0$. This gives $\overline{Z}_M(L) \subseteq \text{Ker}(f)$. Since $\text{Ker}(f) \neq L$ ($f \neq 0$), $\text{Ker}(f) \subseteq N$. Thus $\overline{Z}_M(L) \subseteq N$. Therefore $\overline{Z}_M(L) = \text{Rad}(L)$. \square

Corollary 3.11. *Suppose that R is commutative. Let P be a projective semiperfect module in $\sigma[M]$. Then the following are equivalent:*

- (1) P is \overline{Z}_M -lifting.
- (2) $P = P_1 \oplus P_2$ with $\text{Rad}(P_1) = \overline{Z}_M(P_1)$ and P_2 is semisimple.

Proof (1) \Rightarrow (2) Since P is projective semiperfect in $\sigma[M]$, P is a direct sum of local submodules and $Rad(P) \ll P$ by [9, 42.4]. By Proposition 3.10 and [8, Proposition 2.1], $P = P_1 \oplus P_2$ with $Rad(P_1) = \overline{Z}_M(P_1)$ and P_2 is semisimple.

(2) \Rightarrow (1) Since P_1 is projective semiperfect in $\sigma[M]$, as a direct summand of P , and $Rad(P_1) = \overline{Z}_M(P_1)$, P_1 is \overline{Z}_M -lifting. From Theorem 3.2, we conclude that P is \overline{Z}_M -lifting. \square

4 τ_M -Semiperfect Modules

Let τ_M be any preradical for $\sigma[M]$ and let N be a module in $\sigma[M]$. The module N is called a τ_M -semiperfect module in $\sigma[M]$ if it satisfies one of the following equivalent two conditions (See [7, Proposition 2.1]):

- (1) For every submodule K of N , there is a decomposition $K = A \oplus B$ such that A is a projective direct summand of N in $\sigma[M]$ and $B \subseteq \tau_M(N)$;
- (2) For every submodule K of N , there is a decomposition $N = A \oplus B$ such that A is projective in $\sigma[M]$, $A \leq K$ and $K \cap B \subseteq \tau_M(N)$.

It is easy to see that every τ_M -semiperfect module is τ_M -lifting and any projective module in $\sigma[M]$ is τ_M -lifting if and only if it is τ_M -semiperfect in $\sigma[M]$.

Examples 4.1. (1) Let K be a field and let $R = \prod_{n \geq 1} K_n$ with $K_n = K$ for all $n \geq 1$. Then the ring R is a von Neumann regular ring which is not semisimple (See [3, p. 264]). Hence the R -module R is not semiperfect ($Rad(R) = 0$). On the other hand, since $\overline{Z}_R(R) = R$, the module R is \overline{Z}_R -semiperfect.

(2) If R is a DVR, then the R -module R_R is semiperfect but by Proposition 3.6, the module R_R is not \overline{Z}_R -semiperfect.

Theorem 4.2. Any finite direct sum of τ_M -semiperfect modules is τ_M -semiperfect.

Proof To prove this result it is sufficient by induction to prove that a direct sum of two τ_M -semiperfect modules in $\sigma[M]$ is again τ_M -semiperfect in $\sigma[M]$. Let $N = N_1 \oplus N_2$ such that N_1 and N_2 are τ_M -semiperfect modules in $\sigma[M]$. Let L be any submodule of N . Note that $N_1 \cap (L + N_2)$ is a submodule of N_1 and N_1 is τ_M -semiperfect in $\sigma[M]$. Thus there exists a decomposition $N_1 = A_1 \oplus B_1$ such that A_1 is projective in $\sigma[M]$ with $A_1 \subseteq N_1 \cap (L + N_2)$ and $B_1 \cap (L + N_2) \subseteq \tau_M(N_1)$. Then $N = L + (B_1 \oplus N_2)$. Since $N_2 \cap (L + B_1)$ is a submodule of N_2 , there exists a decomposition $N_2 = A_2 \oplus B_2$ such that A_2 is projective in $\sigma[M]$ with $A_2 \subseteq N_2 \cap (L + B_1)$ and $B_2 \cap (L + B_1) \subseteq \tau_M(N_2)$. Then $N = L + (B_1 \oplus B_2) = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$. Since A_1 and A_2 are projective in $\sigma[M]$, $A_1 \oplus A_2$ is $(B_1 \oplus B_2)$ -projective. Hence by [9, 41.14], $N = L' \oplus (B_1 \oplus B_2)$ for some submodule L' of N with $L' \subseteq L$. Clearly, L' is projective in $\sigma[M]$. Now, $L \cap (B_1 \oplus B_2) \subseteq B_1 \cap (L + B_2) + B_2 \cap (L + B_1)$

implies that $L \cap (B_1 \oplus B_2) \subseteq \tau_M(N_1) \oplus \tau_M(N_2) = \tau_M(N)$. Therefore N is τ_M -semiperfect in $\sigma[M]$. \square

Note that Theorem 4.2 generalizes [7, Theorem 2.10].

Corollary 4.3. *Suppose that R is commutative. Let P be a projective module in $\sigma[M]$ with $\text{Rad}(P) \ll P$ and P has finite hollow dimension. Then the following are equivalent:*

- (1) P is \overline{Z}_M -semiperfect.
- (2) $P = P_1 \oplus P_2 \oplus P_3$ with P_1 is semiperfect and $\text{Rad}(P_1) = \overline{Z}_M(P_1)$, P_2 is semisimple and $\overline{Z}_M(P_3) = P_3$.

Proof (1) \Rightarrow (2) By Corollary 3.8, $P = K \oplus L$ with K is semiperfect and $L = \overline{Z}_M(L)$. Now following Corollary 3.11, $K = K_1 \oplus K_2$ with $\text{Rad}(K_1) = \overline{Z}_M(K_1)$ and K_2 is semisimple. It is clear that K_1 is semiperfect in $\sigma[M]$.

(2) \Rightarrow (1) It is clear that P_1, P_2 and P_3 are all \overline{Z}_M -semiperfect in $\sigma[M]$. So P is \overline{Z}_M -semiperfect by Theorem 4.2. \square

5 \overline{Z}_M -semiperfect Modules and \overline{Z}_M -cover

Let $N \in \sigma[M]$. We call an epimorphism $f : P \longrightarrow N$ a projective τ_M -cover of N in $\sigma[M]$ if P is projective in $\sigma[M]$ and $\text{Ker}(f) \subseteq \tau_M(P)$.

In [2], the authors called a module $L \in \sigma[M]$ τ_M -semiperfect, if every factor module of N has a projective τ_M -cover. In [7, Theorem 2.23], the authors showed that this definition agrees with the one given in this paper for a projective module in $\sigma[M]$ and for the preradical Soc . It is of interest to know whether these two definitions coincide in the case $\tau_M = \overline{Z}_M$ for projective modules. It is not our purpose to answer the question. We will touch on only some special cases.

Lemma 5.1. *Let $N \in \sigma[M]$. If every factor module of N has a projective \overline{Z}_M -cover, then $\text{Rad}(N) \leq \overline{Z}_M(N)$.*

Proof By [2, 2.17], $\frac{N}{\overline{Z}_M(N)}$ is semisimple. Therefore $\text{Rad}(N) \leq \overline{Z}_M(N)$. \square

Proposition 5.2. *Let $N \in \sigma[M]$ be a projective module in $\sigma[M]$. If N is τ_M -semiperfect in $\sigma[M]$, then every factor module of N has a projective τ_M -cover.*

Proof Let A be a submodule of N . By hypothesis, $N = N_1 \oplus N_2$ such that $N_1 \subseteq A$ and $A \cap N_2 \subseteq \tau_M(N_2)$. Now consider the canonical epimorphism $f : N_2 \longrightarrow N/A$ with $\text{Ker}(f) = A \cap N_2$. Clearly, f is a projective τ_M -cover of N/A . \square

Proposition 5.3. *Let M be a module such that every simple module in $\sigma[M]$ is M -small. Then for every module N in $\sigma[M]$, we have $\overline{Z}_M(N) \leq \text{Rad}(N)$.*

Proof By the definition of the radical (see [1, p. 109 and 120]), we have $\text{Rad}(N) = \cap\{\text{Ker}h \mid h \in \text{Hom}_R(N, S) \text{ and } S \text{ is a simple } R\text{-module}\}$. Thus $\text{Rad}(N) = \cap\{\text{Ker}h \mid h \in \text{Hom}_R(N, S) \text{ and } S \text{ is a simple module with } S \in \sigma[M]\}$. By hypothesis and the definition of $\overline{Z}_M(N)$, we have $\overline{Z}_M(N) \leq \text{Rad}(N)$. \square

Corollary 5.4. *Let M be an R -module such that every simple module in $\sigma[M]$ is M -small and let N be any module in $\sigma[M]$. Consider the following conditions:*

- (1) *Every factor module of N has a projective \overline{Z}_M -cover.*
 - (2) *Every factor module of N has a projective Rad -cover.*
- Then (1) implies (2).*

Proof By Proposition 5.3. \square

The following example shows that, in general, the converse of Corollary 5.4 is false.

Example 5.5. *Let R be a local ring with maximal ideal m such that $m \neq 0$. Suppose that every factor of R has a projective \overline{Z}_R -cover. By Lemma 5.1, $m \subseteq \overline{Z}_R(R_R)$. But $\overline{Z}_R(R_R) = \text{ann}_r(m)$. Then $\overline{Z}_R(R_R) = m$. By Proposition 3.6, R_R is \overline{Z}_R -semiperfect. This proves that R_R is \overline{Z}_R -semiperfect if and only if every factor of R has a projective \overline{Z}_R -cover.*

Let R be a DVR with maximal ideal m . Consider the ring $S = \frac{R}{m^3}$. It is clear that S is not a von Neuman regular ring ($m \neq m^3$). Thus the simple S -module $\frac{S}{m/m^3}$ is not an injective S -module. Hence every simple S -module is S -small. Since $(\frac{m}{m^3})^2 \neq 0$, the module S_S is not \overline{Z}_S -semiperfect by Proposition 3.6. Thus the S -module S_S does not satisfy the condition (1) of Corollary 5.4. But it is easily seen that the module S_S satisfies the condition (2).

Lemma 5.6. *If M is a local R -module with maximal submodule K , then $\frac{M}{K}$ is M -small or $\overline{Z}_M(N) = N$ for every module $N \in \sigma[M]$.*

Proof Let I be a right ideal of R such that $M \cong \frac{R}{I}$ and let J be a right maximal ideal of R such that $\frac{M}{K} \cong \frac{R}{J}$. Since M is local, J is the only right maximal ideal over I . Let S be a simple module in $\sigma[M]$. Then S is isomorphic to a submodule of a factor module of a direct sum $M^{(\Lambda)}$ for some index set Λ . Since $MI = 0$, we have $SI = 0$. Hence $I \subseteq \text{ann}_r(S)$. But $\text{ann}_r(S)$ is a right maximal ideal of R . Therefore $\text{ann}_r(S) = J$. So $S \cong \frac{M}{K}$. Suppose that $\frac{M}{K}$ is not M -small. By [6, 5.1.4], every simple module in $\sigma[M]$ is M -injective. Thus the module M is a V -module. Hence $\overline{Z}_M(N) = N$ for every module $N \in \sigma[M]$ by [8, Proposition 2.5]. \square

Proposition 5.7. *Let N be a nonzero module in $\sigma[M]$ which has a projective Rad -cover in $\sigma[M]$. Then N has a maximal submodule.*

Proof Let $f : P \rightarrow N$ be a projective Rad -cover of N in $\sigma[M]$. Then $Ker(f) \subseteq Rad(P)$. By [9, 22.3], P contains a maximal submodule K . Therefore $Ker(f) \subseteq K$. It is easy to check that $f(K)$ is a maximal submodule of N . \square

Proposition 5.8. *Let M be a local projective module in $\sigma[M]$ with maximal submodule K . The following are equivalent for a module N in $\sigma[M]$ which is isomorphic to $M^{(\Lambda)}$ for some index set Λ :*

- (1) *Every factor module of N has a projective \overline{Z}_M -cover.*
- (2) *N is \overline{Z}_M -semiperfect.*

Proof (1) \Rightarrow (2) Since M is local, $\overline{Z}_M(L) = L$ for every module $L \in \sigma[M]$ or $\frac{M}{K}$ is M -small by Lemma 5.6. If $\overline{Z}_M(L) = L$ for every module $L \in \sigma[M]$, then $\overline{Z}_M(N) = N$. So N is \overline{Z}_M -semiperfect. By the proof of Lemma 5.6, every simple module in $\sigma[M]$ is isomorphic to $\frac{M}{K}$. So if $\frac{M}{K}$ is M -small, then Corollary 5.4 shows that every factor module of N has a projective Rad -cover. By Proposition 5.7, every proper submodule of N is contained in a maximal submodule of N . Hence $Rad(N) \ll N$. But N is projective in $\sigma[M]$ and it is a direct sum of local modules. Thus N is semiperfect in $\sigma[M]$ by [9, 42.3(1)] and [10, Satz 1.4(A)]. Therefore N is \overline{Z}_M -semiperfect in $\sigma[M]$ since $Rad(N) \leq \overline{Z}_M(N)$ (See Lemma 5.1).

(2) \Rightarrow (1) By Proposition 5.2. \square

Proposition 5.9. *Let R be a semilocal ring such that $Soc({}_R R) = Soc(R_R)$ and let N be a projective R -module. The following are equivalent:*

- (1) *Every factor module of N has a projective \overline{Z}_R -cover.*
- (2) *N is \overline{Z}_R -semiperfect.*
- (3) *N is Soc-semiperfect.*

Proof Let P be any projective R -module. By Corollary 2.7, we have $\overline{Z}_R(P) = Soc(P)$. The result follows from [7, Theorem 2.23]. \square

Corollary 5.10. *Let R be a QF ring. Let N be a projective R -module. The following are equivalent:*

- (1) *Every factor module of N has a projective \overline{Z}_R -cover.*
- (2) *N is \overline{Z}_R -semiperfect.*
- (3) *N is Soc-semiperfect.*

Proof By Proposition 5.9 and [5, Corollary 15.7]. \square

Proposition 5.11. *Suppose that τ_M is a radical for $\sigma[M]$. Let N be a projective module in $\sigma[M]$ such that N is Soc-lifting. The following are equivalent:*

- (1) *Every factor module of N has a projective τ_M -cover.*
- (2) *N is τ_M -semiperfect in $\sigma[M]$.*

Proof (1) \Rightarrow (2) By [2, 2.17], $\frac{N}{\tau_M(N)}$ is semisimple. Hence $Rad(N) \leq \tau_M(N)$. On the other hand, $N = N_1 \oplus N_2$ where N_1 is semisimple and $\tau_M(N_2)$ is essential in N_2 ([2, 2.2 and 2.16]). Therefore $Soc(N_2) \leq \tau_M(N_2)$. Since N is Soc-lifting, N_2 is Soc-lifting and hence it is τ_M -lifting. By Theorem 3.2, N is τ_M -lifting. Therefore N is τ_M -semiperfect since it is projective in $\sigma[M]$.

(2) \Rightarrow (1) Clear by Proposition 5.2. \square

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