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ON s-IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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Abstract

In this paper we characterize compact-covering (resp. pseudo-sequence-covering, subsequence-covering, sequentially-quotient) s-images of locally separable metric spaces by certain point-countable covers.

1 Introduction

Characterizations of images of metric spaces have attracted many authors (see [1], [6], [7], [10]). Recently, many topologists were engaged in s-images of locally separable metric spaces. In [10] and [12], authors have characterized quotient s-images and sequence-covering s-images of locally separable metric spaces as follows.

Theorem 1.1 ([10], Theorem 2.2). *The following conditions are equivalent for a space X.*

1. X is a quotient s-image of a locally separable metric space,

2. X is a sequential space, and there exists a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ of X where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for some subsequence of S.

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Theorem 1.2 ([12], Theorem 3.4). *The following are equivalent for a space X.*

1. X is a sequence-covering s-image of a locally separable metric space,

2. X has a point-countable cs-network consisting of cosmic subspaces,

3. X has a point-countable cs-network, and an so-cover consisting of $\aleph_0$-subspaces.

Note that, for a map $f : X \longrightarrow Y$, f is compact-covering or sequence-covering $\Rightarrow$ f is pseudo-sequence-covering $\Rightarrow$ subsequence-covering, and f is quotient if and only if f is subsequence-covering such that Y is sequential. Any of each converse implication need not hold. Compact-covering maps and sequence-covering maps are exclusive ([15], [16]). These lead us to be interested in compact-covering (resp. pseudo-sequence-covering, subsequence-covering, sequentially-quotient) s-images of locally separable metric spaces, that is, we are interested in the following question.

Question 1.3. *How are compact-covering (resp. pseudo-sequence-covering, subsequence-covering, sequentially-quotient) s-images of locally separable metric spaces characterized by means of point-countable covers?*

In this paper we give internal characterizations of compact-covering (pseudo-sequence-covering, subsequence-covering, sequentially-quotient) s-images of locally separable metric spaces by means of certain point-countable covers to answer Question 1.3 completely.

Throughout this paper, all spaces are assumed to be regular and T_1, all maps are assumed continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers, ω denotes $\mathbb{N} \cup \{0\}$, and a convergent sequence includes its limit point. Let $f : X \longrightarrow Y$ be a map and $\mathcal{P}$ be a collection of subsets of X, we denote $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$ and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$.

Definition 1.4. Let A be a subset of a space X and $\mathcal{P}$ be a collection of subsets of X.

$\mathcal{P}$ is a *cover for A in X*, if $A \subset \bigcup \mathcal{P}$.

$\mathcal{P}$ is a *network at x in X*, if $x \in P$ for every $P \in \mathcal{P}$ and whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

$\mathcal{P}$ is a *network for A in X*, if whenever $x \in U \cap A$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

$\mathcal{P}$ is a *k-network for A in X*, if whenever $K \subset U \cap A$ with K compact and U open in X, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

$\mathcal{P}$ is a *cfp-cover for A in X*, if $\mathcal{P}$ is finite and for each $P \in \mathcal{P}$ there is a closed set A_P in A with $A_P \subset P$ such that $A \subset \bigcup\{A_P : P \in \mathcal{P}\}$. Note that a such family $\mathcal{P}$ is a *full cover* in [1].

$\mathcal{P}$ is a *cfp-network for A in X*, if whenever $K \subset U \cap A$ with K compact and U open in X, then there is a cfp-cover $\mathcal{F} \subset \mathcal{P}$ for K in X such that $K \subset \bigcup \mathcal{F} \subset U$. Note that a such family $\mathcal{P}$ is a *strong k-network* in [1].

$\mathcal{P}$ is a *cs-network for A in X* (resp. *cs^*-network for A in X*), if whenever S is a convergent sequence in A converging to $x \in A \cap U$ with U open in X, then S is eventually (resp. frequently) in $P \subset U$ for some $P \in \mathcal{P}$. Note that when A is a convergent sequence there are different definitions in [10] but they are the same if $\mathcal{P}$ is a network for A in X.

A cover $\mathcal{P}$ for A in X is an *irreducible cover for A in X*, if whenever $\mathcal{Q} \subset \mathcal{P}$ covers A, then $\mathcal{Q} = \mathcal{P}$.

$\mathcal{P}$ is *point-countable*, if $\{P \in \mathcal{P} : x \in P\}$ is countable for each point $x \in X$.

If $A = X$, then a cover (resp. network, cfp-cover, cfp-network, cs-network, cs^*-network, irreducible cover) $\mathcal{P}$ for A in X is abbreviated to a *cover* (resp. *network, cfp-cover, cfp-network, cs-network, cs^*-network, irreducible cover*) *$\mathcal{P}$ for X* (see [1], [14]).

Definition 1.5 ([14]). Let $f : X \longrightarrow Y$ be a map.

f is a *compact-covering* map, if each compact subset of Y is the image of some compact subset of X.

f is an *s-map*, if whenever $y \in Y$, then $f^{-1}(y)$ is a separable subset of X.

f is a *sequence-covering* map, if each convergent sequence in Y is the image of some convergent sequence in X.

f is a *pseudo-sequence-covering* map, if each convergent sequence in Y is the image of some compact subset of X.

f is a *sequentially-quotient* map, if for each convergent sequence S in Y, there is a convergent sequence L in X such that $f(L)$ is a convergent subsequence of S.

f is a *subsequence-covering* map, if for each convergent sequence S in Y there is a compact subset K of X such that $f(K)$ is a convergent subsequence of S.

Definition 1.6 ([14]). Let X be a space.

X is an *$\aleph_0$-space*, if X has a countable cs-network. Note that "cs-" can be replaced by "k-", "cfp-", or "cs^*-".

X is a *Fréchet* space, if whenever $x \in \overline{A}$ with $A \subset X$, then there is a sequence in A converging to x.

X is a *sequential* space, if whenever A is a non closed subset of X, then there is a sequence in A converging to a point not in A.

For terms which are not defined here, please refer to [10], [12] and [14].

2 Results

Firstly, we give some results on preservations of certain networks under covering-maps and on relations of covering-maps.

Lemma 2.1. *Let $f : X \longrightarrow Y$ be a map.*

1. *If $\mathcal{P}$ is a cs-network for a convergent sequence L in X, then $f(\mathcal{P})$ is a cs-network for $f(L)$ in Y.*
2. *If $\mathcal{P}$ is a cfp-cover for a compact subset K in X, then $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.*
3. *If $\mathcal{P}$ is a cfp-network for a compact subset K in X, then $f(\mathcal{P})$ is a cfp-network for $f(K)$ in Y.*

Proof (1) and (2) are routine.

(3). Let $H \subset f(K) \cap U$ with H compact and U open in Y. Then $G = f^{-1}(H) \cap K$ is compact and $G \subset K \cap f^{-1}(U)$ with $f^{-1}(U)$ open in X. It is easy to see that $H = f(G)$. Since $\mathcal{P}$ is a cfp-network for K in X and $G \subset K \cap f^{-1}(U)$ with $f^{-1}(U)$ open in X, there is a cfp-cover $\mathcal{F} \subset \mathcal{P}$ for G in X such that $G \subset \bigcup \mathcal{F} \subset f^{-1}(U)$. Then $f(\mathcal{F}) \subset f(\mathcal{P})$ is a cfp-cover for $f(G) = H$ in Y by (2) satisfying that $H \subset \bigcup f(\mathcal{F}) = f(\bigcup \mathcal{F}) \subset U$. It implies that $f(\mathcal{P})$ is a cfp-network for $f(K)$ in Y. □

Lemma 2.2. *Let $f : X \longrightarrow Y$ be a map.*

1. *If f is quotient and Y is Fréchet, then f is pseudo-open.*
2. *If f is compact-covering and $\mathcal{P}$ is a k-network for X, then $f(\mathcal{P})$ is a k-network for Y.*
3. *If f is sequence-covering and $\mathcal{P}$ is a cs-network for X, then $f(\mathcal{P})$ is a cs-network for Y.*
4. *If f is sequentially-quotient and $\mathcal{P}$ is a cs^*-network for X, then $f(\mathcal{P})$ is a cs^*-network for Y.*
5. *If f is compact-covering and $\mathcal{P}$ is a cfp-network for X, then $f(\mathcal{P})$ is a cfp-network for Y.*
6. *If X is sequential and f is subsequence-covering, then f is sequentially-quotient.*

Proof (1). See [4], Proposition 2.3.

(2), (3), (4) and (5) are routine.

(6). Let S be a convergent sequence converging to a point $y \in Y$. Since f is subsequence-covering, there is a compact subset K in X such that $f(K)$ is a convergent subsequence of S. Put $f(K) = \{y\} \cup \{y_n : n \in \mathbb{N}\}$ where $\{y_n : n \in \mathbb{N}\}$ converges to y. For each $n \in \mathbb{N}$ pick $x_n \in f^{-1}(y_n) \cap K$, then $\{x_n : n \in \mathbb{N}\} \subset K$. Note that K is a compact subset in a sequential space, K is sequentially compact. So there is a convergent subsequence $\{x\} \cup \{x_{n_k} : k \in \mathbb{N}\}$ of $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ that converges to $x \in f^{-1}(y)$. Then $\{y\} \cup \{f(x_{n_k}) : k \in \mathbb{N}\}$ is a convergent subsequence of $\{y\} \cup \{y_n : n \in \mathbb{N}\}$. Therefore $\{y\} \cup \{f(x_{n_k}) : k \in \mathbb{N}\}$ is a convergent subsequence of S. This proves that f is sequentially-quotient. □

The next three lemmas give some properties on certain networks for compact subsets, particularly, convergent sequences.

Lemma 2.3. *If $\mathcal{P}$ is a cfp-network for a compact subset K in X and $x \in K \cap U$ with U open in X, then there is a cfp-cover $\mathcal{F} \subset \mathcal{P}$ for K in X such that $\bigcup\{F \in \mathcal{F} : x \in F\} \subset U$.*

A such cfp-cover $\mathcal{F}$ for K in X is called having *property* $k(x, U)$.

Proof We get that $x \in W_1 \subset \overline{W_1} \subset W \subset \overline{W} \subset U \cap K$ where both W_1 and W are open in K. Since $\overline{W}$ is compact and $\overline{W} \subset U$, there is a cfp-cover $\mathcal{F}_1$ for $\overline{W}$ in X with $\bigcup \mathcal{F}_1 \subset U$. Since the open set $X - \overline{W_1}$ contains a compact set $K - W$, then there is a cfp-cover $\mathcal{F}_2$ for $K - W$ in X such that $\bigcup \mathcal{F}_2 \subset X - \overline{W}_1$. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a cfp-cover for K in X such that $\bigcup\{F \in \mathcal{F} : x \in F\} \subset U$. □

Similarly, we get the following

Lemma 2.4. *If $\mathcal{P}$ is a cs-network for a convergent sequence S in X with $S \subset U$ and U open in X, then there is a subfamily $\mathcal{F} \subset \mathcal{P}$ satisfying the following.*

1. *$\mathcal{F}$ is finite,*
2. *$\emptyset \neq F \cap S \subset F \subset U$ for every $F \in \mathcal{F}$,*
3. *If $x \in S$, then there is a unique $F \in \mathcal{F}$ such that $x \in F$,*
4. *If $F \in \mathcal{F}$ contains the limit point of S, then $S - F$ is finite.*

A such family $\mathcal{F}$ is called having *property cs*(S, U).

Lemma 2.5. *If $\mathcal{P}$ is a point-countable cs*-network for a convergent sequence S in X with $S \subset U$ and U open in X, then there is a family $\mathcal{F} \subset \mathcal{P}$ satisfying the following.*

1. *$\mathcal{F}$ is finite,*
2. *$\emptyset \neq F \cap S \subset F \subset U$ for every $F \in \mathcal{F}$,*
3. *If $x \in S$, then there is $F \in \mathcal{F}$ such that $x \in F$,*
4. *$F \cap S$ is closed for every $F \in \mathcal{F}$.*

A such family $\mathcal{F}$ is called having *property cs*$^*(S, U)$.

The next result is easy to prove.

Lemma 2.6. *Let S be a convergent sequence in X, and $\mathcal{P}$ be a point-countable cover for S in X. Then $\mathcal{P}$ is a cfp-network for S in X if and only if $\mathcal{P}$ is a cs*-network for S in X.*

The next two lemmas and notations in their proofs will be used frequently in the following parts.

Lemma 2.7. *Let $f : M \longrightarrow X$ be an s-map from a locally separable metric space M onto a space X. Then X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ such that each X_α has a countable network $\mathcal{P}_\alpha$.*

Proof Since M is a locally separable metric space, $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ where all M_α's are separable metric spaces by 4.4.F in [2]. For each $\alpha \in \Lambda$, let $\mathcal{B}_\alpha$ be a countable base for M_α, and put $X_\alpha = f(M_\alpha)$, $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$. Then $\{X_\alpha : \alpha \in \Lambda\}$ is a point-countable cover for X, and $\mathcal{P}_\alpha$ is a countable network for X_α. □

Lemma 2.8. *Let X be a space has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ such that each X_α has a countable network $\mathcal{P}_\alpha$. Then X is an s-image of a locally separable metric space.*

Proof For each $\alpha \in \Lambda$ put $\mathcal{P}_\alpha = \{P_\beta : \beta \in \Gamma^\alpha\}$ where Γ^α is countable. Let Γ_n^α be the set Γ^α with the discrete topology for each $n \in \mathbb{N}$. Denote

$$M_\alpha = \Big\{ b = (\beta_n) \in \prod_{n\in\mathbb{N}} \Gamma_n^\alpha : \{P_{\beta_n} : n \in \mathbb{N}\} \text{ forms a network in } X_\alpha \text{ at some point } x_b \in X_\alpha \Big\}.$$

Then M_α is a hereditarily separable metric space of the space $\prod_{n\in\mathbb{N}} \Gamma_n^\alpha$. It implies that $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ is a locally separable metric space.

For every $b = (\beta_n) \in M_\alpha$, $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at some point x_b in X_α. Put $f_\alpha(b) = x_b$. It is easy to check that $f_\alpha : M_\alpha \longrightarrow X_\alpha$ is a map. For every $b \in M$ put $f(b) = f_\alpha(b)$ whenever $b \in M_\alpha$. Then f is a map from M onto X. Let $x \in X$. Since $\{X_\alpha : \alpha \in \Lambda\}$ is point-countable, $\Lambda_x = \{\alpha \in \Lambda : x \in X_\alpha\}$ is countable. For each $\alpha \in \Lambda_x$ because $f_\alpha^{-1}(x) \subset M_\alpha$ and M_α is hereditarily separable, we get $f_\alpha^{-1}(x)$ is separable. Therefore $f^{-1}(x) = \bigcup\{f_\alpha^{-1}(x) : \alpha \in \Lambda_x\}$ is separable. It implies that f is an s-map from a locally separable metric space M onto X. □

Using notations in Lemma 2.8 we get the following two lemmas which play important roles in our proofs. The first proof is similar to the proof of Theorem 2 in [11], the second one is similar to that of Theorem 1 in [8].

Lemma 2.9. *If $\mathcal{P}_\alpha$ is a countable cfp-network for a compact subset K_α in X_α, then there is a compact subset L_α in M_α such that $f_\alpha(L_\alpha) = K_\alpha$.*

Lemma 2.10. *If $\mathcal{P}_\alpha$ is a countable cs-network for a convergent sequence S_α in X_α, then there is a convergent sequence C_α in M_α such that $f_\alpha(C_\alpha) = S_\alpha$.*

Now, we give a characterization of a subsequence-covering s-image of a locally separable metric space.

Theorem 2.11. *The following are equivalent for a space X.*

1. *X is a subsequence-covering s-image of a locally separable metric space,*
2. *X is a sequentially-quotient s-image of a locally separable metric space,*

3. *X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network in X_α for some subsequence of S.*

Proof (1) $\Rightarrow$ (2). By Lemma 2.2.(6).

(2) $\Rightarrow$ (3). Let $f : M \longrightarrow X$ be a sequentially-quotient s-map from a locally separable metric space M onto X. It follows from Lemma 2.7 that X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each $X_\alpha = f(M_\alpha)$ has a countable network $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$ with $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ and $\mathcal{B}_\alpha$ being a countable base for a separable metric space M_α. Let S be a convergent sequence in X, we shall prove that there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network in X_α for some subsequence of S. Indeed, since f is sequentially-quotient, there is a convergent sequence L of M such that $f(L)$ is a subsequence of S. Since L is a convergent sequence, L is eventually in M_α for some $\alpha \in \Lambda$. It is clear that $\mathcal{B}_\alpha$ is a cs-network for $L \cap M_\alpha$ in M_α and $f(L \cap M_\alpha)$ is a subsequence of S. It follows from Lemma 2.1 that $\mathcal{P}_\alpha$ is a cs-network for $f(L \cap M_\alpha)$ in X_α. It implies that X has a cover $\{X_\alpha : \alpha \in \Lambda\}$ having required properties.

(3) $\Rightarrow$ (1). It follows from Lemma 2.8 that there exists an s-map $f : M \longrightarrow X$ from a locally separable metric space $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ onto X where $f(b) = f_\alpha(b)$ whenever $b \in M_\alpha$ with each $f_\alpha : M_\alpha \longrightarrow X_\alpha$ being a map from a separable metric space M_α onto X_α. We shall prove that f is subsequence-covering. Let S be a convergent sequence in X. Then there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network in X_α for a subsequence T of S. Therefore $T = f_\alpha(C_\alpha) = f(C_\alpha)$ where C_α is a convergent sequence in M_α by Lemma 2.10. It implies that f is a sequentially-quotient map. Then f is subsequence-covering. $\square$

From Theorem 2.11, it is easy to get the following corollary by the fact that sequential spaces and Fréchet spaces are preserved by quotient maps and pseudo-open maps respectively. If we drop the parenthetic part, then we get Theorem 2.2 in [10] (see Theorem 1.1).

Corollary 2.12. *The following are equivalent for a space X.*

1. *X is a quotient (resp. pseudo-open) s-image of a locally separable metric space,*
2. *X is a sequential (resp. Fréchet) space having a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network in X_α for some subsequence of S.*

Next, we give an another result based on the relation between a convergent sequence and some $\mathcal{P}_\alpha$ as follows.

Theorem 2.13. *The following are equivalent for a space X.*

1. *X is a sequence-covering s-image of a locally separable metric space,*

2. *X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for a subsequence T of S in X_α with $S - T$ finite.*

Proof (1) $\Rightarrow$ (2). Let $f : M \longrightarrow X$ be a sequence-covering s-map from a locally separable metric space M onto X. It follows from Lemma 2.7 that X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each $X_\alpha = f(M_\alpha)$ has a countable network $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$ with $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ and $\mathcal{B}_\alpha$ being a countable base for a separable metric space M_α. Let S be a convergent sequence in X, we shall prove that there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for a subsequence T of S in X_α with $S-T$ finite. Indeed, since f is sequence-covering, $S = f(L)$ with some convergent sequence L in M. Since L is a convergent sequence in M, L is eventually in M_α for some $\alpha \in \Lambda$. It is clear that $\mathcal{B}_\alpha$ is a cs-network for $L \cap M_\alpha$ in M_α. It follows from Lemma 2.1 that $\mathcal{P}_\alpha$ is a cs-network for $T = f(L \cap M_\alpha)$ in X_α where T is a subsequence of S. Since L is eventually in M_α, $S - T$ is finite. It implies that X has a cover $\{X_\alpha : \alpha \in \Lambda\}$ having required properties.

(2) $\Rightarrow$ (1). It follows from Lemma 2.8 that there exists an s-map $f : M \longrightarrow X$ from a locally separable metric space $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ onto X where $f(b) = f_\alpha(b)$ whenever $b \in M_\alpha$ with each $f_\alpha : M_\alpha \longrightarrow X_\alpha$ being a map from a separable metric space M_α onto X_α. We shall prove that f is sequence-covering. Let S be a convergent sequence in X. Then there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for a subsequence T of S with $S - T$ finite. It follows from Lemma 2.10 that $T = f_\alpha(C_\alpha)$ where C_α is a convergent sequence in M_α. Since $S - T$ is finite, there is finite set F in M such that $f(F) = S - T$. Put $L = F \cup C_\alpha$, then L is a convergent sequence in M such that $f(L) = S$. It implies that f is sequence-covering. □

The following corollary is routine.

Corollary 2.14. *The following are equivalent for a space X.*

1. *X is a sequence-covering quotient (resp. pseudo-open) s-image of a locally separable metric space,*
2. *X is a sequential (resp. Fréchet) space having a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a cs-network for a subsequence T of S in X_α with $S - T$ finite.*

Definition 2.15 ([12], Definition 2.1). A space X is *sequentially separable* if X has a countable subset D such that for each $x \in X$ there is a sequence $\{x_n : n \in \mathbb{N}\} \subset D$ with $x_n \longrightarrow x$.

By the above notion and results we get a nice characterization of sequence-covering s-images of locally separable metric spaces as follows.

Corollary 2.16. *The following are equivalent for a space X.*

1. *X is a sequence-covering s-image of a locally separable metric space,*
2. *X has a point-countable cs-network consisting of $\aleph_0$-subspaces.*

Proof (1) $\Rightarrow$ (2). It follows from Theorem 2.13 that X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X there is $\alpha \in \Lambda$ such that $\mathcal{P}_\alpha$ is a *cs*-network for a subsequence T of S in X_α with $S - T$ finite. Put $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha \in \Lambda\}$, then $\mathcal{P}$ is a point-countable *cs*-network for X. We need only to prove that each element $P \in \mathcal{P}$ is an $\aleph_0$-subspace. By (1) $\Rightarrow$ (2) in Proof of Theorem 2.13 we can assume that $P = f(B)$ where B is separable metric. Note that every separable metric space is a sequentially separable space and sequential separability is preserved by a map, then P is a sequentially separable subspace having a point-countable *cs*-network $\mathcal{P}_P = \{Q \cap P : Q \in \mathcal{P}\}$. It follows from Lemma 2.4(3) in [12] that P is an $\aleph_0$-space.

(2) $\Rightarrow$ (1). Let $\mathcal{P} = \{X_\alpha : \alpha \in \Lambda\}$ be a point-countable *cs*-network for X consisting of $\aleph_0$-subspaces. So each X_α has a countable *cs*-network $\mathcal{P}_\alpha$. Then $\{X_\alpha : \alpha \in \Lambda\}$ satisfies (2) in Theorem 2.13. It implies that X is a sequence-covering s-image of a locally separable metric space. □

Next, we characterize pseudo-sequence-covering s-images of locally separable metric spaces based on the relation between a convergent sequence and some finite collection of $\mathcal{P}_\alpha$'s as follows.

Theorem 2.17. *The following are equivalent for a space X.*

1. *X is a pseudo-sequence-covering s-image of a locally separable metric space,*
2. *X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is a finite subset Λ_S of Λ such that $S = \bigcup\{S_\alpha : \alpha \in \Lambda_S\}$ where S_α is compact and $\mathcal{P}_\alpha$ is a cs*-network for S_α in X_α for every $\alpha \in \Lambda_S$.*

Proof (1) $\Rightarrow$ (2). Let $f : M \longrightarrow X$ be a pseudo-sequence-covering s-map from a locally separable metric space M onto X. It follows from Lemma 2.7 that X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each $X_\alpha = f(M_\alpha)$ has a countable network $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$ with $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ and each $\mathcal{B}_\alpha$ being a countable base for a separable metric space M_α. Let S be a convergent sequence in X, we shall prove that there is a finite subset Λ_S of Λ such that $S = \bigcup\{S_\alpha : \alpha \in \Lambda_S\}$ where S_α is compact and $\mathcal{P}_\alpha$ is a cs^*-network for S_α in X_α for every $\alpha \in \Lambda_S$. Indeed, since f is pseudo-sequence-covering, $S = f(L)$ with some compact subset L of M. Because L is a compact subset of M, $\Lambda_S = \{\alpha \in \Lambda : M_\alpha \cap L \neq \emptyset\}$ is finite. For each $\alpha \in \Lambda_S$ put $L_\alpha = L \cap M_\alpha$, then L_α is compact and $L = \bigcup\{L_\alpha : \alpha \in \Lambda_S\}$. Denote $S_\alpha = f(L_\alpha)$, then

S_α is compact for each $\alpha \in \Lambda_S$. We get $S = f(L) = f(\bigcup\{L_\alpha : \alpha \in \Lambda_S\}) = \bigcup\{f(L_\alpha) : \alpha \in \Lambda_S\} = \bigcup\{S_\alpha : \alpha \in \Lambda_S\}$. It follows from Claim 4.2 in [1] that $\mathcal{B}_\alpha$ is a cfp-network of M_α for each $\alpha \in \Lambda$, then $\mathcal{B}_\alpha$ is a cfp-network for L_α in M_α for each $\alpha \in \Lambda_S$. From Lemma 2.1.(3), $\mathcal{P}_\alpha$ is a cfp-network for S_α in X_α for each $\alpha \in \Lambda_S$. Since S_α is compact in a convergent sequence S, S_α is a convergent sequence for each $\alpha \in \Lambda_S$. Then $\mathcal{P}_\alpha$ is a cs^*-network for S_α in X_α by Lemma 2.6 for each $\alpha \in \Lambda_S$. It implies that X has a cover $\{X_\alpha : \alpha \in \Lambda\}$ having required properties.

(2) $\Rightarrow$ (1). It follows from Lemma 2.8 that there exists an s-map $f : M \longrightarrow X$ from a locally separable metric space $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ onto X where $f(b) = f_\alpha(b)$ whenever $b \in M_\alpha$ with each $f_\alpha : M_\alpha \longrightarrow X_\alpha$ being a map from a separable metric M_α onto X_α. We shall prove that f is pseudo-sequence-covering. Let S be a convergent sequence in X, then there is a finite subset Λ_S of Λ such that $S = \bigcup\{S_\alpha : \alpha \in \Lambda_S\}$ where S_α is compact and $\mathcal{P}_\alpha$ is a cs^*-network for S_α in X_α for each $\alpha \in \Lambda_S$. Since S_α is compact in a convergent sequence S, S_α is a convergent sequence for each $\alpha \in \Lambda_S$. It follows from Lemma 2.6 that $\mathcal{P}_\alpha$ is a cfp-network for S_α in X_α. Then for each $\alpha \in \Lambda_S$ there is a compact subset L_α of M_α such that $S_\alpha = f_\alpha(L_\alpha)$ by Lemma 2.9. Put $L = \bigcup\{L_\alpha : \alpha \in \Lambda_S\}$, then L is compact and $S = \bigcup\{S_\alpha : \alpha \in \Lambda_S\} = \bigcup\{f_\alpha(L_\alpha) : \alpha \in \Lambda_S\} = f(\bigcup\{L_\alpha : \alpha \in \Lambda_S\}) = f(L)$. It implies that f is a pseudo-sequence-covering map. $\square$

It is easy to get the following from Theorem 2.17.

Corollary 2.18. *The following are equivalent for a space X.*

1. *X is a pseudo-sequence-covering quotient (resp. pseudo-open) s-image of a locally separable metric space,*
2. *X is a sequential (resp. Fréchet) space having a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each convergent sequence S of X, there is a finite subset Λ_S of Λ such that $S = \bigcup\{S_\alpha : \alpha \in \Lambda_S\}$ where S_α is compact and $\mathcal{P}_\alpha$ is a cs^*-network for S_α in X_α for every $\alpha \in \Lambda_S$.*

The following is a general result of Corollary 2.3 in [10].

Corollary 2.19. *If X is a space with a point-countable cs^*-network consisting of $\aleph_0$-subspaces, then X is a pseudo-sequence-covering s-image of a locally separable metric space.*

Finally, we characterize compact-covering s-images of locally separable metric spaces based on the relation between a compact subset and some finite collection of $\mathcal{P}_\alpha$'s.

Theorem 2.20. *The following are equivalent for a space X.*

1. *X is a compact-covering s-image of a locally separable metric space,*

2. *X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each compact subset K of X, there is a finite subset Λ_K of Λ such that $K = \bigcup\{K_\alpha : \alpha \in \Lambda_K\}$ where K_α is compact and $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α for every $\alpha \in \Lambda_K$.*

Proof (1) $\Rightarrow$ (2). Let $f : M \longrightarrow X$ be a compact-covering s-map from a locally separable metric space M onto X. It follows from Lemma 2.7 that X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each $X_\alpha = f(M_\alpha)$ has a countable network $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$ with $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ and $\mathcal{B}_\alpha$ being a countable base for a separable metric space M_α. Let K be a compact subset of X, we shall prove that there is a finite subset Λ_K of Λ such that $K = \bigcup\{K_\alpha : \alpha \in \Lambda_K\}$ where K_α is compact and $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α for every $\alpha \in \Lambda_K$. Indeed, since f is compact-covering, $K = f(L)$ with some compact subset L of M. Because L is a compact subset of M, $\Lambda_K = \{\alpha \in \Lambda : M_\alpha \cap L \neq \emptyset\}$ is finite. For each $\alpha \in \Lambda_K$ put $L_\alpha = L \cap M_\alpha$, then L_α is compact and $L = \bigcup\{L_\alpha : \alpha \in \Lambda\}$. Denote $K_\alpha = f(L_\alpha)$, then K_α is compact for each $\alpha \in \Lambda_K$. We get $K = f(L) = f(\bigcup\{L_\alpha : \alpha \in \Lambda_K\}) = \bigcup\{f(L_\alpha) : \alpha \in \Lambda_K\} = \bigcup\{K_\alpha : \alpha \in \Lambda_K\}$. It follows from Claim 4.2 in [1] that $\mathcal{B}_\alpha$ is a cfp-network of M_α for each $\alpha \in \Lambda$, then $\mathcal{B}_\alpha$ is a cfp-network for K_α in M_α for each $\alpha \in \Lambda_K$. From Lemma 2.1.(3), $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α for each $\alpha \in \Lambda_K$. It implies that X has a cover $\{X_\alpha : \alpha \in \Lambda\}$ having required properties.

(2) $\Rightarrow$ (1). It follows from Lemma 2.8 that there exists an s-map $f : M \longrightarrow X$ from a locally separable metric space $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ onto X where $f(b) = f_\alpha(b)$ whenever $b \in M_\alpha$ with each $f_\alpha : M_\alpha \longrightarrow X_\alpha$ being a map from a separable metric space M_α onto X_α. We shall prove that f is compact-covering. Let K be a compact subset of X. By assumption, there is a finite subset Λ_K of Λ such that $K = \bigcup\{K_\alpha : \alpha \in \Lambda_K\}$ where K_α is compact and $\mathcal{P}_\alpha$ is a cfp-network of K_α for each $\alpha \in \Lambda_K$. Then for each $\alpha \in \Lambda_K$ there is a compact subset L_α in M_α such that $K_\alpha = f_\alpha(L_\alpha)$ by Lemma 2.9. Put $L = \bigcup\{L_\alpha : \alpha \in \Lambda_K\}$, then L is compact and $K = \bigcup\{K_\alpha : \alpha \in \Lambda_K\} = \bigcup\{f_\alpha(L_\alpha) : \alpha \in \Lambda_K\} = f(\bigcup\{L_\alpha : \alpha \in \Lambda_K\}) = f(L)$. It implies that f is a compact-covering map. $\square$

The following corollary is routine.

Corollary 2.21. *The following are equivalent for a space X.*

1. *X is a compact-covering quotient (resp. pseudo-open) s-image of a locally separable metric space,*
2. *X is a sequential (resp. Fréchet) space having a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ where each X_α has a countable network $\mathcal{P}_\alpha$ satisfying that for each compact subset K of X, there is a finite subset Λ_K of Λ such that $K = \bigcup\{K_\alpha : \alpha \in \Lambda_K\}$ where K_α is compact and $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α for every $\alpha \in \Lambda_K$.*

The following is similar to the Corollary 2.19.

Corollary 2.22. *If X is a space with a point-countable cfp-network consisting of $\aleph_0$-subspaces, then X is a compact-covering s-image of a locally separable metric space.*

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