East-West J. of Mathematics: Vol. 20, No 2 (2018) pp. 107-124

TOWARDS QUANTITATIVE CLASSIFICATION OF CAYLEY AUTOMATIC GROUPS

Dmitry Berdinsky* and **Phongpitak Trakuldit**†

Department of Mathematics, Faculty of Science, Mahidol University
Centre of Excellence in Mathematics, Commission on Higher Education
Bangkok, 10400, Thailand
email: * *berdinsky@gmail.com,* † *p.trakuldit@gmail.com*

Abstract

In this paper we address the problem of quantitative classification of Cayley automatic groups in terms of a certain numerical characteristic which we earlier introduced for this class of groups. For this numerical characteristic we formulate and prove a fellow traveler property, show its relationship with the Dehn function and prove its invariance with respect to taking finite extension, direct product and free product. We study this characteristic for nilpotent groups with a particular accent on the Heisenberg group, the fundamental groups of torus bundles over the circle and groups of exponential growth.

1 Introduction and Preliminaries

Strings over a finite alphabet appear a natural way to represent elements of a finitely generated group. Following this way Thurston introduced automatic groups which became an important part of geometric group theory [9]. Trying to extend the class of automatic groups, one can either use more powerful computational models (e.g., asynchronous automata, pushdown automata and etc.) or relax the constraint on the correspondence between strings and group elements (for automatic groups this correspondence is given by the canonical map). The latter approach leads to Cayley automatic groups introduced

* Corresponding author.
Key words: automatic group, Cayley automatic group, nilpotent group, fundamental group, Dehn function, growth function, numerical characteristic.
(2010) Mathematics Subject Classification: 20F10, 20F65.

by Kharlampovich, Khoussainov and Miasnikov [11]. Utilization of both approaches simultaneously leads further to $\mathcal{C}$–graph automatic groups introduced by Elder and Taback [8]. In this paper we focus only on Cayley automatic groups.

Cayley automatic groups utilize exactly the same computational model as automatic groups, so they preserve some key algorithmic features of automatic groups, but the correspondence between strings and group elements can be arbitrary. Another way to define Cayley automatic groups is to say that they are finitely generated groups for which labeled directed Cayley graphs are automatic (FA–presentable) structures [13, 12, 14]. For a recent survey of the theory of automatic structures we refer the reader to [20]. The class of Cayley automatic groups is essentially wider than the class of automatic groups [11]. Also, Cayley automatic groups include important classes of groups such as nilpotent groups of nilpotency class two, fundamental groups of 3–manifolds, Baumslag–Solitar groups, restricted wreath products of Cayley automatic groups by the infinite cyclic group, higher rank lamplighter groups [11, 3, 5].

We assume that the reader is familiar with the definitions of finite automata and regular languages (a concise introduction is given in, e.g., [9, Sections 1.1–2]). For a given finite alphabet Σ we denote by Σ^* the set of all finite strings over Σ and by $\Sigma_\diamond$ the alphabet $\Sigma = \Sigma \cup \{\diamond\}$ (it is assumed that $\diamond \notin \Sigma$). For any $w \in \Sigma^*$, we denote by $|w|$ the length of the string w. Let $w_1, \ldots, w_n \in \Sigma^*$. The convolution $w_1 \otimes \cdots \otimes w_n$ is the string of a length $m = \max\{|w_1|, \ldots, |w_n|\}$ over the alphabet $\Sigma_\diamond^{n\prime} = \Sigma_\diamond^n \setminus \{(\diamond, \ldots, \diamond)\}$ for which the kth symbol, $k = 1, \ldots, m$, is $(\sigma_{1k}, \ldots, \sigma_{nk}) \in \Sigma_\diamond^{n\prime}$, where σ_{ik} is the kth symbol of w_i if $k \leqslant |w_i|$ and $\sigma_{ik} = \diamond$ if $k > |w_i|$ for $i = 1, \ldots, n$. For any relation $R \subseteq \Sigma^{*n}$, we say that R is FA–recognizable (regular) if $\otimes R = \{w_1 \otimes \cdots \otimes w_n \,|\, (w_1, \ldots, w_n) \in R\}$ is a regular language over the alphabet $\Sigma_\diamond^{n\prime}$. Let G be a finitely generated (f.g.) group and $A \subset G$ be a finite generating set of G. Let A^{-1} be the set of the inverses of elements of A and $S = A \cup A^{-1}$. We denote by $\pi : S^* \to G$ the canonical map which maps any given string $w = s_1 \ldots s_n \in S^*$ to the group element $g = s_1 \ldots s_n \in G$.

Definition 1.1. *A group G is called Cayley automatic if there exists a bijection $\psi : L \to G$ between some regular language $L \subseteq \Sigma^*$ and the group G for which the binary relation $R_a = \big\{\big(\psi^{-1}(g), \psi^{-1}(ga)\big) \,|\, g \in G\big\}$ is FA–recognizable for every $a \in A$. Such a bijection $\psi : L \to G$ is called a Cayley automatic representation of G.*

In this paper we assume that $\Sigma = S$, unless otherwise stated. This assumption is needed to correctly define the function $h(n)$ in the formula (1.1) below: if $w \in S^*$, then $\pi(w)$ is in the group G as well as $\psi(w)$, so one can get the distance $d_A(\pi(w), \psi(w))$ between $\pi(w)$ and $\psi(w)$ in the Cayley graph $\Gamma(G, A)$. We recall that for given $g_1, g_2 \in G$, the distance $d_A(g_1, g_2)$ between the elements g_1 and g_2 in G with respect to A is the length of a shortest path from g_1 to g_2 in the Cayley graph $\Gamma(G, A)$. For a given $g \in G$, we denote by $d_A(g)$ the distance $d_A(e, g)$, where e is the identity of the group G. Since the cardinality of S is at least two, it can be verified that Definition 1.1 (either together with the assumption that $\Sigma = S$ or without it) is equivalent to the original definition of Cayley automatic groups [11, Definition 6.4] (they are also referred as Cayley graph automatic or graph automatic groups in the literature). Furthermore, assuming that $\Sigma = S$ and $\psi = \pi$ in Definition 1.1, one gets the definition of automatic groups; it can be also verified that it is equivalent to the original definition given by Thurston, see [9, Definition 2.3.1]. This observation motivated us to introduce a function (1.1) as a measure of deviation of a given Cayley automatic representation ψ from automatic representations [4]:

$$h(n) = \max\left\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\right\}, \tag{1.1}$$

where $L^{\leqslant n} = \{w \in L \,|\, |w| \leqslant n\}$ is the set of strings from L of a length less or equal than n. If a group G is Cayley automatic but not automatic, a Cayley automatic representation ψ for which $\psi = \pi$ does not exist. So, in this case, for every Cayley automatic representation ψ of G the function $h(n)$ defined by (1.1) is not identically equal to zero.

We denote by $\mathfrak{F}$ the set of all nondecreasing functions from some interval $[Q, +\infty) \subseteq \mathbb{N}$ to the set of nonnegative real numbers. Clearly, a function $h(n)$ given in (1.1) is in $\mathfrak{F}$. For any given $g, f \in \mathfrak{F}$, we say that $g \preceq f$ (g is coarsely less or equal than f) if there exist nonnegative integer N and positive integers K and M for which $g(n) \leqslant Kf(Mn)$ for all $n \geqslant N$. We say that $g \asymp f$ (g is coarsely equal to f) if $g \preceq f$ and $f \preceq g$. Similarly, we say that $g \prec f$ (g is coarsely strictly less than f) if $g \preceq f$ and $g \not\asymp f$. Clearly, the coarse equality $\asymp$ gives an equivalence relation on $\mathfrak{F}$. In this paper we will be considering functions from $\mathfrak{F}$ up to this equivalence relation.

Any given Cayley automatic group G admits infinitely many Cayley automatic representations $\psi : L \to G$. So, in general, the problem of finding Cayley automatic representations minimizing coarsely the function (1.1) is nontrivial. In [4, Theorems 11 and 13], we constructed Cayley automatic representations of the Baumslag–Solitar groups $BS(p, q)$, $q > p \geqslant 1$ and the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ which are minimizers of the function (1.1). In both cases the minimum for the function $h(n)$ is the identity function $\mathfrak{i}$: $\mathfrak{i}(n) = n$ for all $n \in \mathbb{N}$. Furthermore, in [4] we introduced classes of Cayley automatic groups $\mathcal{B}_f$ as follows. For a given $f \in \mathfrak{F}$, $G \in \mathcal{B}_f$ if there exists a Cayley automatic representation $\psi : L \to G$ for which $h \preceq f$, where h is given by (1.1). In particular, the Baumslag–Solitar groups $BS(p, q)$, $q > p \geqslant 1$ and the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ are in the class $\mathcal{B}_\mathfrak{i}$ and they cannot be in any class $\mathcal{B}_f$ if $f \prec \mathfrak{i}$.

It is easy to show that the definition of a class $\mathcal{B}_f$ does not depend on the choice of generators [4, Proposition 5]. Clearly, $\mathcal{B}_f \subseteq \mathcal{B}_g$ if $f \preceq g$. Also, for the zero function $\mathbf{z}$, where $\mathbf{z}(n) = 0$ for all $n \in \mathbb{N}$, the class $\mathcal{B}_\mathbf{z}$ coincides with the class of automatic groups. In [4, Theorem 8] we proved that there exists no nonautomatic group in any class $\mathcal{B}_d$, where $d \in \mathfrak{F}$ is a function bounded from above by some constant; that is, $\mathcal{B}_d = \mathcal{B}_{\mathfrak{z}}$ for any such function d. Another group that we considered in [4] was the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$. We showed that $\mathcal{H}_3(\mathbb{Z}) \in \mathcal{B}_{\mathfrak{e}}$, where $\mathfrak{e}$ is the exponential function: $\mathfrak{e}(n) = \exp(n)$. But a lower bound for $h(n)$ which we could find in the case of $\mathcal{H}_3(\mathbb{Z})$ is far from being exponential, it is $\sqrt[3]{n}$ [4, Theorem 15].

For a given $G \in \mathcal{B}_f$ we treat $f \in \mathfrak{F}$ as a numerical characteristic of G. We especially interested in those f which are sharp lower bounds for (1.1). The fact that the sharp lower bounds can be obtained for some groups sounds promising. Numerical characteristics of groups, e.g. growth functions, Dehn functions, drifts of simple random walks and etc., and relations between them are very important in group theory, see, e.g., [22]. Another motivation to study this numerical characteristic is to address the problem of characterization of Cayley automatic groups; see also [1], where this problem is addressed in terms of numerical characteristics of Turing transducers.

In this paper we continue studying this numerical characteristic of Cayley automatic groups and its relation to other numerical characteristics initiated in [4]. In Section 2 we propose a fellow traveler property for Cayley automatic groups in Theorem 2.1 and show a relation with the Dehn function in Theorem 2.3. The fellow traveler property is well known for automatic groups but its analog for Cayley automatic groups had not been formulated before. In Section 3 we prove invariance of classes $\mathcal{B}_f$ under taking finite extension, direct product and free product in Theorems 3.1, 3.2 and 3.3, respectively; in the latter case we require the function f to satisfy a certain inequality.

In Section 4 we show that the semidirect products $\mathbb{Z}^n \rtimes_A \mathbb{Z}$, unitriangular matrix groups $UT_n(\mathbb{Z})$ and all f.g. nilpotent groups of nilpotency class two are in the class $\mathcal{B}_{\mathfrak{e}}$, see Theorem 4.2. However, this result is obtained from certain Cayley automatic representations of these groups and we do not know whether they are minimizers of the function (1.1) or not. We partly address this issue in Theorem 4.4 by showing that if a virtually nilpotent group G is in a class $\mathcal{B}_p$ for some polynomial p, then the language L of a Cayley automatic representation $\psi : L \to G$, for which $h \preceq p$, must be simply starred.

In Section 5 we address the problem of sharp lower bounds of the function (1.1) specifically for the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$. In Theorem 5.1 we show that under a certain condition on a Cayley automatic representation of $\mathcal{H}_3(\mathbb{Z})$ the growth of the function (1.1) must be at least exponential. We note that the proof of Theorem 5.1 does not use any knowledge about growth of the Dehn function, which is very often used to show that a given group is not automatic. We believe that Theorem 5.1 can be useful for proposing new approaches to proving nonautomaticity of groups. Section 6 concludes the paper by showing that for any Cayley automatic representation $\psi : L \to G$ of a group of exponential growth a linear upper bound $d_A(\pi(w), \psi(w)) \leqslant C|w|$ holds for almost all $w \in L$ in a certain sense, see Theorem 6.1. However, in

Remark 6.3 we explain that one should be careful with this simple observation made in Theorem 6.1 by constructing Cayley automatic representations of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the function (1.1) grows faster than any tower of exponents.

All questions that we posed in [4, §7] remain open. Let us pose an additional question here: is there any Cayley automatic representation of a group of polynomial growth (which is not virtually abelian) or a fundamental group of a 3–manifold (which is not automatic) for which the function (1.1) is coarsely strictly less than the exponential function $\mathfrak{e}$?

2 Fellow Traveler Property and Connection with Dehn Functions

In this section we formulate a fellow traveler property for Cayley automatic groups and obtain a relation between the Dehn function of a group $G \in \mathcal{B}_f$ and a function f. For any word $w \in S^*$ and nonnegative integer t we put $w(t)$ to be the prefix of w of a length t if $t \leqslant |w|$ and w if $t > |w|$. We denote by $\widehat{w} : [0, \infty) \to \Gamma(G, A)$ the corresponding path in the Cayley graph $\Gamma(G, A)$: if t is an integer, then $\widehat{w}(t) = \pi(w(t))$ and if t is not an integer, $\widehat{w}(t)$ is obtained by moving along the edge $(\widehat{w}(\lfloor t \rfloor), \widehat{w}(\lceil t \rceil))$ with unit speed; we will use only integer values of t. Let $\psi : L \to G$ be any Cayley automatic representation of a group G. We denote by s be the following function:

$$s(n) = \max\{d_A(\widehat{w_1}(t), \widehat{w_2}(t)) | \psi(w_1)g = \psi(w_2), g \in A, t \leqslant n\}. \tag{2.1}$$

That is, for every two words $w_1, w_2 \in L$ representing neighboring vertices in the Cayley graph $\Gamma(G, A)$ (i.e., for some $g \in A$, $\psi(w_1)g = \psi(w_2)$) the distance between $\widehat{w_1}(t)$ and $\widehat{w_2}(t)$ for all $t \leqslant n$ is bounded from above by $s(n)$. If G is automatic and ψ is an automatic representation of G, then $s(n)$ must be a bounded function due to the fellow traveler property for automatic groups [9, Lemma 2.3.2].

Theorem 2.1. *Assume that $G \in \mathcal{B}_f$ for some nonzero function $f \in \mathfrak{F}$. Then there is a Cayley automatic representation $\psi : L \to G$ such that for the function $s(n)$ given by (2.1), $s \preceq f$.*

Proof. Since $G \in \mathcal{B}_f$, there exists a Cayley automatic representation $\psi : L \to G$ such that for the function $h(n) = \max\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\}$, $h \preceq f$. Let t, n be some nonnegative integers for which $t \leqslant n$ and $w_1, w_2 \in L$ be some words representing neighboring vertices in $\Gamma(G, A)$ (i.e., $\psi(w_1)g = \psi(w_2)$ for some $g \in A$). The convolution $w_1 \otimes w_2$ is in a regular language $\otimes R_g$ accepted by some two–tape synchronous automaton M_g. Let T be a maximal number of states in the automata M_g for all $g \in A$. We assume that

$t > T$. If $t \leqslant \max\{|w_1|, |w_2|\}$, there exist strings u_1, v_1, u_2, v_2 for which u_1 and u_2 are prefixes of $w_1(t)$ and $w_2(t)$ such that $|u_1|, |u_2| \geqslant t - T$ and for the strings $w_1' = u_1 v_1$ and $w_2' = u_2 v_2$, $|w_1'|, |w_2'| \leqslant t$ and the convolution $w_1' \otimes w_2' \in \otimes R_g$. If $t \geqslant \max\{|w_1|, |w_2|\}$, then we simply put $u_1 = w_1$, $u_2 = w_2$ and $v_1 = v_2 = \epsilon$, where ϵ is the empty string. We have: $d_A(\widehat{w_1}(t), \widehat{w_2}(t)) \leqslant d_A(\pi(u_1), \pi(u_2)) + 2T \leqslant d_A(\pi(w_1'), \pi(w_2')) + |v_1| + |v_2| + 2T \leqslant d_A(\pi(w_1'), \pi(w_2')) + 4T$. Moreover, $d_A(\pi(w_1'), \pi(w_2')) \leqslant d_A(\pi(w_1'), \psi(w_1')) + d_A(\psi(w_1'), \psi(w_2')) + d_A(\psi(w_2'), \pi(w_2')) \leqslant h(|w_1'|) + 1 + h(|w_2'|) \leqslant 2h(t) + 1$. Therefore, $d_A(\widehat{w_1}(t), \widehat{w_2}(t)) \leqslant 2h(t) + 4T + 1 \leqslant 2h(n) + 4T + 1$. If $t \leqslant T$, $d_A(\widehat{w_1}(t), \widehat{w_2}(t))$ can be bounded from above by $2T$. Since $h \preceq f$ and f is a nonzero function, then $s \preceq f$. $\square$

Remark 2.2. *Clearly, we have $d_A(\widehat{w_1}(t), \widehat{w_2}(t)) \leqslant d_A(\widehat{w_1}(t)) + d_A(\widehat{w_2}(t)) \leqslant 2t$. Therefore, $s \preceq \mathfrak{i}$ for any function s given by (2.1). So, Theorem 2.1 is of interest if $f \prec \mathfrak{i}$. It is not known whether there exists any Cayley automatic group in a class $\mathcal{B}_f$, for $f \prec \mathfrak{i}$, which is not automatic. If such groups do not exist, Theorem 2.1 might be a first step to prove it. At least, Theorem 2.1 can serve as an argument to prove that a given group $G \notin \mathcal{B}_f$ for some $f \prec \mathfrak{i}$.*

Let G be a group $G = \langle A|R \rangle$ defined by a finite set of generators A and a finite set of relators R. Let $S = A \cup A^{-1}$. The Dehn function $D(n)$ of G given by A and R is defined as $D(n) = \max\{\mathrm{Area}(w) | w \in S^{\leqslant n} \wedge \pi(w) = e\}$, where $\mathrm{Area}(w)$ is the minimal integer k for which $w = \prod_{i=1}^{k} v_i r_i^{\pm 1} v_i^{-1}$, $r_i \in R$, in the free group $F(A)$. Let us assume that $G \in \mathcal{B}_f$ for some nonzero function $f \in \mathfrak{F}$. Theorem 2.3 and Corollary 2.4 below extend the results we obtained in [4, Theorems 11 and 15].

Theorem 2.3. *Assume that we are given two functions $p, q \in \mathfrak{F}$ for which $p(n) \preceq D(n) \preceq q(n)$. Then $p(n) \leqslant Cn^2 q(Kf(Mn))$ for all $n \geqslant N$ for some constants C, K, M and N. In particular, if $p = q = n^d$ for some $d > 2$, then $n^{\frac{d-2}{d}} \preceq f$. If $p = q = \mathfrak{e}$, then $\mathfrak{i} \preceq f$.*

Proof. Let $\psi : L \to G$ be a Cayley automatic representation of G such that for the function $h(n) = \max\{d_A(\psi(w), \pi(w)) | w \in L^{\leqslant n}\}$, $h \preceq f$. Let $w = a_1 \dots a_n \in S^*$ be a word representing the identity in G, where $a_i \in S$. For a given $j = 1, \dots, n-1$, we put $g_j = a_1 \dots a_j$ and $g_0 = g_n = e$. We first divide a loop given by the word w into n subloops as follows. For any $i = 0, \dots, n-1$ let $u_i \in S^*$ be the following concatenation of words: $u_i = \eta_i \xi_i a_{i+1} \xi_{i+1}^R \eta_{i+1}^R$, where $\eta_i = \psi^{-1}(g_i)$, ξ_i is some fixed word traversing a shortest path from $\pi(\eta_i)$ to g_i, ξ_{i+1}^R and η_{i+1}^R are the inverses of ξ_{i+1} and η_{i+1}, respectively; e.g., if $\xi = abbc^{-1}a^{-1}$, then $\xi^R = acb^{-1}b^{-1}a^{-1}$. Clearly $\pi(u_i) = e$, so we obtain a loop.

By the bounded difference lemma (see, e.g., [11, Lemma 14.1]), the length of each string η_i is bounded by Cn for some constant C. Then each of the subloops given by u_i, $i = 0, \dots, n-1$ we divide into at most Cn smaller subloops as follows. For every $1 \leqslant j \leqslant \max\{|\eta_i|, |\eta_{i+1}|\}$ we construct a loop starting at the point $\widehat{\eta_i}(j-1)$ as follows. For $1 \leqslant j < \max\{|\eta_i|, |\eta_{i+1}|\}$ the loop defined by the word $v_{ij} = p_{ij}\zeta_{ij}p^R_{(i+1)j}\zeta^R_{i(j-1)}$, where p_{ij} is the string for which $\eta_i(j) = \eta_i(j-1)p_{ij}$ (so p_{ij} is either a single–letter string or the empty string) and ζ_{ij} is some word traversing a shortest path from $\widehat{\eta_i}(j)$ to $\widehat{\eta_{i+1}}(j)$; clearly, the length of this loop is bounded by $s(j) + s(j-1) + 2$, where s is the function given (2.1). For $j = \max\{|u_i|, |u_{i+1}|\}$ the loop is defined by the word $v_{ij} = p_{ij}\xi_i a_{i+1}\xi^R_{i+1}p^R_{(i+1)j}\zeta^R_{i(j-1)}$; the length of this loop is bounded by $(2h(j)+1) + s(j-1) + 2$. Let $\ell'(k) = \max\{2s(k)+2, 2h(k)+s(k)+3\}$ and $\ell(k) = \ell'(Ck)$. So, the length of each of these smaller subloops is bounded by $\ell(n) = \ell'(Cn)$. By the inequalities $h \preceq f$ and $s \preceq f$ (see Theorem 2.1), we have $\ell \preceq f$. The total number of these smaller subloops is at most Cn^2. Thus we obtain the inequality $D(n) \leqslant Cn^2 D(\ell(n))$. Therefore, $D(n) \preceq n^2 D(\ell(n))$.

From the inequalities $D(n) \preceq n^2 D(\ell(n))$, $\ell \preceq f$ and $p(n) \preceq D(n) \preceq q(n)$ we obtain that: $p(n) \leqslant C_1 D(C_2 n) \leqslant C_3 n^2 D(\ell(C_4 n)) \leqslant Cn^2 q(C_5 \ell(C_4 n))) \leqslant Cn^2 q(Kf(Mn))$ for all $n \geqslant N$ for some constants C, K, M, N and $C_i, i = 1, \ldots, 5$. If $p = q = n^d$, then $n^d \leqslant Cn^2(Kf(Mn))^d$ for all $n \geqslant N$. Therefore, $n^{\frac{d-2}{d}} \leqslant C^{\frac{1}{d}} Kf(Mn)$ for all $n \geqslant N$, i.e., $n^{\frac{d-2}{d}} \preceq f$. If $p = q = \mathfrak{e}$, then $\exp(n) \leqslant Cn^2 \exp(Kf(Mn))$ for all $n \geqslant N$. Therefore, $n \leqslant \log C + 2\log n + Kf(Mn)$ for all $n \geqslant N$, which implies that $\mathfrak{i} \preceq f$. □

Corollary 2.4. *For a given function $f \in \mathfrak{F}$ we have:*

- *if the Baumslag–Solitar group $BS(p,q) \in \mathcal{B}_f$ for some $q > p \geqslant 1$, then $\mathfrak{i} \preceq f$;*
- *if the Heisenberg group $\mathcal{H}_3(\mathbb{Z}) \in \mathcal{B}_f$, then $\sqrt[3]{n} \preceq f$;*
- *if the group $\mathbb{Z}^2 \rtimes_A \mathbb{Z} \in \mathcal{B}_f$ for a matrix $A \in GL(2,\mathbb{Z})$ with two real eigenvalues not equal to ± 1, then $\mathfrak{i} \preceq f$.*

Proof. This follows from Theorem 2.3 and the facts that for the groups $BS(p,q)$, $1 \leqslant p < q$, $\mathcal{H}_3(\mathbb{Z})$ and $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, for a matrix $A \in GL(2,\mathbb{Z})$ with two real eigenvalues not equal to ± 1, the Dehn functions are exponential, cubic and exponential, respectively (see [7] and, e.g., [9, §7.4–§8.1]). □

Remark 2.5. *We recall that the groups $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ are the fundamental groups of 3–manifolds which are 2–dimensional torus bundles over the circle. The Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ is isomorphic $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ for some unipotent matrix A; see also Section 5.*

Remark 2.6. *The examples of Dehn functions for Cayley automatic groups, which are known to us, are quadratic (e.g, for the higher Heisenberg groups $\mathcal{H}_{2k+1}(\mathbb{Z}), k > 1$), cubic (e.g., for the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$), n^d for any integer $d > 3$ (e.g., for some semidirect products $\mathbb{Z}^m \rtimes_A \mathbb{Z}$, see [6, 7]), and the exponential function $\mathfrak{e}$ (e.g., for the Baumslag–Solitar groups $BS(p,q)$, $1 \leqslant p < q$).*

3 Finite Extensions, Direct Products, Free Products

In this section we show that classes $\mathcal{B}_f$ are invariant with respect to taking finite extension, direct product and free product. For the latter case we require that f satisfies the inequality $f(x)+f(y) \leqslant f(x+y)$ for all $x, y \geqslant n_0$, where n_0 is some constant. Let H be a subgroup of finite index in a f.g. group G. It is known that if H is automatic, then G is automatic. Moreover, by [11, Theorem 10.1], if H is Cayley automatic, then G is Cayley automatic[1].

Theorem 3.1. *Let H be a subgroup of finite index of a group G. If $H \in \mathcal{B}_f$, then $G \in \mathcal{B}_f$.*

Proof. Let us fix a finite set of generators of H: $A_1 = \{h_1, \dots, h_n\}$, and a set of unique representatives of the right cosets Hg of the subgroup H in G, where $g \notin H$: $A_2 = \{k_1, \dots, k_m\}$. We put $S_1 = A_1 \cup A_1^{-1}$. Since $H \in \mathcal{B}_f$, there exist a Cayley automatic representation $\psi_1 : L_1 \to H$, $L_1 \subseteq S_1^*$ such that, for the function $h_1(n) = \max\{d_{A_1}(\pi(u), \psi_1(u)) \,|\, u \in L_1^{\leqslant n}\}$, $h_1(n) \preceq f(n)$. Let L_2 be the finite language consisting of m single–letter strings $k_1, \dots, k_m$ and the empty sting ϵ. We put ψ_2 to be the natural embedding of these strings into the group G: a string k_i maps to the group element k_i and the empty string ϵ maps to the identity of the group G. We put L to be the concatenation of L_1 and L_2. Clearly, $L \subseteq S^*$, where $S = A \cup A^{-1}$ and $A = A_1 \cup A_2$. Now, we define the map $\psi : L \to G$ as follows. Let $w = uv \in L$, where $u \in L_1$ and $v \in L_2$. We put $\psi(w) := \psi_1(u)\psi_2(v)$. It is easy to verify that the constructed map ψ is a Cayley automatic representation of the group G (see [11, Theorem 10.1]). Furthermore, $d_A(\pi(w), \psi(w)) \leqslant d_A(\pi(w), \pi(u)) + d_A(\pi(u), \psi(u)) + d_A(\psi(u), \psi(w)) \leqslant 1 + h_1(|u|) + 1 \leqslant h_1(|w|) + 2$. This immediately implies that for the function $h(n) = \max\{d_A(\pi(w), \psi(w)) \,|\, w \in L^{\leqslant n}\}$, $h \preceq h_1$. Therefore, $h \preceq f$. □

It is known that the direct product of two automatic groups is automatic. The direct product of Cayley automatic groups is also Cayley automatic [11, Corollary 10.4].

Theorem 3.2. *If $G_1, G_2 \in \mathcal{B}_f$, then $G_1 \times G_2 \in \mathcal{B}_f$.*

Proof. Let A_1 and A_2 be some sets of generators of the groups G_1 and G_2 for which $A_1 \cap A_2 = \varnothing$; we put $S_1 = A_1 \cup A_1^{-1}$ and $S_2 = A_2 \cup A_2^{-1}$. Since $G_1, G_2 \in \mathcal{B}_f$, there exist Cayley automatic representation $\psi_1 : L_1 \to G_1$ and $\psi_2 : L_2 \to G_2$ for which the functions $h_1(n) = \max\{d_{A_1}(\pi(w), \psi_1(w)) \,|\, w \in L_1^{\leqslant n}\}$ and $h_2(n) = \max\{d_{A_2}(\pi(w), \psi_2(w)) \,|\, w \in L_2^{\leqslant n}\}$ satisfy the inequalities $h_1 \preceq f$ and $h_2 \preceq f$, where $L_1 \subseteq S_1^*$ and $L_2 \subseteq S_2^*$.

[1] A complete analog of [9, Theorem 4.1.4] for automatic groups, claiming that a subgroup H of finite index of a group G is automatic iff G is automatic, is not known for Cayley automatic groups. We remark that in the original [11, Theorem 10.1] the assumption that H is a normal subgroup of G can be omitted; see, e.g., [2, Theorem 2.2.4].

Let $L = L_1 L_2$. We construct the map $\psi : L \to G_1 \times G_2$ as follows. For a given $w = uv$, where $u \in L_1$ and $v \in L_2$, we put $\psi(w) = (\psi_1(u), \psi_2(v)) \in G_1 \times G_2$. It is easy to verify that the constructed map ψ provides a Cayley automatic representation of $G_1 \times G_2$. The groups G_1 and G_2 are naturally embedded in $G_1 \times G_2$, so we have $\pi(w) = \pi(u)\pi(v) = (\pi(u), \pi(v)) \in G_1 \times G_2$. Therefore, $d_A(\pi(w), \psi(w)) \leqslant d_A(\pi(u), \psi_1(u)) + d_A(\pi(v), \psi_2(v)) \leqslant h_1(|u|) + h_2(|v|) \leqslant h_1(|w|) + h_2(|w|) = s(|w|)$, where $s(n) = h_1(n) + h_2(n)$ for all $n \in \operatorname{dom} h_1 \cap \operatorname{dom} h_2$. Clearly, the inequalities $h_1 \preceq f$ and $h_2 \preceq f$ imply that $s \preceq f$. Therefore, for the function $h(n) = \max\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\}$, we have $h \preceq f$. □

It is known that the free product of automatic groups is automatic. Therefore, if $G_1, G_2 \in \mathcal{B}_d$, then $G_1 \star G_2 \in \mathcal{B}_d$, where d is a bounded function (recall that in this case, by [4, Theorem 8], $\mathcal{B}_d$ is the class of automatic groups). Moreover, the free product of Cayley automatic groups is Cayley automatic [11, Theorem 10.8]. In the following theorem we consider the case when $G_1, G_2 \in \mathcal{B}_f$ for some unbounded function $f \in \mathfrak{F}$.

Theorem 3.3. *Let $f \in \mathfrak{F}$ be a function for which $f(x) + f(y) \leqslant f(x + y)$ for all $x, y \geqslant n_0$, where n_0 is a constant. If $G_1, G_2 \in \mathcal{B}_f$, then $G_1 \star G_2 \in \mathcal{B}_f$.*

Proof. For initial settings we use the same notation as in the first paragraph of the proof of Theorem 3.2. Without loss of generality we may assume that the empty word $\epsilon \in L_1, L_2$, and $\psi_1(\epsilon)$ and $\psi_2(\epsilon)$ are the identities in the groups G_1 and G_2, respectively. We put $L_1' = L_1 \setminus \{\epsilon\}$ and $L_2' = L_2 \setminus \{\epsilon\}$. Let $A = A_1 \cup A_2$. Let L be defined by the following regular expression $L = (L_1' L_2')^* \vee (L_1' L_2')^* L_1' \vee (L_2' L_1')^* \vee (L_2' L_1')^* L_2' \vee \epsilon$. That is, L is the regular language consisting of the empty string ϵ and the strings of the form $u_1 \dots u_k$, where each substring u_i, $i = 1, \dots, k$ either $u_i \in L_1'$ or $u_i \in L_2'$, and no consecutive strings u_i, u_{i+1} are elements of the same language L_1' or L_2'. Let us construct the map $\psi : L \to G_1 \star G_2$ as follows: $\psi(\epsilon) = e$ and $\psi(u_1 \dots u_k) = \psi(u_1) \dots \psi(u_k)$, where for each u_i, $i = 1, \dots, k$, $\psi(u_i) = \psi_1(u_i)$ or $\psi(u_i) = \psi_2(u_i)$ if $u_i \in L_1'$ or $u_i \in L_2'$, respectively. It is easy to verify that the constructed map ψ provides a Cayley automatic representation of $G_1 \star G_2$ (see also [11, Theorem 10.8]).

Now, let $w = u_1 \dots u_k \in L$. Then, $d_A(\pi(w), \psi(w)) \leqslant d_A(\pi(w)) + d_A(\psi(w)) \leqslant |w| + \sum_{i=1}^k d_A(\psi(u_i))$. For each u_i, $i = 1, \dots, k$, we have $d_A(\psi(u_i)) \leqslant d_A(\pi(u_i)) + d_A(\pi(u_i), \psi(u_i)) \leqslant |u_i| + Kf(M|u_i|)$, if $|u_i| \geqslant N$ for some positive integer constants K, M and N; here we also assume that $MN \geqslant n_0$. For all $|u_i| < N$ we can bound $d_A(\psi(u_i))$ from above by some constant C since there exist only finitely many such u_i; we also assume that $C \geqslant 1$. Therefore, by the assumption that $f(x) + f(y) \leqslant f(x + y)$ for all $x, y \geqslant n_0$, we obtain $\sum_{i=1}^k d_A(\psi(u_i)) \leqslant C|w| + Kf(M|w|)$. Thus, $d_A(\pi(w), \psi(w)) \leqslant (C + 1)|w| + Kf(M|w|)$ for all $w \in L$. We note that the inequality $f(x) + f(y) \leqslant f(x + y)$ for all $x, y \geqslant n_0$ implies that $\mathfrak{i} \preceq f$, unless f is identically equal to zero. So, for the function $h(n) = \max\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\}$, we have $h \preceq f$. □

Corollary 3.4. *If $G_1, G_2 \in \mathcal{B}_{\mathfrak{i}}$ or $G_1, G_2 \in \mathcal{B}_{\mathfrak{e}}$, then $G_1 \star G_2$ is also in the class $\mathcal{B}_{\mathfrak{i}}$ or $\mathcal{B}_{\mathfrak{e}}$, respectively.*

Proof. It is enough to notice that for the functions $f = \mathfrak{i}$ and $f = \mathfrak{e}$, the inequality $f(x) + f(y) \leqslant f(x+y)$ holds for all $x, y \geqslant 1$. □

4 Nilpotent Groups and Fundamental Groups of n–dimensional Torus Bundles over The Circle

In this section we show that some classes of nilpotent groups and the fundamental groups of n–dimensional torus bundles over the circle are in the class $\mathcal{B}_{\mathfrak{e}}$. In the second half of the section we address the problem of finding sharp lower bounds of the function (1.1) for virtually nilpotent groups. Before we proceed with the main result of the section let us prove the following technical lemma which is needed, in particular, for the proof of Theorem 4.2. Let $\varphi : L_\Sigma \to G$ be a Cayley automatic representation of G, where $L_\Sigma \subseteq \Sigma^*$ now is a regular language over some alphabet Σ^* (here we do not assume that $\Sigma = S$). We denote by h_φ the function $h_\varphi(n) = \max\{d_A(\varphi(w)) | w \in L^{\leqslant n}\}$.

Lemma 4.1. *Suppose that $h_\varphi \preceq f$ for some function $f \in \mathfrak{F}$. Then $G \in \mathcal{B}_{\tilde{f}}$, where $\tilde{f} = f + \mathfrak{i}$.*

Proof. For every $\sigma \in \Sigma$ let us choose a string $w_\sigma \in S^*$ such that the lengths $|w_\sigma|$ are equal to some constant ℓ for all $\sigma \in \Sigma$. Then we define a monoid homomorphism $\xi : \Sigma^* \to S^*$ as follows: $\xi(\sigma_1 \dots \sigma_k) = w_{\sigma_1} \dots w_{\sigma_k}$. We define $L = \xi(L_\Sigma)$ and $\psi = \varphi \circ \xi^{-1} : L \to G$. Clearly, $\psi : L \to G$ is a Cayley automatic representation of G. Moreover, for any $w \in L$ we have $d_A(\pi(w), \psi(w)) \leqslant d_A(\pi(w)) + d_A(\psi(w)) = |w| + d_A(\varphi \circ \xi^{-1}(w)) \leqslant |w| + h_\varphi(|\xi^{-1}(w)|) = |w| + h_\varphi\left(\frac{1}{\ell}|w|\right) \leqslant |w| + h_\varphi(|w|)$. Therefore, for the function $h(n) = \max\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\}$, we clearly have $h \preceq \tilde{f}$. □

Theorem 4.2. *The following groups are all in the class $\mathcal{B}_{\mathfrak{e}}$:*

- *fundamental groups of n–dimensional torus bundles over the circle $\mathbb{Z}^n \rtimes_A \mathbb{Z}$,*
- *unitriangular matrices $UT_n(\mathbb{Z})$,*
- *f.g. nilpotent groups of nilpotency class 2.*

Proof. Let β be a representation of $\mathbb{Z}$ for which every $z \in \mathbb{Z}$ is represented as a signed binary number. Let γ be a representation of $\mathbb{Z}$ for which every $y \in \mathbb{Z}$ is represented as the concatenation of $|y|$ identical single–letter strings; for positive and negative integers we use different letters. See also the representation of the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ that we constructed in [4, Section 6]. For any given $\overline{z} = (z_1, \dots, z_n) \in \mathbb{Z}^n$ we represent it as the convolution $v = w_1 \otimes \dots \otimes w_n$, where $w_i = \beta^{-1}(z_i)$, $i = 1, \dots, n$. Then we represent an element $g = (y, \overline{z}) \in \mathbb{Z}^n \rtimes_A \mathbb{Z}$ as the concatenation $w = uv$, where $u = \gamma^{-1}(y)$. By [11, Theorem 10.3], it provides a Cayley automatic representation φ of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$. In the group $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ the element $g = (y, \overline{z})$ is equal to the product $g = (y, \overline{0}) \cdot (0, \overline{z})$, where 0 is the identity of $\mathbb{Z}$ and $\overline{0}$ is the identity of $\mathbb{Z}^n$. It is easy to see now that the condition of Lemma 4.1 is satisfied for the representation φ, the function $f = \mathfrak{e}$ and a natural set of generators $(1, \overline{0})$ and $(0, \overline{e}_i)$, $i = 1, \dots, n$, where $\overline{e}_i \in \mathbb{Z}^n$ has the jth element equal to δ_{ij}, $j = 1, \dots, n$. Therefore, $\mathbb{Z}^n \rtimes_A \mathbb{Z} \in \mathcal{B}_{\mathfrak{e}}$.

Any element g of the unitriangular matrix group $UT_n(\mathbb{Z})$ is given by a $n \times n$ matrix M with all elements below the main diagonal equal to 0 and all elements of the main diagonal equal to 1. Let $m_{ij} \in \mathbb{Z}$, $i < j$ be the element of M in row i and column j. We denote by $t_{ij} \in UT_n(\mathbb{Z})$ the transvection given by a $n \times n$ matrix with all elements on the main diagonal and the element in row i and column j equal to 1 and all other elements equal to 0. In the group $UT_n(\mathbb{Z})$ the element g is equal to the product of transvections $g = t_{1n}^{m_{1n}} \dots t_{(n-1)n}^{m_{(n-1)n}} \dots t_{13}^{m_{13}} t_{23}^{m_{23}} t_{12}^{m_{12}}$. We represent g as the convolution $s_{12} \otimes \dots \otimes s_{(n-1)n}$, where $s_{ij} = \beta^{-1}(m_{ij})$, $1 \leqslant i < j \leqslant n$. Clearly, the condition of Lemma 4.1 is satisfied for this representation, the function $f = \mathfrak{e}$ and the set of generators $\{t_{ij} | 1 \leqslant i < j \leqslant n\}$. Therefore, $UT_n(\mathbb{Z}) \in \mathcal{B}_{\mathfrak{e}}$.

It is known that for every f.g. nilpotent group its torsion subgroup is finite. Moreover, every f.g. nilpotent group is residually finite. Therefore, every f.g. nilpotent group has a torsion–free subgroup of finite index. So, by Theorem 3.1, it is enough for us to show that any given torsion–free f.g. nilpotent group G of nilpotency class 2 is in $\mathcal{B}_{\mathfrak{e}}$. In [11, Theorem 12.4] the authors used Mal'cev coordinates to construct Cayley automatic representation of the group G. Below we use their representation to show that $G \in \mathcal{B}_{\mathfrak{e}}$. Let $\overline{a} = (a_1, \dots, a_n) \in G^n$ be any Mal'cev basis for G associated with the upper central series of G. We recall that the factors of the upper central series of a torsion–free nilpotent group are torsion–free. So, for any given $g \in G$, we have a unique presentation of g in G as a product: $g = a_1^{k_1} \dots a_n^{k_n}$, where $(k_1, \dots, k_n) \in \mathbb{Z}^n$ is a tuple of the Mal'cev coordinates of g with respect to the basis $\overline{a}$. We represent g as the convolution $s_1 \otimes \dots \otimes s_n$, where $s_i = \beta^{-1}(k_i)$, $i = 1, \dots, n$. The condition of Lemma 4.1 is satisfied for this representation, the function $f = \mathfrak{e}$ and the set of generators $\{a_1, \dots, a_n\}$. Thus, $G \in \mathcal{B}_{\mathfrak{e}}$. $\square$

Can any of the groups from Theorem 4.2 be in the class $\mathcal{B}_f$ for some $f \prec \mathfrak{e}$? The greatest lower bound for the function f that we can obtain from Theorem 2.3 is $\mathfrak{i}$, see, e.g., Corollary 2.4. However, for some groups, e.g. the higher Heisenberg groups $\mathcal{H}_{2k+1}$, $k > 1$, Theorem 2.3 does not give any lower bound (recall that they are nilpotent groups of nilpotency class 2 and their Dehn functions are quadratic). Thurston proved that automatic nilpotent groups must be virtually abelian (see, e.g., [9, Theorem 8.2.8]). So, by [4, Theorem 8], for any class $\mathcal{B}_f$ containing a Cayley automatic nilpotent group (which is not virtually abelian) the function f must be unbounded. Moreover, while for the Baumslag–Solitar groups $BS(p,q)$, $q > p \geqslant 1$ and the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ we obtain the sharp lower bounds [4, Theorem 11 and 13], we do not know whether the lower bounds, which we can obtain from Theorem 2.3 for other groups mentioned in this paper, are sharp. To address this issue we make a simple observation in Theorem 4.4 that might, potentially, be useful in the search for the sharp lower bounds for virtually nilpotent groups. Furthermore, in Theorem 5.1 we show that, for the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$, the exponential function $\mathfrak{e}$ is a lower bound of the function (1.1), if one puts some additional constraints on a Cayley automatic representation ψ. We recall that a regular language L is called simply starred if a regular expression for L is of the form: $R_1 \vee \cdots \vee R_I$, where $R_i = v_{i,0}u_{i,1}^* v_{i,1} \dots v_{i,P_{i-1}} u_{i,P_i}^* v_{i,P_i}$ for $i = 1, \dots, I$. We have the following proposition.

Proposition 4.3 (polynomial growth condition). *A regular language L has polynomial growth if it is simply starred and exponential growth otherwise.*

Proof. For the proof see, e.g., [9, Theorem 8.2.8]. □

Let $\psi : L \to G$ be a Cayley automatic representation of a virtually nilpotent group G; as usual, $L \subseteq (A \cup A^{-1})^*$ for some set of generators $A \subset G$. Let h be the function defined by (1.1) corresponding to the representation ψ.

Theorem 4.4. *Suppose that $h \preceq p$ for some polynomial p. Then the language L is simply starred.*

Proof. For any given $w \in L^{\leqslant n}$ we have $d_A(\psi(w)) \leqslant d_A(\pi(w)) + d_A(\pi(w), \psi(w)) \leqslant n + h(n)$. Therefore, since $h \preceq p$, there exists a polynomial q for which $\psi(w)$ must be in the ball $B_{q(n)} \subset G$ of radius $q(n)$. Recall that a growth function of any virtually nilpotent group is bounded by a polynomial. Therefore, the cardinality of $B_{q(n)}$ must be bounded by $r(n)$ for some polynomial r so the cardinality of the set $L^{\leqslant n}$. By Proposition 4.3 we obtain the statement of the theorem. □

5 In The Search for Alternative Approaches to Proving Nonautomaticity

In this section we focus on the problem of finding a sharp lower bound of the function (1.1) for the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$. Another motivation of this section is to propose alternative methods for proving nonautomaticity of groups. Clearly, if a group $G \notin \mathcal{B}_f$ for some function $f \in \mathfrak{F}$, then G is not automatic. We already know two ways to show that a group is not in a class $\mathcal{B}_f$ if $f \prec f_0$ for some nonzero function f_0 (see Theorem 2.3 and the proof that the lamplighter group is not in the class $\mathcal{B}_f$ for any $f \prec \mathfrak{i}$ [4, Theorem 13]). In the first approach we use the Dehn function (when it grows faster than the quadratic function), while in the second approach we implicitly use a fact that the lamplighter group is not finitely presented. However, in both cases one straightforwardly gets nonautomaticity by [9, Theorem 2.3.12]. Is there any alternative method to show that a given group G is not in $\mathcal{B}_f$ for some $f \in \mathfrak{F}$? Such a method could potentially provide a new way to prove nonautomaticity. In this part we make a first tiny step in this direction focusing on the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$.

It was first noticed by Sénizergues that the Heisenberg group is not automatic, but its Cayley graph is FA–presentable; also, it was one of the first examples of such groups. Another motivation to focus on $\mathcal{H}_3(\mathbb{Z})$ is the "Heisenberg alternative" – each f.g. group G of polynomial growth is either virtually abelian or $\mathcal{H}_3(\mathbb{Z})$ can be embedded into G. In [17], Nies and Thomas used this alternative to give a new proof of the theorem that every f.g. FA–presentable group is virtually abelian; this was first proved by Oliver and Thomas in [18]. We recall that $\mathcal{H}_3(\mathbb{Z})$ is the group of all matrices of the form: $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, where x, y and z are integers; so, every element $g \in \mathcal{H}_3(\mathbb{Z})$ corresponds to a triple (x, y, z). We denote by s, p and q the group elements corresponding to the triples $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. If g corresponds to a triple (x, y, z), then gs, gp and gq correspond to the triples $(x + 1, y, z)$, $(x, y + 1, x + z)$ and $(x, y, z + 1)$, respectively. We put $A = \{e, s, p, q\}$ and $S = \{e, s, p, q, s^{-1}, p^{-1}, q^{-1}\}$.

It is straightforward to verify that $\mathcal{H}_3(\mathbb{Z})$ is isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes_T \mathbb{Z}$, where $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$: an isomorphism is given by the following mapping $(x, y, z) \mapsto \left(y, \begin{bmatrix} x \\ z \end{bmatrix}\right)$. We denote by H the normal subgroup of $\mathcal{H}_3(\mathbb{Z})$ generated by s and q, and by N the cyclic subgroup of $\mathcal{H}_3(\mathbb{Z})$ generated by p. Clearly, $H \cong \mathbb{Z}^2$, $N \cong \mathbb{Z}$ and $\mathcal{H}_3(\mathbb{Z}) = NH$. We denote by φ, p_1 and p_2 the endomorphisms of the group H given by the matrices $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. The endomorphisms p_1 and p_2 are the projectors of H on the cyclic subgroups generated by s and q, respectively. We denote these subgroups by H_1 and H_2: $H_1 = p_1(H) = \langle s \rangle$ and $H_2 = p_2(H) = \langle q \rangle$. Let $\psi : L \to \mathcal{H}_3(\mathbb{Z})$ be a Cayley automatic representation of $\mathcal{H}_3(\mathbb{Z})$, where $L \subseteq S^*$. We denote by L_H the language $L_H = \psi^{-1}(H) \subset L$ and by w_0 the string $w_0 = \psi^{-1}(e)$, where e is the identity of the group $\mathcal{H}_3(\mathbb{Z})$.

Let $R_\varphi = \{\langle w, \psi^{-1} \circ \varphi \circ \psi(w)\rangle \mid w \in L_H\}, R_{p_1} = \{\langle w, \psi^{-1} \circ p_1 \circ \psi(w)\rangle \mid w \in L_H\}, R_{p_2} = \{\langle w, \psi^{-1} \circ p_2 \circ \psi(w)\rangle \mid w \in L_H\} \subset L_H \times L_H$ be the binary relations on L_H defined by the endomorphisms φ, p_1, p_2, respectively, and the Cayley automatic representation ψ. For a given binary relation $R \subseteq S^* \times S^*$, we denote by $L_H \triangleleft R$ and $R \triangleright L_H$ the left– and right–restrictions of R on L_H: $L_H \triangleleft R = \{\langle u, v\rangle \in R \mid u \in L_H\}$ and $R \triangleright L_H = \{\langle u, v\rangle \in R \mid v \in L_H\}$. We denote by L_{H_1} and L_{H_2} the languages $L_{H_1} = \psi^{-1}(H_1) \subset L_H$ and $L_{H_2} = \psi^{-1}(H_2) \subset L_H$.

Theorem 5.1. *Assume that there exist some FA–recognizable relations R_0, R_1, $R_2 \subseteq S^* \times S^*$ for which $L_H \triangleleft R_0 = R_\varphi$, $R_1 \triangleright L_{H_1} = R_{p_1}$, $L_H \triangleleft R_2 = R_{p_2}$ and $R_2 \triangleright \{w_0\} = R_{p_2} \triangleright \{w_0\}$. Then, for the function $h(n) = \max\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\}$, $\mathfrak{e} \preceq h$. In particular, $h \npreceq f$ for any $f \prec \mathfrak{e}$.*

Proof. Let $\eta(a, b, c)$ be the following first–order formula:

$$\eta(a, b, c) \equiv \exists r, s_1, s_2, t_1, t_2, t_3 \{R_1(r, a) \wedge (R_0(b, s_1) \wedge R_2(s_1, s_2) \wedge R_2(r, s_2)) \wedge (R_0(r, t_1) \wedge R_2(t_1, t_2)) \wedge (R_2(c, w_0) \wedge R_0(c, t_3) \wedge R_2(t_3, t_2))\}.$$

Let us verify that for any $a, b \in L_{H_1}$ the formula $\eta(a, b, c)$ is true if and only if $c \in L_{H_1}$ and $\psi(a) + \psi(b) = \psi(c)$ in the cyclic group H_1. Suppose that, for some $a, b \in L_{H_1}$, $\eta(a, b, c)$ is true. Let $\psi(a) = \begin{bmatrix} k \\ 0 \end{bmatrix}$, $\psi(b) = \begin{bmatrix} \ell \\ 0 \end{bmatrix}$ for some $k, \ell \in \mathbb{Z}$. Since $R_1(r, a)$ is true and $R_1 \triangleright L_{H_1} = R_{p_1}$, then $r \in L_H$ and $\psi(r) = \begin{bmatrix} k \\ \star \end{bmatrix}$. Furthermore, since $R_0(b, s_1) \wedge R_2(s_1, s_2) \wedge R_2(r, s_2)$ is true and $L_H \triangleleft R_0 = R_\varphi$, $L_H \triangleleft R_2 = R_{p_2}$, then $s_1 \in L_H$, $s_2 \in L_{H_2}$ and $\psi(s_1) = \begin{bmatrix} \ell \\ \ell \end{bmatrix}$, $\psi(s_2) = \begin{bmatrix} 0 \\ \ell \end{bmatrix}$, $\psi(r) = \begin{bmatrix} \star \\ \ell \end{bmatrix}$. Therefore, $\psi(r) = \begin{bmatrix} k \\ \ell \end{bmatrix}$. Moreover, since $R_0(r, t_1) \wedge R_2(t_1, t_2)$ is true and $L_H \triangleleft R_0 = R_\varphi$, $L_H \triangleleft R_2 = R_{p_2}$, then $\psi(t_1) = \begin{bmatrix} k \\ k+\ell \end{bmatrix}$, $\psi(t_2) = \begin{bmatrix} 0 \\ k+\ell \end{bmatrix}$. Finally, since $R_2(c, w_0) \wedge R_0(c, t_3) \wedge R_2(t_3, t_2)$ and $R_2 \triangleright \{w_0\} = R_{p_2} \triangleright \{w_0\}$, $L_H \triangleleft R_0 = R_\varphi$, $L_H \triangleleft R_2 = R_{p_2}$, then $c = \begin{bmatrix} m \\ 0 \end{bmatrix}$, $\psi(t_3) = \begin{bmatrix} m \\ m \end{bmatrix}$, $\psi(t_2) = \begin{bmatrix} 0 \\ m \end{bmatrix}$. Thus, $c \in L_{H_1}$ and $m = k + \ell$ which implies that $\psi(a) + \psi(b) = \psi(c)$. The reverse is straightforward.

Let $R \subseteq S^* \times S^* \times S^*$ be the relation defined by η, that is, $R(a, b, c)$ is true iff $\eta(a, b, c)$ is true. Since R_0, R_1, R_2 are FA–recognizable, R is FA–recognizable. Let $(M, \times)$ be a monoid generated by s, where $M = \{s^n \mid n \geqslant 0\} \subset H_1$ and $\times$ is the group multiplication in H_1. Clearly, $(M, \times) \cong (\mathbb{N}, +)$. Let $u_n = \psi^{-1}(s^n) \in L$ and $L_M = \{u_n | n \geqslant 0\} \subset L$. It follows directly from [17, Lemma 6] (this lemma was originally proved in [15] for automatic monoids) that there exist constants C, N_0 for which $|u_n| \leqslant C \log n$ for all $n \geqslant N_0$, where

$|u_n|$ is the length of the string u_n. It follows from the metric inequalities for the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$, see, e.g., [19, Proposition 1.38], that there exist a constant $C_1 > 0$ for which $d_A(s^n) \geqslant C_1 n$ for all $n \geqslant 0$. We have: $d_A(\pi(u_n), \psi(u_n)) \geqslant d_A(\psi(u_n)) - d_A(\pi(u_n)) \geqslant C_1 n - |u_n| \geqslant C_1 n - C \log n$ for all $n \geqslant N_0$. Therefore, there exist some constants $C_2 > 0$ and $N_1 \geqslant N_0$ for which $d_A(\pi(u_n), \psi(u_n)) \geqslant C_2 n \geqslant C_2 \exp\left(\frac{1}{C}|u_n|\right)$ for all $n \geqslant N_1$. Clearly, the set $\{|u_n| \,|\, u_n \in L_M\} \subseteq \mathbb{N}$ is infinite. Moreover, by the finite difference lemma (see, e.g., [11, Lemma 14.1]) , $||u_{n+1}| - |u_n|| \leqslant D$ for every $n \geqslant 0$ and some constant D. Therefore, there exists a constant D_0 such that for every $j \geqslant D_0$ there is $u_n \in L_M$ for which $j \geqslant |u_n| > j - D$. Thus, for the function $h(n) = \max\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\}$, $\mathfrak{e} \preceq h$. The last statement of the theorem is straightforward. $\square$

Remark 5.2. *We note that the conditions of Theorem 5.1 are clearly satisfied for the Cayley automatic representation of the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ constructed in [4, Section 6]. As for FA–recognizable relation $R_0 \subset S^* \times S^*$ for which $L_H \triangleleft R_0 = R_\varphi$, it exists if, for example, one additionally requires that the left multiplication by p^{-1} in the group $\mathcal{H}_3(\mathbb{Z})$ is FA–recognizable; it follows from the fact that for any $h \in H$: $p^{-1}hp = \varphi(h)$.*

6 Linear Upper Bounds for Almost All Elements in Groups of Exponential Growth

In this section we show that for an arbitrary bijection $\psi : L \to G$ between a language $L \subseteq (A \cup A^{-1})^*$ and a group G of exponential growth a linear upper bound $d_A\left(\pi\left(\psi^{-1}(g)\right), g\right) \leqslant C|\psi^{-1}(g)|$ holds for almost all $g \in G$ in a certain sense, see Theorem 6.1 and Remark 6.2. However, in the following Remark 6.3 we show how to construct Cayley automatic representations of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the function (1.1) grows faster than any tower of exponents.

Theorem 6.1. *Let us assume that $\psi : L \to G$ is a Cayley automatic representation of a group G which has exponential growth. Then there exist constants $\lambda_1, \lambda_2 > 0$ such that for almost all $g \in G$: $\lambda_1 d_A(g) \leqslant |w| \leqslant \lambda_2 d_A(g)$, where $\psi(w) = g$. The term almost all here means that $\lim\limits_{n\to\infty} \frac{\#Q_n}{\#B_n} = 1$, where $B_n = \{g \in G \,|\, d_A(g) \leqslant n\}$ is the ball of radius n in G and $Q_n \subseteq B_n$ is defined as $Q_n = \{g \in B_n \,|\, \lambda_1 d_A(g) \leqslant |w| \leqslant \lambda_2 d_A(g)\}$. In particular, for every $g \in Q_n$, $d_A(\pi(w), \psi(w)) \leqslant \left(1 + \frac{1}{\lambda_1}\right)|w|$.*

Proof. The inequality $|w| \leqslant \lambda_2 d_A(g)$ always holds for some $\lambda_2 > 0$ due to the bounded difference lemma. Since G has exponential growth, there exists $\lambda > 1$ for which $\#B_n \geqslant \lambda^n$ for all $n \geqslant n_0$. For a given integer $k > 0$ we denote by

R_k the following finite subset of G: $R_k = \{g \in G \,|\, g = \psi(w), w \in L^{<k}\}$; where $L^{<k} = \{w \in L \,|\, |w| < k\}$. Since $L \subseteq S^*$, $\#R_k \leqslant |S|^{k-1}$. We denote by $T_{n,k}$ the set $T_{n,k} = B_n \setminus R_k$. For every $g \in T_{n,k}$, $|w| \geqslant k$. Therefore, if $\lambda_1 \leqslant \frac{k}{n}$, for every $g \in T_{n,k}$ we have that $\lambda_1 d_A(g) \leqslant |w|$; so $T_{n,k} \subseteq Q_n$. We notice that $\frac{\#T_{n,k}}{\#B_n} \geqslant 1 - \frac{\#R_k}{\#B_n} \geqslant 1 - \frac{|S|^{k-1}}{\lambda^n}$ for all $n \geqslant n_0$. So, it is enough to provide λ_1 and a sequence $k_n, n \geqslant n_0$ for which $\lambda_1 \leqslant \frac{k_n}{n}$ for all $n \geqslant n_0$ and $\lim\limits_{n\to\infty} \frac{|S|^{k_n-1}}{\lambda^n} = 0$. We note that $\frac{|S|^{k_n-1}}{\lambda^n} = \frac{|S|^{k_n-1}}{|S|^{(\log_{|S|}\lambda)n}} = \frac{1}{|S|^{(\log_{|S|}\lambda)n - k_n + 1}}$. Let us put $k_n = \lceil \frac{1}{2}(\log_{|S|}\lambda)n \rceil$ for all $n \geqslant n_0$ and $\lambda_1 = \frac{1}{2}(\log_{|S|}\lambda)$. Therefore, $(\log_{|S|}\lambda)n - k_n + 1 \geqslant \frac{1}{2}(\log_{|S|}\lambda)n$, so $\lim\limits_{n\to\infty} \frac{|S|^{k_n-1}}{\lambda^n} = 0$. Moreover, $\frac{k_n}{n} \geqslant \lambda_1$ for all $n \geqslant n_0$. In order to prove the last inequality, we observe that $d_A(\pi(w), \psi(w)) \leqslant |w| + d_A(g) \leqslant |w| + \frac{1}{\lambda_1}|w|$. □

Remark 6.2. *It is easy to see that Theorem 6.1 holds for any bijection* $\psi : L \to G$ *such that* $||\psi^{-1}(ga)| - |\psi^{-1}(g)|| \leqslant C$ *for all* $g \in G$ *and every generator* $a \in A$*, where* C *is a constant. Moreover, for the inequality* $\lambda_1 d_A(g) \leqslant |w|$ *and, accordingly, the inequality* $d_A(\pi(w), \psi(w)) \leqslant \left(1 + \frac{1}{\lambda_1}\right)|w|$*, no assumption is needed – it holds for almost all* $g \in G$ *for any bijection* ψ *between a language* L *and a group* G *of exponential growth. Since in this paper we focus mainly on Cayley automatic representations of groups, in Theorem 6.1 we assume that* $\psi : L \to G$ *is a Cayley automatic representation of* G*.*

Remark 6.3. *We note that although for any Cayley automatic representation* $\psi : L \to G$ *of a group of exponential growth* G *the inequality* $d_A(\pi(w), \psi(w)) \leqslant C|w|$ *holds for some constant* C *for almost all* $\psi(w) = g \in G$ *in the sense of Theorem 6.1, it does not hold for all* $g \in G$*. For example, let us consider the following Cayley automatic representation* $\varphi : L_\Sigma \to \mathbb{Z}_2 \wr \mathbb{Z}$ *over the alphabet* $\Sigma = \{+, -, 0, 1, C_0, C_1, \#\}$*. For any given pair* $(f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$*, we represent it as the string:* $u\#f(s)\dots C_{f(z)}\dots f(t)$*, where* s *and* t *are the minimum and the maximum integers of the set* $\{i|f(i) = 1\} \cup \{z\}$*,* $C_{f(z)}$ *is* C_0 *or* C_1 *if* $f(z) = 0$ *or* $f(z) = 1$*, respectively, and the string* u *is a binary representation of the integer* s*. For example, let us consider a pair* $(f, 3)$*, where* $f(1) = 1, f(2) = 1$ *and* $f(i) = 0$ *if* $i \neq 1, 2$*, it is represented as the string:* $+1\#11C_0$*. Let us consider a pair* $(f, -3)$*, where* $f(-4) = 1, f(-3) = 1$*,* $f(-2) = 1$*,* $f(i) = 0$ *if* $i \neq -4, -3, -2$*, it is represented as the string:* $-100\#1C_11$*. We also refer the reader to [2, Example 4.2.1].*

One can then convert this representation φ*, in a same way as in Lemma 4.1, into some representation* $\psi : L \to \mathbb{Z}_2 \wr \mathbb{Z}$ *over the alphabet* $S = A \cup A^{-1}$*, where* $A = \{a, t\}$ *is the standard set of generators of* $\mathbb{Z}_2 \wr \mathbb{Z}$*:* a *is the nontrivial element of* $\mathbb{Z}_2$ *and* t *is a generator of* $\mathbb{Z}$ *(here we treat* $\mathbb{Z}_2$ *and* $\mathbb{Z}$ *as the subgroups of* $\mathbb{Z}_2 \wr \mathbb{Z}$*). Although* $\mathbb{Z}_2 \wr \mathbb{Z} \in \mathcal{B}_\mathfrak{i}$ *[4, Theorem 13], for the representation* ψ *the inequality* $d_A(\pi(w), \psi(w)) \leqslant C|w|$ *does not hold for all* $w \in L$ *and any constant* C*. In order to see that, let us consider the representatives* $w_i = \psi^{-1}(g_i)$ *of the* *elements* $g_i = (f_0, i) \in \mathbb{Z}_2 \wr \mathbb{Z}$, $i > 0$ *with respect to* ψ, *where* $f_0(j) = 0$ *for all* $j \in \mathbb{Z}$. *Apparently,* $d_A(g_i) = i$ *but the function* $\ell(i) = |w_i|$ *grows, coarsely, as* $\log i$. *So, the function* $h(n) = \max\{d_A(\pi(w), \psi(w))|w \in L^{\leqslant n}\}$ *grows at least as fast as the exponential function.*

Moreover one can construct a Cayley automatic representation $\psi : L \to \mathbb{Z}_2 \wr \mathbb{Z}$ *for which the function* $h(n) = \max\{d_A(\pi(w), \psi(w))|w \in L^{\leqslant n}\}$ *grows faster than any tower of exponents* $e^{e^{\cdots^{e}}}$. *This follows from the result shown by Frank Stephan:*

Theorem 6.4 (Frank Stephan [21]). *There exists an automatic representation* $\tau : L_\tau \to \mathbb{N}$ *of the structure* $(\mathbb{N}, S)$, *where* S *is the successor function, for which the function* $r(n) = \max\{\tau(w)|w \in L_\tau^{\leqslant n}\}$ *grows faster than any tower of exponents* $e^{e^{\cdots^{e}}}$.

Clearly, one cannot directly generalize Theorem 6.1 for Cayley automatic groups of subexponential growth. Moreover, it simply does not hold for many Cayley automatic groups of subexponential growth – consider, for example, a binary representation of the infinite cyclic group $\mathbb{Z}$. A f.g. group of subexponential growth has either intermediate growth or polynomial growth. Miasnikov and Savchuk constructed a FA–presentable graph of intermediate growth [16]. However, it is still unknown whether there exists any Cayley automatic group of intermediate growth. As for f.g. groups of polynomial growth, due to celebrated Gromov's theorem [10], any such group is virtually nilpotent.

Acknowledgment

The authors thank Murray Elder, Bakhadyr Khoussainov and Frank Stephan for useful comments.

References

[1] Berdinsky, D.: Cayley automatic groups and numerical characteristics of Turing transducers. In: Brlek, S., Reutenauer, C. (eds.) Developments in Language Theory 2016, Lecture Notes in Computer Science, vol. 9840, pp. 26–37. Springer International Publishing (2016)

[2] Berdinsky, D.: Representations of transitive graphs by automats. PhD Thesis, The University of Auckland (2017)

[3] Berdinsky, D., Khoussainov, B.: On automatic transitive graphs. In: Shur, A., Volkov, M. (eds.) Developments in Language Theory 2014, Lecture Notes in Computer Science, vol. 8633, pp. 1–12. Springer International Publishing (2014)

[4] Berdinsky, D., Trakuldit, P.: Measuring closeness between Cayley automatic groups and automatic groups. In: Klein, S., Martn-Vide, C., Shapira, D. (eds.) Language and Automata Theory and Applications 2018, Lecture Notes in Computer Science, vol. 10792, pp. 245–257. Springer International Publishing (2018)

[5] Bérubé, S., Palnitkar, T., Taback, J.: Higher rank lamplighter groups are graph automatic. Journal of Algebra 496, 315–343 (2018)

[6] Bridson, M.R., Gersten, S.M.: The optimal isoperimetric inequality for torus bundles over the circle. The Quarterly Journal of Mathematics 47(1), 1–23 (1996)

[7] Bridson, M.R., Pittet, C.: Isoperimetric inequalities for the fundamental groups of torus bundles over the circle. Geometriae Dedicata 49(2), 203–219 (1994)

[8] Elder, M., Taback, J.: $\mathcal{C}$–graph automatic groups. Journal of Algebra 413, 289–319 (2014)

[9] Epstein, D.B.A., Cannon, J.W., Holt, D.F., Levy, S.V.F., Paterson, M.S., Thurston, W.P.: Word Processing in Groups. Jones and Barlett Publishers. Boston, MA (1992)

[10] Gromov, M.: Groups of polynomial growth and expanding maps. Publ. Math. Inst. Hautes tudes Sci. 53, 53–78 (1981)

[11] Kharlampovich, O., Khoussainov, B., Miasnikov, A.: From automatic structures to automatic groups. Groups, Geometry, and Dynamics 8(1), 157–198 (2014)

[12] Khoussainov, B., Minnes, M.: Three lectures on automatic structures. Proceedings of Logic Colloquium, 132–176 (2007)

[13] Khoussainov, B., Nerode, A.: Automatic presentations of structures. In: Leivant, D. (ed.) Logic and Computational Complexity, Lecture Notes in Computer Science, vol. 960, pp. 367–392. Springer Berlin Heidelberg (1995)

[14] Khoussainov, B., Nerode, A.: Open questions in the theory of automatic structures. Bulletin of the EATCS 94, 181–204 (2008)

[15] Khoussainov, B., Nies, A., Rubin, S., Stephan, F.: Automatic structures: richness and limitations. Logical Method in Computer Science 3(2:2), 1–18 (2007)

[16] Miasnikov, A., Savchuk, D.: An example of an automatic graph of intermediate growth. Ann. Pure Appl. Logic 166(10), 1037–1048 (2015)

[17] Nies, A., Thomas, R.M.: FA–presentable groups and rings. Journal of Algebra 320, 569–585 (2008)

[18] Oliver, G.P., Thomas, R.M.: Automatic presentations for finitely generated groups. In: Diekert, V., Durand, B. (eds.) STACS 2005, Lecture Notes in Computer Science, vol. 3404, pp. 693–704. Springer Berlin Heidelberg (2005)

[19] Roe, J.: Lectures on Coarse Geometry. University Lecture Series, vol. 31. American Mathematical Society (2003)

[20] Stephan, F.: Automatic structures — recent results and open questions. Journal of Physics: Conference Series 622, 012013 (jun 2015)

[21] Stephan, F.: Private communication

[22] Vershik, A.: Numerical characteristics of groups and corresponding relations. Journal of Mathematical Sciences 107(5), 4147–4156 (2001)