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# MULTIFRACTAL STRUCTURE OF FRACTAL MEASURES

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**Abstract**

In [HN], the authors showed that if $2 \leq q \leq m \leq 2q-2$ then the set $E$ of the attainable local dimensions of fractal measure $\mu$ is an interval. In this paper we will prove that this result is not true if we replace the probabilistic system $p_0 = p_1 = \ldots = p_m$ by the system $p_j = C_m^j/2^m, j = 0, 1, \ldots, m$. More precisely, the set $E$ has an isolated point. Hence the multifractal formalism fails in this case.

The special of our case when $q = 3$, the results was obtained earlier in [HL].

## 1 Introduction

Let $\{F_1, \ldots, F_m\}$ be an *iterated function system* ( IFS ) of $m$ contractive similitudes on $\mathbb{R}^d$:

$$F_j(x) = \rho_j R_j x + b_j,\ j = 1, \ldots, m,$$

where $0 < \rho_j < 1, R_j$ is a $d \times d$ orthogonal matrix and $b_j$ is a vector in $\mathbb{R}^d$. It is well known that there exists a unique nonempty compact subset $E$ in $\mathbb{R}^d$ such that

$$E = \bigcup_{j=1}^{m} F_j(E).$$

The set $E$ is called the *self-similar set* or the *invariant set* of the IFS (see [Hut]). If further, we associate the IFS with a set of probability weights $p_1, \ldots, p_m,\ 0 \leq$

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$p_j \le 1$ and $\sum_{j=1}^{m} p_j = 1$, then it will generate a unique invariant Borel probability measure such that

$$\mu = \sum_{j=1}^{m} p_j \mu \circ F_j^{-1}. \tag{1.1}$$

We call $\mu$ a *self-similar measure* or *invariant measure.*

The invariant sets and measures play a central role in theory of fractals. Jessen and Winter [JW] showed that this measure is either purely singular or absolutely continuous. If $0 < \rho < 1/m$, then the measure $\mu$ is purely singular. Otherwise, the different choice of the values $b_1, \dots, b_m$ and the probability weights $p_1, \dots, p_m$ will produce different type of the measure $\mu$. The determination of which type, in general, is very difficult.

When the measure $\mu$ is purely singular, the local dimension measures the degree of singularities of $\mu$ locally.

Recall that for $s \in \text{supp } \mu$, the *lower local dimension* and *upper local dimension* of $\mu$ at $s$ are defined as

$$\underline{\alpha}(s) = \lim_{h\to 0^+} \inf \frac{\log \mu(B(s,h))}{\log h};$$

$$\overline{\alpha}(s) = \lim_{h\to 0^+} \sup \frac{\log \mu(B(s,h))}{\log h},$$

where $B(s,h)$ is the closed interval $[s-h, s+h]$. When $\underline{\alpha}(s) = \overline{\alpha}(s)$ we refer to the common value as the *local dimension* of $\mu$ at $s$, and we denote it by $\alpha(s)$.

Put

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\};\ \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\};$$
$$E = \{\alpha : \alpha(s) = \alpha,\ s \in \text{ supp } \mu\} \ \text{ and } \ E_\alpha = \{s \in \text{ supp } \mu : \alpha(s) = \alpha\}.$$

One of the main objectives in fractal geometry is to study the multifractal structure of a measure $\mu$ such as the local dimension spectrum defined by

$$f(\alpha) = \dim_H E_\alpha,$$

the Hausdoff dimension of the level sets $E_\alpha$. It was first proposed by physicists to investigate various chaotic models arising from natural phenomena (see [FP], [HJKPS], [M]).

A direct computation of $f(\alpha)$ in general is rather difficult. Based on some physical intuition and analogous to the *thermodynamic formalism* in statistical mechanics, it was suggested that $f(\alpha)$ can be determined using the $L^q-$spectrum and the Legendre transformation (see [HP], [HJKPS], [FP]). Namely,

$$f(\alpha) = \tau^*(\alpha) := \inf\{\alpha p - \tau(p) : p \in \mathbb{R}\}, \tag{1.2}$$

where

$$\tau(p) = \lim \inf_{\delta \to 0} \frac{\log \sup \sum_j \mu(B(x_j, \delta))^p}{\log \delta},$$

and the supremum is over all families of disjoint closed $\delta$-balls $B(x_j, \delta)$ centered at $x_j \in$ supp $\mu$. The function $\tau(p)$ is called a $L^q-$ *spectrum* of the measure $\mu$.

The formula (1.2), known as *multifractal formalism*, holds for fractal measures associated with probabilistic systems satisfying the open set condition (see [CM], [Ols], [AP]). And more generally, for fractal measures associated with probabilistic systems possessing the weak separation property (see [LN]). More recently, D. J. Feng and E. Olivier proved that the multifractal formalism holds under a so-called “weak-Gibbs” condition (see [FO]). Without separation, however, much less is known, and almost all that is known refers to the portion of the $L^q$-spectrum corresponding to $p \geq 0$, see [LN] and [PS] for some of the deep results obtained.

In order for the multifractal formalism to hold, $f(\alpha)$ must be a concave function and the domain is an interval (i.e., the set of local dimensions of $\mu$ forms an interval). Therefore, the main question was proposed that: what condition on the chooses of parameters will ensure the domain of $f(\alpha)$ to contain an isolated point or ensure its domain to be an interval. In [HL], a first investigation was made for the $m$-fold convolution of the Cantor measure for $m \geq 3$. The authors proved that the set $E$ contains an isolated point. This result was proved by two other ways by Feng, Lau and Wang in [FLW]. In [HN], the authors considered the measure $\mu$ induced by IFS $\{F_j(x) = \frac{1}{q}(x+j) : j = 0, 1, \ldots, m\}$ and probabilistic system $\{p_j = 1/(m+1) : j = 0, 1, \ldots, m\}$. They showed that the maximum of the set $E$ is an isolated point of it for $m > 2q-2$. For the Bernoulli convolutions associated with the PV-number, Lau, Ngai and Feng gave a detailed study on the multifractal formalism (see [LN1-2], [F1-2]).

On the other hand, also in [HN], the authors showed that for $2 \leq q \leq m \leq 2q-2$ the set $E$ is an interval. Now we will prove that if we replace $p_j = 1/(m+1)$ in [HN] by $p_j = C_m^j/2^m$ for $j = 0, 1, \ldots, m$, then $E$ has an isolated point. Therefore the formula (1.2) is not true in this case.

In this paper we will consider the measure $\mu$ induced by the IFS $\{F_j = \frac{1}{q}(x+j) : j = 0, 1, \ldots, m\}$ and the associated probability system $\{p_j = \frac{C_m^j}{2^m} : j = 0, 1, \ldots, m\}$ for $3 \leq q \leq m \leq 2q-2$.

Denote $[x]$ be the largest integer not exceeding $x$. Our main result is stated as follows.

**Main Theorem .** *Let* $2q-2 \geq m \geq q \geq 3,\ m, q$ *be integers. Then*

*1.* $\overline{\alpha} = \dfrac{m \log 2}{\log q}$.

*2.* $\underline{\alpha} = \dfrac{m \log 2}{logq} - \dfrac{\log C_m^{[\frac{m+1}{2}]}}{\log q}$.

*3.* $\overline{\alpha}$ *is an isolated point of* $E$*. More precisely,* $E_\alpha = \emptyset$ *for all* $\alpha \in (\hat{\alpha}, \overline{\alpha})$ *where* $\hat{\alpha} = \dfrac{m \log 2 - \log C_m^{[(m-q+1)/2]}}{\log q}$.

The paper is organized as follows. In section 2 we will give some preliminaries and prove some basic lemmas for counting. The Main Theorem is proved in Section 3.

# 2 Notation and Primarily Results

Let $\mathbb{N}$ denote the set of all nonnegative integers. Let $3 \leq q \leq m \leq 2q-2$; $q, m \in \mathbb{N}$. We denote

$$\mathbb{D}_m = \{0, 1, \ldots, m\} \text{ and } \mathbb{D}_m^n = \{0, 1, \ldots, m\}^n, \text{ where } n \leq \infty,$$

and let

$$S = \sum_{k=1}^{\infty} q^{-k} X_k, \text{ and } S_n = \sum_{k=1}^{n} q^{-k} X_k$$

be functions defined on $\mathbb{D}_m^\infty$ and $\mathbb{D}_m^n$ respectively. Then for $x = (x_0, x_1, \ldots) \in \mathbb{D}_m^\infty$, we have $S(x) = \sum_{k=1}^{\infty} q^{-k} x_k$ and $S_n(x) = \sum_{k=1}^{n} q^{-k} x_k$.

Given $\rho \in (0, 1)$, let $\mu$ be a probability measure induced by IFS

$$\{F_j = \rho(x + b_j) : j = 1, \ldots, m\}$$

and the associated probability system $\{p_j : \ j = 1, \ldots, m\}$. Then this measure can be viewed as generated by a sequence of independent identically distributed (i.i.d) random variables as follows.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables each taking real values $b_1, \ldots, b_m$ with probability $p_1, \ldots, p_m$ respectively. Given $\rho \in (0, 1)$, we define a random variable

$$S = S_\rho = \sum_{i=1}^{\infty} \rho^i X_i.$$

Let $\mu_\rho$ be the probability measure induced by $S$, i.e.,

$$\mu_\rho(A) = \text{ Prob } \{\omega : S(\omega) \in A\}.$$

We call $\mu_\rho$ a *fractal measure* and $\{X_1, X_2, ...\}$ a *probabilistic system.* The range of $S$, or the support of $\mu_\rho$, is given by

$$\begin{aligned} F &= \{\sum_{i=1}^{\infty} \rho^i x_i : x_n \in \{b_1, b_2, \ldots, b_m\}\} \\ &= \{\rho(x_1 + \sum_{i=1}^{\infty} \rho^i x_i) : x_i \in \{b_1, b_2, \ldots, b_m\}\} \\ &= \bigcup_{j=1}^{m} \rho(x_j + F) \\ &= \bigcup_{j=1}^{m} F_j(F). \end{aligned}$$

Thus, $F$ is exactly the invariant compact set under the IFS $\{F_1, \ldots, F_m\}$. It can be verified that the measure $\mu_\rho$ also satisfies equation (1.1). In fact, we have $S = \rho(X_1 + S')$, where $S'$ has the same distribution as $S$ and it is independent of $X_1$. So we have

$$\begin{aligned} \mu_\rho(A) &= \text{ Prob } \{\omega : S(\omega) \in A\} \\ &= \text{ Prob } \{\rho(X_1 + S') \in A\} \\ &= \sum_{j=1}^{m} \text{Prob}(X_1 = b_j)\text{Prob}(\rho(b_j + S') \in A) \\ &= \sum_{j=1}^{m} p_j \text{Prob}(S' \in F_j^{-1}(A)) \\ &= \sum_{j=1}^{m} p_j \mu_\rho(F_j^{-1}(A)). \end{aligned}$$

By uniqueness we obtain $\mu = \mu_\rho$. So we will write $\mu$ for $\mu_\rho$ if no confusion will occur.

In this paper we consider the measure $\mu$ generated by a probabilistic system $\{X_J\}_{j=0}^{\infty}$ each taking real values $0, 1, \ldots, m$ with probability $p_j = p(X = j) = \frac{C_m^i}{2^m}$ respectively.

Let $\mu$ and $\mu_n$ be the probability measures induced by $S$ and $S_n$ respectively. By $\#A$ we denote the cardinal of the set $A$. We have

**Proposition 2.1 ([HN]).** *For any two consecutive points $s_n, t_n \in$ supp $\mu_n$, we have*

$$\#S_n^{-1}(s_n)/\#S_n^{-1}(t_n) \leq n + 1.$$

From Proposition 2.1 it follows that

**Corollary 2.1.** *If $s_n, t_n \in$ supp $\mu_n$ and $|s_n - t_n| \le kq^{-n}$, then*

$$\# S_n^{-1}(s_n)/\# S_n^{-1}(t_n) \le (n+1)^k.$$

By Proposition 2.1 and Corollary 2.1 we can see easily the following result.

**Proposition 2.2 ([HN]).** *Let $m \ge 2$, then*

$$\alpha(s) = \lim_{n\to\infty} |\frac{\log \mu_n(s_n)}{n \log q}|,$$

*provided that the limit exists. Otherwise, we can replace $\alpha(s)$ by $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$ and consider the upper and the lower limits.*

**Proposition 2.3.** *Let $s = \sum_{j=1}^{\infty} q^{-j}x_j$, $s' = \sum_{j=1}^{\infty} q^{-j}x'_j$ and $s - s' = \sum_{j=1}^{\infty} q^{-j}y_j$.*

*(i) If $s_n = s'_n$ then $x_n \equiv x'_n$ (mod $q$), and $(y_1, \ldots, y_n)$ can be decomposed as segments of the forms*

$$(0, \ldots, 0),\ \pm(-1, q),\ \pm(-1, q-1, q-1, \ldots, q-1, q) \tag{2.5}$$

*(ii) Conversely, if $(y_1, y_1, \ldots)$ can be decomposed as segments as in (2.5) or $\pm(-1, q-1, q-1, \ldots)$, then $s = s'$.*

*Proof.* (i) If $s_n = s'_n$, then $q^{n-1}(x_1 - x'_1) + \ldots + q(x_{n-1} - x'_{n-1}) + (x_n - x'_n) = 0$. Hence $x_n \equiv x'_n$ (mod $q$).

For the second statement in (i), we note that the last non-zero term of $y_1, \ldots, y_n$ must be congruent to 0 module $q$. Since $|y_j| \le 2q-2$, we can assume without loss of generality that $y_n = q$. We have

$$\sum_{j=1}^{n-2} q^{-j}y_j + q^{-(n-1)}(y_{n-1} + 1) = 0, \tag{2.6}$$

hence $y_{n-1} + 1 \equiv 0$ (mod $q$). Since $|y_j| \le 2q-2$, either $y_{n-1} = -1$ or $y_{n-1} = q-1$.

If $y_{n-1} = -1$, then $(y_{n-1}, y_n) = (-1, q)$ as asserted. Therefore, $\sum_{j=1}^{n-2} q^{-j}y_j = 0$ and we repeat the same argument to this sum.

If $y_{n-1} = q-1$, then we can write (2.6) as

$$\sum_{j=1}^{n-3} q^{-j}y_j + q^{-(n-2)}(y_{n-2} + 1) = 0.$$

This is the same form as (2.6) and the process can be repeated. Thus, we have the result as asserted.

(ii) The proof of this assertion is trivial. □

**Lemma 2.1.** *Let* $s = \sum_{j=1}^{\infty} q^{-j} x_j \in (0, \frac{m}{q-1})$. *Then for any fixed* $q \leq r \leq m$, *there exists* $k$ *and another representation* $s = \sum_{j=1}^{\infty} q^{-j} x'_j$ *such that*

$$0 \leq x'_j \leq r-1 \quad \text{for all} \quad j \geq k.$$

*Proof.* If there is an index $j_0$ such that $x_j \leq r-1$ for all $j \geq j_0$, then the lemma is true for $k = j_0$ and $x'_j = x_j$. Otherwise, we put

$$a = \max \{x_j : j > 1\},$$

then $a \geq r$. We can assume that $x_1 \neq a$ and there are infinitely many $x_j = a$. We will repeat the following procedure to reduce the values of $a$ until $a \leq r-1$. We consider two following cases.
**Case 1:** There exists $j_0$ such that $x_j = a$ for all $j > j_0$ and $x_{j_0} < a$. Let

$$x'_j = \begin{cases} x_j + 1 & \text{if } j = j_0 \\ x_j & \text{if } j < j_0 \\ x_j - (q-1) = a - q + 1 & \text{if } j > j_0. \end{cases}$$

Put

$$\sum_{j=1}^{\infty} q^{-j} y_j = \sum_{j=1}^{\infty} q^{-j} x_j - \sum_{j=1}^{\infty} q^{-j} x'_j, .$$

Then

$$y_j = \begin{cases} 0 & \text{if } j < j_0 \\ -1 & \text{if } j = j_0 \\ q-1 & \text{if } j > j_0. \end{cases}$$

Thus, $(y_1, \ldots, y_n)$ is decomposed as segments of the forms

$$(0, \ldots, 0), \ (-1, q-1, q-1, \ldots).$$

Therefore, by Proposition 2.3

$$s = \sum_{j=1}^{\infty} q^{-j} x_j = \sum_{j=1}^{\infty} q^{-j} x'_j.$$

Since $a - q + 1 \leq m - q + 1 \leq (2q-2) - q + 1 = q - 1 \leq r - 1$,

$$\max_{j > j_0} \{x'_j\} = a - q + 1 \leq r - 1.$$

Thus, the lemma is true for $k = j_0$.
**Case 2:** If $x_j < a$ for infinitely many $j$. Without loss of generality we can assume that $x_1 < a-1$. Let $n$ be the smallest integer such that $x_n = a$. Let $j_0$ be the largest integer less than $n$ such that $x_{j_0} < a-1$. We put

$$x'_j = \begin{cases} x_j + 1 & \text{if } \ j = j_0 \\ x_j - q = a - q & \text{if } \ j = n \\ x_j - (q-1) = a - q & \text{if } \ j_0 < j < n \\ x_j & \text{if } \ j > n \text{ or } j < j_0. \end{cases}$$

Then

$$y_j = \begin{cases} 0 & \text{if } \ j < j_0 \text{ or } \ j > n \\ -1 & \text{if } \ j = j_0 \\ q-1 & \text{if } \ j_0 < j < n \\ q & \text{if } \ j = n, \end{cases}$$

where

$$\sum_{j=1}^{\infty} q^{-j} x_j - \sum_{j=1}^{\infty} q^{-j} x'_j = \sum_{j=1}^{\infty} q^{-j} y_j.$$

Thus, $(y_1, \ldots, y_n)$ is decomposed as segments of the forms

$$(0, \ldots, 0), \ (-1, q-1, q-1, \ldots, q).$$

Consequently, by Proposition 2.3 we have

$$s = \sum_{j=1}^{\infty} q^{-j} x_j = \sum_{j=1}^{\infty} q^{-j} x'_j \quad \text{and} \quad \max_{n \geq j \geq 1} \{x'_j\} \leq a - 1.$$

We repeat this procedure to have all $x'_j \leq r-1$. The lemma is proved. □

**Lemma 2.2.** *Let* $s = \sum\limits_{j=1}^{\infty} q^{-j} x_j \ \in (0, \frac{m}{q-1})$*. Then for any fixed* $0 \leq r \leq m - q$*, there exists* $k$ *and another representation* $s = \sum\limits_{j=1}^{\infty} q^{-j} x'_j$ *such that* $x'_j \in \{r, r+1, \ldots, r+q-1\}$ *for all* $j \geq k$*.*

*Proof.* In the Lemma 2.1, if we replace $r$ by $r+q$, then for $0 \leq r \leq m-q$ there exists $k$ such that $s = \sum\limits_{j=1}^{\infty} q^{-j} x'_j$ and $0 \leq x'_j \leq r+q-1$ for all $j > k$. We can assume without loss of generality that $0 \leq x'_j \leq r+q-1$ for all $j$.

If $r = 0$, then the lemma is true. So we assume that $r \geq 1$. We will replace $0 \leq x'_j \leq r+q-1$ by $1 \leq x''_j \leq r+q-1$. Therefore, after $r$ steps we have the result as in the lemma.

In fact, assume that there exists some $x_n = 0$. We consider the following cases.

(i) If $x_j = 0$ or $x_j = 1$ for all $j$, then let $j_0 = \min\{j : x_j = 1\}$. We put

$$x'_j = \begin{cases} 0 & \text{if } \ j \leq j_0 \\ x_j + q - 1 & \text{if } \ j > j_0. \end{cases}$$

It means

$$x'_j = \begin{cases} 0 & \text{if } \ j \leq j_0 \\ q & \text{if } \ j > j_0 \ \text{ and } x_j = 1 \\ q-1 & \text{if } \ j > j_0 \ \text{ and } x_j = 0 \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } \ j < j_0 \\ 1 & \text{if } \ j = j_0 \\ -(q-1) & \text{if } \ j > j_0, \end{cases}$$

where

$$\sum_{j=1}^{\infty} q^{-j} y_j = \sum_{j=1}^{\infty} q^{-j} x_j - \sum_{j=1}^{\infty} q^{-j} x'_j.$$

By Proposition 2.3 we have

$s = \sum_{j=1}^{\infty} q^{-j} x'_j$ and $x'_j \in \{q-1, q\} \subset \{r, r+1, \dots, r+q-1\}$ for all $j > j_0$.

Thus, the lemma is true for $k = j_0 + 1$.

(ii) Otherwise, we consider a segment of the form $(x_i, x_{i+1}, \dots, x_n)$ with $x_i > 1$, $x_n = 0$ and $x_j = 0$ or $x_j = 1$ for all $i + 1 \leq i \leq n - 1$. Put

$$x'_j = \begin{cases} x_j - 1 > 0 & \text{if } \ j = i \\ x_j + q - 1 & \text{if } \ i < j < n \\ x_j + q = q & \text{if } \ j = n \\ x_j & \text{if } \ j > n \text{ or } \ j < i. \end{cases}$$

It implies that

$$y_j = \begin{cases} 0 & \text{if } \ j > n \text{ or } \ j < i \\ 1 & \text{if } \ j = i \\ -(q-1) & \text{if } \ n > j > i \\ -q & \text{if } \ j = n, \end{cases}$$

where

$$\sum_{j=1}^{\infty} q^{-j} y_j = \sum_{j=1}^{\infty} q^{-j} x_j - \sum_{j=1}^{\infty} q^{-j} x'_j.$$

By Proposition 2.3 we have $s = \sum\limits_{j=1}^{\infty} q^{-j} x'_j$ and $0 < x'_j \leq r + q - 1$ for $i \leq j \leq n$. We repeat this process until all the 0 after $x_n$ are replaced.

After having $1 \leq x'_j \leq q + r - 1$, we repeat the same process until obtain the representation $s = \sum\limits_{j=1}^{\infty} q^{-j} z_j$ and $r \leq z_j \leq r + q - 1$ for all $j > k = i$. The lemma is proved. □

# 3 The proof of the Main Theorem

**Theorem 1 .** *For $m \geq q \geq 2$ we have $\overline{\alpha} = \frac{m \log 2}{\log q}$ and the value is attained at $s = 0$ or $s = \frac{m}{q-1}$.*

*Proof.* Let $s = \sum\limits_{j=1}^{\infty} q^{-j} x_j \in [0, \frac{m}{q-1}]$. Then for every $n \in \mathbb{N}$

$$\mu_n(s_n) \geq \prod_{j=1}^{n} P(X = x_j) \geq \big(\frac{C_m^0}{2^m}\big)^n = 2^{-mn}.$$

By Proposition 2.2 we have

$$\overline{\alpha}(s) = \overline{\lim_{n\to\infty}} \Big|\frac{\log \mu_n(s_n)}{n \log q}\Big| \leq \overline{\lim_{n\to\infty}} \Big|\frac{\log 2^{-mn}}{n \log q}\Big| = \frac{m \log 2}{\log q}. \tag{3.1}$$

Observer that when $s_0 = 0$ or $s_0 = \frac{m}{q-1}$, they have the unique representation $s_0 = \sum\limits_{j=1}^{\infty} q^{-j} x_j$, where $(x_1, x_2, \dots) = (0, 0, \dots)$ or $(x_1, x_2, \dots) = (m, m, \dots)$ respectively. From Proposition 2.2 it follows that $\overline{\alpha}(s_0) = \frac{m \log 2}{\log q}$. Associating the latter with (3.1) we have the proof of the proposition. □

**Theorem 1 .** *Put $E_\alpha = \{s \in \text{ supp } \mu : \alpha(s) = \alpha\}$. Then $E_\alpha = \emptyset$ for all $\alpha \in (\hat{\alpha}, \overline{\alpha})$, where $\hat{\alpha} = \frac{m \log 2 - \log C_m^{[(m-q+1)/2]}}{\log q}$ and $\overline{\alpha} = \frac{m \log 2}{\log q}$. Therefore, $\overline{\alpha}$ is an isolated point of $E$.*

*Proof.* Let $r = [\frac{m-q+1}{2}]$. For any $s = \sum\limits_{j=1}^{\infty} q^{-j} x_j \in (0, \frac{m}{q-1})$, by Lemma 2.2 then there exist $k$ and a representation $s = \sum\limits_{j=1}^{\infty} q^{-j} x'_j$, where

$$x'_j \in \{[\frac{m-q+1}{2}], [\frac{m-q+1}{2}] + 1, \dots, [\frac{m-q+1}{2}] + q - 1\}$$

for all $j \geq k$. Therefore

$$\mu_n(s_n) \geq \prod_{j=1}^{n} P(X = x'_j) \geq \mathrm{C}\Big(\frac{C_m^{[\frac{m-q+1}{2}]}}{2^m}\Big)^n$$

(C only depends on $k$). It implies that

$$\begin{aligned}\alpha(s) =\leq \overline{\lim_{n\to\infty}}\Big|\frac{\log \mu_n(s_n)}{n\log q}\Big| \leq \overline{\lim_{n\to\infty}}\Big|\frac{\log C\big(\frac{C_m^{[\frac{m-q+1}{2}]}}{2^m}\big)^n}{n\log q}\Big|\\ = \frac{m\log 2 - \log C_m^{[\frac{m-q+1}{2}]}}{\log q} = \hat{\alpha}.\end{aligned}$$

The theorem is proved. □

The following simple property seems to be known, however, we were not able to find in the literature.

**Lemma 3.1.** *Let* $3 \leq q \leq m \leq 2q-2$. *Then* $C_m^i + C_m^{i+q} \leq C_m^{[\frac{m+1}{2}]}$ *for all* $0 \leq i \leq m-q$.

*Proof.* We only consider for the case $m$ is even. The other case is proved similarly. Put $m = 2n$. Then $2n \geq q \geq n+1 \geq 2$, since $m \leq 2q-2$. Observe that

$$C_{2n}^0 < C_{2n}^1 < \ldots < C_{2n}^n > C_{2n}^{n+1} > \ldots > C_{2n}^{2n}.$$

Since $n < n+i+1 < i+q$,

$$C_{2n}^{i+q} \leq C_{2n}^{n+i+1}. \tag{3.2}$$

We will show that

$$C_{2n}^i + C_{2n}^{n+i+1} \leq C_{2n}^n \tag{3.3}$$

for all $n \geq 1,\ n-1 \geq i \geq 0$.

In fact,

$$\begin{aligned}(3.3) \quad &\Leftrightarrow \quad \frac{1}{i!(2n-i)!} + \frac{1}{(i+n+1)!(n-i-1)!} \leq \frac{1}{(n!)^2}\\ &\Leftrightarrow \quad \frac{(i+1)(i+2)\ldots n}{(n+1)\ldots(2n-i)} + \frac{(n-i)(n-i+1)\ldots n}{(n+1)\ldots(n+i+1)} \leq 1.\end{aligned}$$

Since $n-1 \geq i \geq 0$, the left of the last inequality in (3.3) does not exceed $\frac{i+1}{2n-i} + \frac{n-i}{n+i+1}$. Therefore, to show the last inequality we need to check that

$$\frac{i+1}{2n-i} + \frac{n-i}{n+i+1} \leq 1 \text{ or } 3i^2 + 3i + 1 \leq 3in + n.$$

In fact, since $n \geq i+1$,

$$3in + n = (3i+1)n \geq (3i+1)(i+1) \geq 3i^2 + 3i + 1.$$

From (3.2), (3.3) the assertion follows. □

Using this lemma, we have the following result.

**Theorem 1 .** *Let* $3 \leq q \leq m \leq 2q-2$. *Then the greatest lower local dimensions is* $\underline{\alpha} = \frac{m\log 2}{logq} - \frac{\log C_m^{[\frac{m+1}{2}]}}{\log q}$. *Moreover, the infimum is attained at* $s = \sum_{j=1}^{\infty} q^{-j}\left[\frac{m+1}{2}\right] = \frac{[\frac{m+1}{2}]}{q-1}$.

*Proof.* Let $t = \sum_{j=1}^{\infty} q^{-j}[\frac{m+1}{2}]$ and $t_n = \sum_{j=1}^{n} q^{-j}[\frac{m+1}{2}]$. We claim that $t_n$ has the unique representation $t_n = \sum_{j=1}^{n} q^{-j}[\frac{m+1}{2}]$ for all $n$. Indeed, if $t_n = \sum_{j=1}^{n} q^{-j}y_j$, then Proposition 2.3, $y_n - [\frac{m+1}{2}] \equiv 0 \pmod q$. Hence $y_n = [\frac{m+1}{2}]$ since $q \leq m \leq 2q-2$. Thus, $t_{n-1} = \sum_{j=1}^{n-1} q^{-j}y_{n-1}$. By repeating this argument we have the claim. Therefore,

$$\mu_n(t_n) = \prod_{j=1}^{n} P(X = [\frac{m+1}{2}]) = \Big(\frac{C_m^{[\frac{m+1}{2}]}}{2^m}\Big)^n.$$

It implies

$$\underline{\alpha}(t) = \underline{\lim}_{n\to\infty}\Big|\frac{\log\mu_n(t_n)}{n\log q}\Big| = \frac{m\log 2}{logq} - \frac{\log C_m^{[\frac{m+1}{2}]}}{\log q} = \underline{\alpha}. \tag{3.5}$$

Now we will show that $\underline{\alpha}(s) \geq \underline{\alpha}$ for any $s \in \operatorname{supp}\mu$. Indeed, assume that $s = \sum_{j=1}^{\infty} q^{-j}x_j$ and $s_n = \sum_{j=1}^{n} q^{-j}x_j$. We will prove by induction that $\mu_n(s_n) \leq \mu_n(t_n)$ for all $n$. It is easy to see that the assertion is true for $n = 1$. Assume that it is true up to $n-1$. In the case $n$, we consider three following cases.

**Case 1:** If $x_n = [\frac{m+1}{2}]$, then

$$\begin{aligned}\mu_n(s_n) &= \mu_{n-1}(s_{n-1})P(X = [\frac{m+1}{2}])\\ &\leq \mu_{n-1}(t_{n-1})\frac{C_m^{[\frac{m+1}{2}]}}{2^m} = \mu_n(t_n).\end{aligned}$$

**Case 2:** $m-q \leq x_n \leq q$. From Proposition 2.3 and $q \leq m \leq 2q-2$, it follows that if $s_n$ has an another representation $s_n = \sum_{j=1}^{n} q^{-j}y_j$, then $y_n = x_n$. Hence

by the argument as above we have $\mu_n(s_n) \le \mu_n(t_n)$.
**Case 3:** $x_n + q \le m$ or $x_n - q \ge 0$. Since $q \le m \le 2q-2$, $s_n$ has an another representation $s_n = \sum_{j=1}^{n} q^{-j} y_j$, where $y_n = x_n + q$ or $y_n = x_n - q$. Without loss of generality we assume that $y_n = x_n + q$. Then $s_n$ has two representations

$$s_n = s_{n-1} + q^{-n}x_n = s'_{n-1} + q^{-n}(x_n + q).$$

By Lemma 3.1 and the induction hypothesis we have

$$\begin{aligned} \mu_n(s_n) &= \mu_{n-1}(s_{n-1})P(X = x_n) + \mu_{n-1}(s'_{n-1})P(X = x_n + q) \\ &\le \mu_{n-1}(t_{n-1})P(X = x_n) + \mu_{n-1}(t_{n-1})P(X = x_n + q) \\ &= \mu_{n-1}(t_{n-1})[P(X = x_n) + P(X = x_n + q)] \\ &\le \mu_{n-1}(t_{n-1})\frac{C_m^{[\frac{m+1}{2}]}}{2^m} = \mu_n(t_n). \end{aligned}$$

By Proposition 2.2, $\underline{\alpha}(s) \ge \alpha$ for any $s \in \text{supp } \mu$. From the latter and (3.5) the assertion follows. □

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