

A P-ADIC GENERALIZED BOREL'S LEMMA

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Abstract

In this paper, by using p-adic Nevanlinna theory, we prove a generalized Borel's lemma in the p-adic case.

1 Introduction

Let us start by recalling Borel's lemma in the complex case:

Theorem 1.1. *Let $f_1, \dots, f_n, n \geq 3$ be non-zero holomorphic functions on \mathbb{C} such that*

$$f_1 + \dots + f_n = 0.$$

Then the function $\{f_1, \dots, f_{n-1}\}$ are linearly dependent.

It is well - known that Borel's lemma plays an important role in the study of hyperbolic spaces. For different purposes some generalizations of the lemma are given. We mention here a result of Y.T. Siu and S.K. Yeung.

Theorem 1.2 ([13]). *Let $g_j(x_0, \dots, x_n)$ be a homogeneous polynomial of degree δ_j for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ so that its image lies in*

$$\sum_{j=0}^n x_j^{k-\delta_j} g_j(x_0, \dots, x_n) = 0,$$

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and $k > (n + 1)(n - 1) + \sum_{j=0}^n \delta_j$. Then there is a nontrivial linear relation among $x_1^{k-\delta_1} g_1(x_0, \dots, x_n), \dots, x_n^{k-\delta_n} g_n(x_0, \dots, x_n)$ on the image of f .

In [12], by using p-adic Nevanlinna-Cartan's theorem, Nguyen Thanh Quang and Phan Duc Tuan proved a p-adic version of Siu-Yeung's lemma as the follows.

Theorem 1.3. *Let $g_j(x_0, \dots, x_n)$ be a homogeneous polynomial of degree δ_j for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C}_p \rightarrow \mathbb{P}^n(\mathbb{C}_p)$ so that its image lies in*

$$\sum_{j=0}^n x_j^{k-\delta_j} g_j(x_0, \dots, x_n) = 0,$$

and

$$k \geq (n + 1)(n - 1) + \sum_{j=0}^n \delta_j.$$

Then the following functions are linearly dependent on \mathbb{C}_p if they are have no common zeros:

$$f_1^{k-\delta_1} g_1(f_0, \dots, f_n), \dots, f_n^{k-\delta_n} g_n(f_0, \dots, f_n).$$

In this paper, by using p-adic Nevanlinna theory, we prove an analog of a generalized *abc*-conjecture for p-adic entire functions and we then apply the result to prove the hypothesis that the functions

$$f_1^{k-\delta_1} g_1(f_0, \dots, f_n), \dots, f_n^{k-\delta_n} g_n(f_0, \dots, f_n)$$

have no common zeros in Theorem 1.3 is not necessary.

2 An analog of a generalized abc-conjecture for p-adic entire functions

Let p be a prime number, \mathbb{Q}_p the field of p-adic number, and let \mathbb{C}_p be the p-adic completion of the algebraic closure of \mathbb{Q}_p . The absolute value in \mathbb{C}_p is normalized so that $|p|_p = p^{-1}$.

Let $a \in \mathbb{C}_p$ and f is a p-adic meromorphic function, we write f in the form:

$$f = (x - a)^l \frac{g}{h}$$

Where g, h are entire functions and $g(a)h(a) \neq 0$, then l is called the *order of f at a* and is denoted by μ_f^a , we also denote $\mu_{f,k}^a = \min(k, \mu_f^a)$. We have the following easily proved properties of μ_f^a .

Lemma 2.1. *Let f, g be two meromorphic functions and $a \in \mathbb{C}_p$, we have*

- a) $\mu_{f+g}^a \geq \min(\mu_f^a, \mu_g^a)$,
- b) $\mu_{fg}^a = \mu_f^a + \mu_g^a$,
- c) $\mu_{\frac{f}{g}}^a = \mu_f^a - \mu_g^a$,
- d) $\mu_{f^{(k)}}^a \geq \mu_f^a - k$.

Let f be a nonconstant p -adic analytic function on \mathbb{C}_p ,

$$f = \sum_{n=0}^{\infty} a_n z^n \quad (a_n \in \mathbb{C}_p)$$

is well-defined whenever

$$|a_n z^n|_p \rightarrow 0.$$

Define the maximum term

$$\mu(r, f) = \max_{n \geq 0} |a_n| r^n.$$

Let f is a meromorphic function, we can uniquely extend μ to meromorphic function $f = \frac{g}{h}$ by defining

$$\mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)} \quad (0 \leq r < \infty).$$

Define the *compensation function* by

$$m(r, f) = \max\{0, \log \mu(r, f)\}.$$

Define the *counting function* by

$$N\left(r, \frac{1}{f}\right) = \mu_f^0 \log r + \sum_{0 < a \leq r, f(a)=0} \mu_f^a \log \left|\frac{r}{a}\right|,$$

and

$$\overline{N}\left(r, \frac{1}{f}\right) = \mu_{f,1}^0 \log r + \sum_{0 < a \leq r, f(a)=0} \mu_{f,1}^a \log \left|\frac{r}{a}\right|.$$

As usual, we define the *characteristic function* by

$$T(r, f) = m(r, f) + N(r, f),$$

where

$$N(r, f) = N\left(r, \frac{1}{h}\right).$$

We have *Jensen Formular* (see [3])

$$N(r, \frac{1}{f}) - N(r, f) = \log \mu(r, f) + 0(1).$$

Now we give an analog of a generalized *abc*-conjecture for p-adic entire functions.

Theorem 2.1. *Let f_0, \dots, f_{n+1} be $(n+2)$ entire functions have no common zeros and g_0, \dots, g_{n+1} be $(n+2)$ entire functions such that $f_0g_0, \dots, f_n g_n$ be linearly independent over \mathbb{C}_p , and*

$$f_0g_0 + \dots + f_n g_n = f_{n+1}g_{n+1}$$

Then

$$\max_{0 \leq j \leq n+1} T(r, f_j g_j) \leq n \sum_{j=0}^{n+1} \overline{N}(r, \frac{1}{f_j}) + \sum_{j=0}^{n+1} N(r, \frac{1}{g_j}) - \frac{n(n+1)}{2} \log r + 0(1).$$

Proof. Since $f_0g_0, \dots, f_n g_n$ are linearly independent, then the Wronskian W of $f_0g_0, \dots, f_n g_n$ does not vanish. We set

$$P = \frac{W(f_0g_0, \dots, f_n g_n)}{f_0g_0 \dots f_n g_n},$$

$$Q = \frac{f_0g_0 \dots f_{n+1}g_{n+1}}{W(f_0g_0, \dots, f_n g_n)}.$$

Hence we have

$$f_{n+1}g_{n+1} = PQ.$$

By Jensen formular, we have

$$T(r, f_{n+1}g_{n+1}) = N(r, \frac{1}{f_{n+1}g_{n+1}}) + 0(1) \leq N(r, \frac{1}{Q}) + N(r, \frac{1}{P}) + 0(1). \quad (1)$$

We have

$$N(r, \frac{1}{Q}) = N\left(r, \frac{1}{\frac{f_0 \dots f_{n+1}}{W(f_0g_0, \dots, f_n g_n)}}\right) + \sum_{j=0}^{n+1} N(r, \frac{1}{g_j}) + 0(1). \quad (2)$$

Suppose that α is a zero of $f_0 f_1 \dots f_{n+1}$, by the hypothesis there exists $\nu, 0 \leq \nu \leq n+1$ such that $f_\nu \neq 0$. By the hypothesis $f_0g_0 + \dots + f_n g_n = f_{n+1}g_{n+1}$ we have

$$\begin{aligned} \mu^\alpha \frac{f_0 \dots f_{n+1}}{W(f_0g_0, \dots, f_n g_n)} &= \mu^\alpha \frac{f_0 \dots f_{\nu-1} f_{\nu+1} \dots f_{n+1}}{W(f_0g_0, \dots, f_{\nu-1}g_{\nu-1}, f_{\nu+1}g_{\nu+1}, \dots, f_n g_n)} \\ &= \sum_{j=0}^{n+1} \mu_{f_j}^\alpha - \mu_{f_0g_0, \dots, f_{\nu-1}g_{\nu-1}, f_{\nu+1}g_{\nu+1}, \dots, f_n g_n}^\alpha \end{aligned}$$

$W(f_0g_0, \dots, f_{\nu-1}g_{\nu-1}, f_{\nu+1}g_{\nu+1}, \dots, f_{n+1}g_{n+1})$ is the sum of follow terms

$$\delta f_{\alpha_0}g_{\alpha_0}(f_{\alpha_1}g_{\alpha_1})' \cdots (f_{\alpha_n}g_{\alpha_n})^{(n)},$$

Where $\alpha_i \in \{0, \dots, n+1\} \setminus \{\nu\}$, $\delta = \pm 1$. By using Lemma 2.1 we have

$$\begin{aligned} & \mu_{f_{\alpha_0}g_{\alpha_0}(f_{\alpha_1}g_{\alpha_1})' \cdots (f_{\alpha_n}g_{\alpha_n})^{(n)}}^{\alpha} \\ \geq & \sum_{0 \leq j \leq n, f_{\alpha_j}(\alpha)=0} \mu_{f_{\alpha_j}g_{\alpha_j}}^{\alpha} - n \left(\sum_{0 \leq j \leq n, f_{\alpha_j}(a)=0} 1 \right) \\ = & \sum_{0 \leq j \leq n, f_{\alpha_j}(\alpha)=0} \mu_{f_{\alpha_j}}^{\alpha} + \sum_{0 \leq j \leq n, f_{\alpha_j}(\alpha)=0} \mu_{g_{\alpha_j}}^{\alpha} - n \left(\sum_{0 \leq j \leq n, f_{\alpha_j}(a)=0} 1 \right) \\ \geq & \sum_{0 \leq j \leq n, f_{\alpha_j}(\alpha)=0} \mu_{f_{\alpha_j}}^{\alpha} - n \left(\sum_{0 \leq j \leq n+1, f_{\alpha_j}(a)=0} 1 \right) \\ = & \sum_{j=0}^{n+1} \mu_{f_j}^{\alpha} - n \left(\sum_{0 \leq j \leq n+1, f_j(a)=0} 1 \right). \end{aligned}$$

By Lemma 2.1 we have

$$\mu_{W(f_0g_0, \dots, f_{\nu-1}g_{\nu-1}, f_{\nu+1}g_{\nu+1}, \dots, f_{n+1}g_{n+1})}^{\alpha} \geq \sum_{j=0}^{n+1} \mu_{f_{\alpha_j}}^{\alpha} - n \left(\sum_{0 \leq j \leq n+1, f_j(a)=0} 1 \right).$$

Hence

$$\mu_{\frac{f_0 \cdots f_{n+1}}{W(f_0g_0, \dots, f_{n+1}g_{n+1})}}^{\alpha} \leq n \left(\sum_{0 \leq j \leq n+1, f_j(a)=0} 1 \right).$$

By the definition of counting function, we have:

$$N \left(r, \frac{1}{\frac{f_0 \cdots f_{n+1}}{W(f_0g_0, \dots, f_{n+1}g_{n+1})}} \right) \leq n \sum_{j=0}^{n+1} \bar{N} \left(r, \frac{1}{f_j} \right) \quad (3)$$

By Jensen formular we have

$$N \left(r, \frac{1}{P} \right) \leq \log \mu(r, P) + 0(1). \quad (4)$$

The value of P is clearly

$$\sum \pm \frac{f_{\alpha_0}g_{\alpha_0}}{f_{\alpha_0}g_{\alpha_0}} \cdot \frac{(f_{\alpha_1}g_{\alpha_1})'}{f_{\alpha_1}g_{\alpha_1}} \cdots \frac{(f_{\alpha_n}g_{\alpha_n})^{(n)}}{f_{\alpha_n}g_{\alpha_n}}$$

summed for the $(n+1)!$ permutations $(\alpha_0, \dots, \alpha_n)$ of $(0, \dots, n)$, the positive sign being taken for a positive permutation, the negative sign for a negative

permutation. We have

$$\begin{aligned}\mu(r, P) &\leq \max \mu \left(r, \frac{(f_{\alpha_1} g_{\alpha_1})'}{f_{\alpha_1} g_{\alpha_1}} \cdots \frac{(f_{\alpha_n} g_{\alpha_n})^{(n)}}{f_{\alpha_n} g_{\alpha_n}} \right) \\ &= \max \mu \left(r, \frac{(f_{\alpha_1} g_{\alpha_1})'}{f_{\alpha_1} g_{\alpha_1}} \right) \cdots \mu \left(r, \frac{(f_{\alpha_n} g_{\alpha_n})^{(n)}}{f_{\alpha_n} g_{\alpha_n}} \right)\end{aligned}$$

By using the lemma of logarithmic derivative (see [2]) which states

$$\mu \left(r, \frac{f^{(i)}}{f} \right) \leq \frac{1}{r^i}$$

for a nonconstant meromorphic function f in C_p , we obtain

$$\mu(r, P) \leq \frac{1}{r} \cdot \frac{1}{r^2} \cdots \frac{1}{r^n} = r^{-\frac{n(n+1)}{2}}. \quad (5)$$

From (4) and (5) we have

$$N(r, \frac{1}{P}) \leq -\frac{n(n+1)}{2} \log r + 0(1). \quad (6)$$

From (1), (2), (3) and (6) we have

$$T(r, f_{n+1} g_{n+1}) \leq n \sum_{j=0}^{n+1} \bar{N}(r, \frac{1}{f_j}) + \sum_{j=0}^{n+1} N(r, \frac{1}{g_j}) - \frac{n(n+1)}{2} \log r + 0(1).$$

By a similar argument applying to the functions $f_0 g_0, \dots, f_n g_n$, we have

$$\max_{0 \leq j \leq n+1} T(r, f_j g_j) \leq n \sum_{j=0}^{n+1} \bar{N}(r, \frac{1}{f_j}) + \sum_{j=0}^{n+1} N(r, \frac{1}{g_j}) - \frac{n(n+1)}{2} \log r + 0(1).$$

Theorem 2.1 is proved. \square

3 A P-adic generalized Borel's lemma

The following theorem is more general than Theorem 1.3.

Theorem 3.1. *Let $g_j(x_0, \dots, x_n)$ be a homogeneous polynomial of degree δ_j for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C}_p \rightarrow \mathbb{P}^n(\mathbb{C}_p)$ so that its image lies in*

$$\sum_{j=0}^n x_j^{k-\delta_j} g_j(x_0, \dots, x_n) = 0,$$

and $k \geq (n+1)(n-1) + \sum_{j=0}^n \delta_j$. Then there is a nontrivial linear relation among $x_1^{k-\delta_1} g_1(x_0, \dots, x_n), \dots, x_n^{k-\delta_n} g_n(x_0, \dots, x_n)$ on the image of f .

Proof. By the hypothesis we have

$$\sum_{j=0}^n f_j^{k-\delta_j} g_j(f_0, \dots, f_n) = 0,$$

Assume on the contrary that the functions

$$f_1^{k-\delta_1} g_1(f_0, \dots, f_n), \dots, f_n^{k-\delta_n} g_n(f_0, \dots, f_n)$$

are linearly independent. By using Theorem 2.1 we have

$$\begin{aligned} & \max\left(T(r, f_0^{k-\delta_0} g_1(f_0, \dots, f_n)), \dots, T(r, f_n^{k-\delta_n} g_n(f_0, \dots, f_n))\right) \\ & \leq (n-1) \sum_{j=0}^{n+1} \overline{N}\left(r, \frac{1}{f_j^{k-\delta_j}}\right) + \sum_{j=0}^{n+1} N\left(r, \frac{1}{g_j(f_0, \dots, f_n)}\right) - \frac{n(n-1)}{2} \log r + 0(1) \\ & = (n-1) \sum_{j=0}^{n+1} \overline{N}\left(r, \frac{1}{f_j}\right) + \sum_{j=0}^{n+1} N\left(r, \frac{1}{g_j(f_0, \dots, f_n)}\right) - \frac{n(n-1)}{2} \log r + 0(1) \\ & \leq (n-1) \sum_{j=0}^{n+1} N\left(r, \frac{1}{f_j}\right) + \sum_{j=0}^{n+1} N\left(r, \frac{1}{g_j(f_0, \dots, f_n)}\right) - \frac{n(n-1)}{2} \log r + 0(1) \end{aligned}$$

We set for simplicity

$$\max_{0 \leq j \leq n} T(r, f_j) = T(r, f_{i_0}).$$

Then we have

$$\begin{aligned} & T(r, f_{i_0}^{k-\delta_{i_0}} g_{i_0}(f_0, \dots, f_n)) \\ & \leq (n-1) \sum_{j=0}^{n+1} N\left(r, \frac{1}{f_j}\right) + \sum_{j=0}^{n+1} N\left(r, \frac{1}{g_j(f_0, \dots, f_n)}\right) - \frac{n(n-1)}{2} \log r + 0(1). \end{aligned}$$

Hence

$$\begin{aligned} & N\left(r, \frac{1}{f_{i_0}^{k-\delta_{i_0}}}\right) + N\left(r, \frac{1}{g_{i_0}(f_0, \dots, f_n)}\right) \\ & \leq (n-1) \sum_{j=0}^{n+1} N\left(r, \frac{1}{f_j}\right) + \sum_{j=0}^{n+1} N\left(r, \frac{1}{g_j(f_0, \dots, f_n)}\right) - \frac{n(n-1)}{2} \log r + 0(1). \end{aligned}$$

Thus

$$\begin{aligned} & (k - \delta_{i_0}) N\left(r, \frac{1}{f_{i_0}}\right) \\ & \leq (n-1)(n+1) N\left(r, \frac{1}{f_{i_0}}\right) + \sum_{0 \leq j \leq n, j \neq i_0}^{n+1} N\left(r, \frac{1}{g_j(f_0, \dots, f_n)}\right) - \frac{n(n-1)}{2} \log r + 0(1) \\ & \leq (n-1)(n+1) N\left(r, \frac{1}{f_{i_0}}\right) + \sum_{0 \leq j \leq n, j \neq i_0}^{n+1} \delta_j N\left(r, \frac{1}{f_{i_0}}\right) - \frac{n(n-1)}{2} \log r + 0(1). \end{aligned}$$

So

$$\left(k - \sum_{j=0}^n \delta_j - (n-1)(n+1)\right)N\left(r, \frac{1}{f_{i_0}}\right) \leq -\frac{(n-1)n}{2} \log r + o(1).$$

By the hypothesis $k \geq \sum_{j=0}^n \delta_j - (n-1)(n+1)$ we have a contradiction when $r \rightarrow +\infty$. \square

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