

ON TRUNCATED DEFECT RELATION FOR NON-ARCHIMEDEAN ANALYTIC CURVES INTERSECTING HYPERSURFACES

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Abstract

Let \mathbb{K} be an algebraically closed field of arbitrary characteristic, completed with respect to a non-Archimedean absolute value “ $|\cdot|$ ”. In this paper, we will estimate truncated defect relation for non-Archimedean analytic curves intersecting hypersurfaces in general position.

1 Introduction

We first introduce some standard notations in Nevanlinna theory. Let f be an entire function on \mathbb{K} , defined by a convergent series

$$f(z) = \sum_{n=m}^{\infty} a_n z^n, \quad (a_m \neq 0; m \geq 0).$$

For each real number $r \geq 0$, we define

$$\begin{aligned} |f|_r &= \sup_n |a_n| r^n = \sup\{|f(z)| : z \in \mathbb{K} \text{ with } |z| \leq r\} \\ &= \sup\{|f(z)| : z \in \mathbb{K} \text{ with } |z| = r\}. \end{aligned}$$

Let $f : \mathbb{K} \rightarrow \mathbb{P}^n(\mathbb{K})$ be a analytic map, $f = (f_0 : \dots : f_n)$ be a reduced representative of f , where f_0, \dots, f_n are entire functions on \mathbb{K} without common zeros,

Key words: Nevanlinna-Cartan theory, Truncated defect relation.
2000 AMS Mathematics Subject Classification: 30D30, 30D35

at least one of which is non-constant. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \log \|f\|_r,$$

where $\|f\|_r = \max\{|f_0|_r, \dots, |f_n|_r\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f .

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{K})$ of degree d . Let G be the homogeneous polynomial in $n+1$ variables with coefficients in \mathbb{K} of degree d defining D . The proximity function of f is defined by

$$m_f(r, D) = m_f(r, G) = \log \frac{\|f\|_r^d}{|G \circ f|_r}.$$

Note that up to a constant term, $m_f(r, D)$ is independent of the choice of defining form G . Let $n_f(r, G)$ be the number of zeros of $G \circ f$ in the disk $|z| < r$, counting multiplicity, and $n_f^\Delta(r, G)$ be the number of zeros of $G \circ f$ in the disk $|z| < r$, truncated multiplicity by a positive integer Δ . The counting function and truncated function are defined by

$$N_f(r, D) = N_f(r, G) = \int_0^r \frac{n_f(t, G) - n_f(0, G)}{t} dt + n_f(0, G) \log r;$$

$$N_f^\Delta(r, D) = N_f^\Delta(r, G) = \int_0^r \frac{n_f^\Delta(t, G) - n_f^\Delta(0, G)}{t} dt + n_f^\Delta(0, G) \log r.$$

It is clear that for any positive integer Δ , $N_f^\Delta(r, D) \leq N_f(r, D)$.

Let X be an n -dimensional (not necessarily smooth) projective subvariety of $\mathbb{P}^N(\mathbb{K})$. A collection of $q \geq n+1$ hypersurfaces D_1, \dots, D_q in $\mathbb{P}^N(\mathbb{K})$ is said to be *in general position with X* if for any subset $\{i_0, \dots, i_n\}$ of $\{1, \dots, q\}$ of cardinality $n+1$,

$$\{x \in X : G_{i_j}(x) = 0, j = 0, \dots, n\} = \emptyset,$$

where $G_j, 1 \leq j \leq q$, be the homogeneous polynomials in $\mathbb{K}[x_0, \dots, x_n]$ defining D_j .

For a hypersurface D , which is defined by homogeneous polynomial G , we define the defect

$$\delta_f(D) = \delta_f(G) = 1 - \limsup_{r \rightarrow +\infty} \frac{N_f(r, G)}{(\deg G)T_f(r)},$$

and the truncated defect

$$\delta_f^\Delta(D) = \delta_f^\Delta(G) = 1 - \limsup_{r \rightarrow +\infty} \frac{N_f^\Delta(r, G)}{(\deg G)T_f(r)},$$

where Δ be a positive integer. It is easy to see that

$$0 \leq \delta_f(D) \leq \delta_f^\Delta(D) \leq 1$$

for any positive integer Δ and hypersurface D .

In [1] (also [11] for the special case when $X = \mathbb{P}^N(\mathbb{K})$), the author showed that

Theorem A. *Let $X \subset \mathbb{P}^N(\mathbb{K})$ be a projective sub-variety of dimension $n \geq 1$ over \mathbb{K} . Let D_1, \dots, D_q be hypersurfaces of degree d_1, \dots, d_q resp. in $\mathbb{P}^N(\mathbb{K})$ in general position with X . Let $f : \mathbb{K} \rightarrow X$ be a non-constant analytic map whose image is not completely contained in any of the hypersurfaces D_1, \dots, D_q . Then*

$$\sum_{j=1}^q \delta_f(D_j) \leq n = \dim X. \quad (1.1)$$

Let D_1, \dots, D_q be hypersurfaces of degree d_1, \dots, d_q resp. in $\mathbb{P}^n(\mathbb{K})$, for any $\varepsilon > 0$, we now define *bound of truncated level*, which is denoted by $\mathcal{B}_\varepsilon(D_1, \dots, D_q)$, of the hypersurfaces D_1, \dots, D_q associating ε as follows: Let d be the least common multiple of d_j 's and let N be the smallest natural number such that $N \geq nd$, divisible by d and $\frac{(N+1)\dots(N+n)}{(N-d)\dots(N-nd)} \leq 1 + \frac{\varepsilon}{n+1}$, then

$$\mathcal{B}_\varepsilon(D_1, \dots, D_q) = \frac{(N+n)!}{N!n!}.$$

In this paper, we will estimate truncated defect relation for non-Archimedean analytic curves to projective space over \mathbb{K} . Our result is stated as follows.

Main Theorem. *Let $f : \mathbb{K} \rightarrow \mathbb{P}^n(\mathbb{K})$ be an algebraically non-degenerate analytic map, and let $D_j, 1 \leq j \leq q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{K})$ of degree d_j in general position. Then for every $\varepsilon > 0$, there exists a positive integer $\Delta = \mathcal{B}_\varepsilon(D_1, \dots, D_q)$ such that*

$$\sum_{j=1}^q \delta_f^\Delta(D_j) \leq n + 1 + \varepsilon. \quad (1.2)$$

Theorem A as above gave a better bound than our result, but it seems impossible to get a truncated defect relation from the approach which is given in the paper [1]. The proof of our Main Theorem is based on the method which was first introduced by Corvaje and Zannier for number fields [6], then by Ru [10] and An-Phuong [3] for the complex field. By our method explicit bound truncation level of Δ in our result is the smallest as possible.

Unfortunately, Δ in Main Theorem depends on ε . It would be interesting if one can find a Δ term independent on $\varepsilon > 0$. It is very important, because we can cut term ε in the right side in (1.2) in that case.

2 Some Preparations

In this section, we show and recall some lemmas and theorem, which are necessary for proof of the our main result. (For detail, readers can find in [1], [5], [6], [7], [10], [12] and also [2]).

Throughout of this paper, we shall use the *lexicographic ordering* on the m -tuples $(i_1, \dots, i_m) \in \mathbb{N}^m$ of natural numbers. Namely, $(j_1, \dots, j_m) > (i_1, \dots, i_m)$ if and only if for some $b \in \{1, \dots, m\}$ we have $j_l = i_l$ for $l < b$ and $j_b > i_b$. With the n -tuples $(\mathbf{i}) = (i_1, \dots, i_n)$ of non-negative integers, we denote $\sigma(\mathbf{i}) = \sum_{j=1}^n i_j$.

Let $g_1, \dots, g_n \in \mathbb{K}[x_0, \dots, x_n]$ be homogeneous polynomials of degree d , such that they define a subvariety of $\mathbb{P}^n(\mathbb{K})$ of dimension 0. For a fixed positive integer N , denote by V_N the space of homogeneous polynomials of degree N in $\mathbb{K}[x_0, \dots, x_n]$. We define a filtration of V_N as follows. Arrange, by the lexicographic order, the n -tuples $(\mathbf{i}) = (i_1, \dots, i_n)$ of non-negative integers such that $\sigma(\mathbf{i}) \leq N/d$. Define the spaces $W_{(\mathbf{i})} = W_{N,(\mathbf{i})}$ by

$$W_{(\mathbf{i})} = \sum_{(\mathbf{e}) \geq (\mathbf{i})} g_1^{e_1} \dots g_n^{e_n} V_{N-d\sigma(\mathbf{e})}.$$

Clearly, $W_{(0, \dots, 0)} = V_N$ and $W_{(\mathbf{i})} \supset W_{(\mathbf{i}')} if $(\mathbf{i}') > (\mathbf{i})$, so $W_{(\mathbf{i})}$ is a filtration of V_N .$

For any pair (\mathbf{i}') follows (\mathbf{i}) in ordering, as in Corvaja-Zannier's original proof we may choose a basis of quotient $\frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}}$ from the set containing all equivalence classes of the form: $\gamma_1^{i_1} \dots \gamma_n^{i_n} \eta$ modulo $W_{(\mathbf{i}')}$ with η being a monomial in x_0, \dots, x_n with total degree $N - d\sigma(\mathbf{i})$. Now we evaluate the dimensions, denoted by $\delta_{(\mathbf{i})}$, of the quotients of successive spaces in the filtration.

Lemma 1. *If $\sigma(\mathbf{i}) \leq N/d - n$ then*

$$\delta_{(\mathbf{i})} := \dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} = d^n.$$

The proof of the above lemma was give in [3].

Next, we will recall the following lemma which is well-know in p -adic Nevanlinna theory. The proof can be found, for example, in [8].

Lemma 2. *Let f be a nonconstant meromorphic function in \mathbb{K} , then*

$$m\left(r, \frac{f'}{f}\right) = O(1).$$

Let f_1, \dots, f_n be meromorphic functions over \mathbb{K} . Their Wronskian is

$$W(f) = W(f_1, \dots, f_n) := \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}.$$

We denote

$$L = L(f) = L(f_1, \dots, f_n) := \begin{vmatrix} 1 & \dots & 1 \\ f_1'/f_1 & \dots & f_n'/f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}/f_1 & \dots & f_n^{(n-1)}/f_n \end{vmatrix}.$$

Lemma 3. *Let f_1, \dots, f_n be meromorphic functions over \mathbb{K} . Then for any $r > 0$ we have*

$$m_L(r) = \log^+ |L|_r = O(1).$$

Proof. Let g be a meromorphic function over \mathbb{K} , for any integer $k \geq 1$, we can write a logarithmic derivative of high order as product

$$\frac{g^{(k)}}{g} = \frac{g^{(k)}}{g^{(k-1)}} \cdots \frac{g'}{g},$$

since Lemma 2, we have

$$m\left(r, \frac{g^{(k)}}{g}\right) = O(1). \quad (2.1)$$

Applies (2.1) to $f_s, s = 1, \dots, n$, we have

$$m\left(r, \prod_{s=1}^n \frac{f_s^{(\mu(s))}}{f_s}\right) = O(1),$$

for any surjective $\mu : \{1, \dots, n\} \rightarrow \{0, \dots, (n-1)\}$. Therefore

$$m_L(r) = O(1).$$

This finishes the proof. \square

In [8], H.H. Khoai and M.V. Tu gave a form of inequality second main theorem type for an analytic curve intersecting hyperplanes, with ramification. For a convenience of readers, we will give here a simple proof of this result. The

method in the proof of the following theorem bases on the method of Vojta, which is shown in [13], over \mathbb{K} .

Theorem 1. *Let $f = (f_0 : \dots : f_n) : \mathbb{K} \rightarrow \mathbb{P}^n(\mathbb{K})$ be a analytic map whose image is not contained in any proper linear subspace. Let H_1, \dots, H_q be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{K})$. Let $L_j, 1 \leq j \leq q$, be the linear forms defining H_1, \dots, H_q . Denote by $W(f_0, \dots, f_n)$ the Wronskian of f_0, \dots, f_n . Then,*

$$\max_K \log \prod_{j \in K} \frac{\|f(z)\|_r}{|L_j(f)(z)|_r} + N_W(r, 0) \leq (n + 1)T_f(r) + O(1),$$

where the maximum is taken over all subsets K of $\{1, \dots, q\}$ such that the linear forms $L_j, j \in K$, are linearly independent. $N_W(r, 0)$ is the counting function for the zeros of Wronskian W of f .

Proof. Without loss of generality, we may assume (by adding more hyperplanes) that $H_1 \cap \dots \cap H_q = \emptyset$. Then the subsets K can be further restricted to subsets having exactly $n + 1$ elements.

Give such a subset K , write $K = \{s_0, \dots, s_n\}$. Also write $\gamma_j = H_j \circ f$ for all $j \in K$. As in Cartan’s original proof we have

$$\frac{W}{\gamma_{s_0} \dots \gamma_{s_n}} = C_K \begin{vmatrix} 1 & \dots & 1 \\ \gamma'_{s_0} & \dots & \gamma'_{s_n} \\ \gamma_{s_0} & \dots & \gamma_{s_n} \\ \vdots & \ddots & \vdots \\ \gamma_{s_0}^{(n)} & \dots & \gamma_{s_n}^{(n)} \\ \gamma_{s_0} & \dots & \gamma_{s_n} \end{vmatrix}, \tag{2.2}$$

where C_K is a constant. Let M_K denote the determinant on the right-hand side.

Obviously, W is a analytic function. Since Jensen’s formula, we have

$$N_W(r, 0) = \log |W|_r + O(1).$$

Hence

$$\begin{aligned} N_W(r, 0) - \sum_{i=0}^n \log |L_{s_i}(f)(z)|_r &= \log \left| \frac{W}{\prod_{i=0}^n L_{s_i}(f)(z)} \right|_r + O(1) \\ &= \log \left| \frac{W}{\gamma_{s_0} \dots \gamma_{s_n}} \right|_r + O(1), \end{aligned}$$

so

$$\sum_{i=0}^n \log \frac{\|f(z)\|_r}{|L_{s_i}(f)(z)|_r} - (n + 1)T_f(r) + N_W(r, 0) = \log \left| \frac{W}{\gamma_{s_0} \dots \gamma_{s_n}} \right|_r + O(1).$$

It then follows, by (2.2), that

$$\begin{aligned} \max_K \sum_{j \in K} \log \frac{\|f(z)\|_r}{|L_j(f)(z)|_r} - (n+1)T_f(r) + N_W(r, 0) \\ = \max_K \log \left| \frac{W}{\gamma_{s_0} \cdots \gamma_{s_n}} \right|_r + O(1) \\ = \max_K \log |M_K|_r + O(1) \\ \leq \max_K \log^+ |M_K|_r + O(1). \end{aligned}$$

Since Lemma 3, we have

$$\max_K \log^+ |M_K|_r = O(1)$$

which concludes the proof of the theorem. \square

3 Proof of Main Theorem

We first recall the Nevanlinna first main theorem in non-Archimedean fields.

First Main Theorem. *Let $f : \mathbb{K} \rightarrow \mathbb{P}^n(\mathbb{K})$ be an analytic curve and let Q be a homogeneous polynomial of degree d . If $Q(f) \not\equiv 0$, then every positive real number r ,*

$$m_f(r, Q) + N_f(r, Q) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r .

Now let $f = (f_0 : \dots : f_n) : \mathbb{K} \rightarrow \mathbb{P}^n(\mathbb{K})$ be an algebraically non-degenerate analytic map, and let D_1, \dots, D_q be hypersurfaces in $\mathbb{P}^n(\mathbb{K})$ of degree d_j in general position. Let $G_j, 1 \leq j \leq q$, be the homogeneous polynomials in $\mathbb{K}[x_0, \dots, x_n]$ of degree d_j defining D_j . Let d is the least common multiple of d_j 's and let $Q_j = G_j^{d/d_j} \forall j = 1, \dots, q$, then $Q_j, 1 \leq j \leq q$ are homogeneous polynomials of the same degree of d .

Note that if $z \in \mathbb{K}$ is a zero of $Q_j \circ f = G_j^{d/d_j} \circ f$ with multiplicity α then z is a zero of $Q_j \circ f$ with multiplicity $\alpha \frac{d_j}{d}$. It implies that for every positive integer Δ and for all positive real number $r > 0$

$$N_f^\Delta(r, Q_j) = N_f^\Delta(r, G_j^{d/d_j}) = \frac{d}{d_j} N_f^{[\Delta \frac{d_j}{d}]}(r, G_j) = \frac{d}{d_j} N_f^\Delta(r, G_j),$$

so for all $j = 1, \dots, q$

$$1 - \frac{N_f^\Delta(r, G_j)}{(\deg G_j)T_f(r)} = 1 - \frac{N_f^\Delta(r, Q_j)}{dT_f(r)}.$$

Hence

$$\sum_{j=1}^q \delta_f^\Delta(D_j) = \sum_{j=1}^q \delta_f^\Delta(G_j) = \sum_{j=1}^q \delta_f^\Delta(Q_j).$$

Given $z \in \mathbb{K}$, there exists a renumbering $\{i_1, \dots, i_q\}$ of the indices $\{1, \dots, q\}$ such that

$$|Q_{i_1} \circ f(z)| \leq |Q_{i_2} \circ f(z)| \leq \dots \leq |Q_{i_q} \circ f(z)|. \quad (3.1)$$

Since $Q_j, 1 \leq j \leq n$ are in general position, by Hilbert's Nullstellensatz [12], for any integer $k, 0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} \delta_{jk}(x_0, \dots, x_n) Q_{i_j}(x_0, \dots, x_n),$$

where $\delta_{jk}, 1 \leq j \leq n+1, 0 \leq k \leq n$, are the homogeneous forms with coefficients in \mathbb{K} of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k-d} \max\{|Q_{i_1} \circ f(z)|, \dots, |Q_{i_{n+1}} \circ f(z)|\}, \quad (3.2)$$

where $\|f(z)\| := \max\{|f_0(z)|, \dots, |f_n(z)|\}$, c_1 is a positive constant depends only on the coefficients of $\delta_{jk}, 1 \leq j \leq n+1, 0 \leq k \leq n$, thus depends only on the coefficients of $Q_i, 1 \leq i \leq q$. Note that, (3.2) holds for all $k = 0, \dots, n$, so

$$\begin{aligned} \|f(z)\|^{m_k} &= \max_{k=0 \dots n} \{|f_k(z)|^{m_k}\} \\ &\leq c_1 \|f(z)\|^{m_k-d} \max\{|Q_{i_1} \circ f(z)|, \dots, |Q_{i_{n+1}} \circ f(z)|\}, \end{aligned}$$

therefore,

$$\|f(z)\|^d \leq c_1 \max\{|Q_{i_1} \circ f(z)|, \dots, |Q_{i_{n+1}} \circ f(z)|\}. \quad (3.3)$$

By (3.1) and (3.3),

$$\prod_{j=1}^q \frac{\|f(z)\|^d}{|Q_j \circ f(z)|} = \left(\prod_{k=1}^n \frac{\|f(z)\|^d}{|Q_{i_k} \circ f(z)|} \right) \left(\prod_{k=n+1}^q \frac{\|f(z)\|^d}{|Q_{i_k} \circ f(z)|} \right) \leq c_1^{q-n} \prod_{k=1}^n \frac{\|f(z)\|^d}{|Q_{i_k} \circ f(z)|}.$$

Hence, by the definition,

$$\begin{aligned} \sum_{j=1}^q m_f(r, Q_j) &= \log \prod_{j=1}^q \frac{\|f(z)\|_r^d}{|Q_j \circ f(z)|_r} \\ &\leq \max_{\{i_1, \dots, i_n\}} \log \prod_{k=1}^n \frac{\|f(z)\|_r^d}{|Q_{i_k} \circ f(z)|_r} + (q-n) \log c_1. \end{aligned} \quad (3.4)$$

Pick n distinct polynomials $g_1, \dots, g_n \in \{Q_1, \dots, Q_q\}$. By the general position assumption, they define a subvariety of dimension 0 in $\mathbb{P}^n(\mathbb{K})$. For a fixed large

integer N , which will be chosen later, let V_N be the space of homogeneous polynomials of degree N in $\mathbb{K}[x_0, \dots, x_n]$. We have constructed a filtration $W_{(\mathbf{i})}$ of V_N and

$$\delta_{(\mathbf{i})} := \dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} = d^n,$$

for any $(\mathbf{i}') > (\mathbf{i})$, which are consecutive n -tuples.

Set $\Delta := \dim V_N$. We now choose a suitable basis $\{\psi_1, \dots, \psi_\Delta\}$ for V_N as the following way. We start with the last nonzero $W_{(\mathbf{i})}$ and pick any basis of it. Then we continue inductively as follows: suppose $(\mathbf{i}') > (\mathbf{i})$ are consecutive n -tuples such that $d\sigma(\mathbf{i}), d\sigma(\mathbf{i}') \leq N$ and assume that we have chosen a basis of $W_{(\mathbf{i}')}$. It follows directly from the definition that we may pick representatives in $W_{(\mathbf{i})}$ for the quotient space $W_{(\mathbf{i})}/W_{(\mathbf{i}')}$, of the form $g_1^{i_1} \dots g_n^{i_n} \eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. We extend the previously constructed basis in $W_{(\mathbf{i}')}$ by adding these representatives. In particular, we have obtained a basis for $W_{(\mathbf{i})}$ and our inductive procedure may go on unless $W_{(\mathbf{i})} = V_N$, in which case we stop. In this way, we have obtained a basis $\{\psi_1, \dots, \psi_\Delta\}$ for V_N . Let $\phi_1, \dots, \phi_\Delta$ be a fixed basis of V_N . Then $\{\psi_1, \dots, \psi_\Delta\}$ can be written as linear forms L_1, \dots, L_Δ in $\phi_1, \dots, \phi_\Delta$ so that $\psi_t(f) = L_t(F)$, where $F = (\phi_1(f) : \dots : \phi_\Delta(f)) : \mathbb{K} \rightarrow \mathbb{P}^{\Delta-1}(\mathbb{K})$. The linear forms L_1, \dots, L_Δ are linearly independent, and we know, from the assumption of algebraically non-degeneracy of f , that F is linearly non-degenerate.

For $z \in \mathbb{K}$, we now estimate $\log \prod_{t=1}^{\Delta} |L_t(F)(z)| = \log \prod_{t=1}^{\Delta} |\psi_t(f)(z)|$. Let ψ be an element of the basis, constructed with respect to $W_{(\mathbf{i})}/W_{(\mathbf{i}')}$, then we have $\psi = g_1^{i_1} \dots g_n^{i_n} \eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. Then we have a bound

$$\begin{aligned} |\psi(f)(z)| &\leq |g_1(f)(z)|^{i_1} \dots |g_n(f)(z)|^{i_n} |\eta(f)(z)| \\ &\leq c_2 |g_1(f)(z)|^{i_1} \dots |g_n(f)(z)|^{i_n} \|f(z)\|^{N-d\sigma(\mathbf{i})}, \end{aligned}$$

where c_2 is the positive constant depending only on ψ , not on f and z . Observe that there are precisely $\delta_{(\mathbf{i})}$ such functions ψ in our basis. Hence,

$$\begin{aligned} \log |\psi_t(f)(z)| &\leq i_1 \log |g_1(f)(z)| + \dots + i_n \log |g_n(f)(z)| + (N - d\sigma(\mathbf{i})) \log \|f(z)\| + c_3 \\ &\leq i_1 \left(\log |g_1(f)(z)| - \log \|f(z)\|^d \right) + \dots \\ &\quad + i_n \left(\log |g_n(f)(z)| - \log \|f(z)\|^d \right) + N \log \|f(z)\| + c_3 \\ &\leq -i_1 \log \frac{\|f(z)\|^d}{|g_1(f)(z)|} - \dots - i_n \log \frac{\|f(z)\|^d}{|g_n(f)(z)|} + N \log \|f(z)\| + c_3, \end{aligned}$$

where c_3 is the positive constant, which does not depend on f and r . Therefore,

$$\begin{aligned}
\log \prod_{t=1}^{\Delta} |L_t(F)(z)| &= \log \prod_{t=1}^{\Delta} |\psi_t(f)(z)| \\
&\leq - \sum_{(\mathbf{i})} \delta_{(\mathbf{i})} \left(i_1 \log \frac{\|f(z)\|^d}{|g_1(f)(z)|} + \cdots + i_n \log \frac{\|f(z)\|^d}{|g_n(f)(z)|} \right) \\
&\quad + \Delta N \log \|f(z)\| + \Delta c_3 \\
&= - \sum_{j=1}^n \log \frac{\|f(z)\|^d}{|g_j(f)(z)|} \left(\sum_{(\mathbf{i})} \delta_{(\mathbf{i})} i_j \right) + \Delta N \log \|f(z)\| + \Delta c_3,
\end{aligned} \tag{3.5}$$

where the summations are taken over the n -tuples with $\sigma(\mathbf{i}) \leq N/d$. Clearly that $\delta := \sum_{(\mathbf{i})} \delta_{(\mathbf{i})} i_j$ does not depend on j , $1 \leq j \leq n$. Hence (3.5) becomes

$$\log \prod_{t=1}^{\Delta} |L_t(F)(z)| \leq -\delta \log \prod_{j=1}^n \frac{\|f(z)\|^d}{|g_j(f)(z)|} + \Delta N \log \|f(z)\| + \Delta c_3.$$

This implies

$$\begin{aligned}
\log \prod_{j=1}^n \frac{\|f(z)\|^d}{|g_j(f)(z)|} &\leq \frac{1}{\delta} \log \prod_{t=1}^{\Delta} \frac{\|F(z)\|}{|L_t(F)(z)|} - \frac{\Delta}{\delta} \log \|F(z)\| \\
&\quad + \frac{\Delta N}{\delta} \log \|f(z)\| + \frac{\Delta c_3}{\delta}.
\end{aligned} \tag{3.6}$$

Since there are only finitely many choices $\{g_1, \dots, g_n\} \subset \{Q_1, \dots, Q_q\}$, we have a finite collection of linear forms L_1, \dots, L_u . From (3.6) we have

$$\begin{aligned}
\max_{\{i_1, \dots, i_n\}} \log \prod_{k=1}^n \frac{\|f(z)\|_r^d}{|Q_{i_k}(f)(z)|_r} &\leq \frac{1}{\delta} \max_K \log \prod_{j \in K} \frac{\|F(z)\|_r}{|L_j(F)(z)|_r} - \frac{\Delta}{\delta} T_F(r) \\
&\quad + \frac{\Delta N}{\delta} T_f(r) + c_4,
\end{aligned}$$

where \max_K is taken over all subsets K of $\{1, \dots, u\}$ such that linear forms $L_j, j \in K$, are linearly independent, c_4 is positive constant independent of r . Applying Theorem 1 to analytic map $F : \mathbb{K} \rightarrow \mathbb{P}^{\Delta-1}(\mathbb{K})$ and linear forms L_1, \dots, L_u , and together with (3.4) we have

$$\begin{aligned}
\sum_{j=1}^q m_f(r, Q_j) &\leq \max_{\{i_1, \dots, i_n\}} \log \prod_{k=1}^n \frac{\|f(z)\|_r^d}{|Q_{i_k}(f)(z)|_r} + (q-n) \log c_1 \\
&\leq -\frac{1}{\delta} N_W(r, 0) + \frac{\Delta N}{\delta} T_f(r) + O(1),
\end{aligned}$$

where W is the Wronskian of F_1, \dots, F_Δ . By the First Main Theorem, we have

$$(qd - \frac{\Delta N}{\delta})T_f(r) \leq \sum_{j=1}^q N_f(r, Q_j) - \frac{1}{\delta}N_W(r, 0) + O(1). \quad (3.7)$$

We will estimate $\sum_{j=1}^q N_f(r, Q_j) - \frac{1}{\delta}N_W(r, 0)$ on the right hand side of the above inequality. For each $z \in \mathbb{K}$, without loss of generality, we may assume that $Q_j \circ f$ vanishes at z for $1 \leq j \leq q_1$ and $Q_j \circ f$ does not vanish at z for $j > q_1$. By the hypothesis Q_j are "in general position", we know $q_1 \leq n$. There are integers $k_j \geq 0$ and nowhere vanishing analytic functions γ_j in a neighborhood U of z such that

$$Q_j \circ f = (\zeta - z)^{k_j} \gamma_j, \text{ for } j = 1, \dots, q,$$

where $k_j = 0$ if $q_1 < j \leq q$. For $\{Q_1, \dots, Q_n\} \subset \{Q_1, \dots, Q_q\}$, we can obtain a basis $\{\psi_1, \dots, \psi_\Delta\}$ of V_N and linearly independent linear forms L_1, \dots, L_Δ such that $\psi_t(f) = L_t(F)$. By the property of Wronskian,

$$\begin{aligned} W &= W(F_1, \dots, F_\Delta) = CW(L_1(F), \dots, L_\Delta(F)) \\ &= C \begin{vmatrix} \psi_1(f) & \dots & \psi_\Delta(f) \\ (\psi_1(f))' & \dots & (\psi_\Delta(f))' \\ \vdots & \ddots & \vdots \\ (\psi_1(f))^{(\Delta-1)} & \dots & (\psi_\Delta(f))^{(\Delta-1)} \end{vmatrix}. \end{aligned}$$

Let ψ be an element of basis, constructed with respect to $W_{(i)}/W_{(i')}$, so we may write $\psi = Q_1^{i_1} \dots Q_n^{i_n} \eta$, $\eta \in V_{N-d\sigma(i)}$. We have

$$\psi(f) = (Q_1(f))^{i_1} \dots (Q_n(f))^{i_n} \eta(f),$$

where $(Q_j(f))^{i_j} = (\zeta - z)^{i_j \cdot k_j} \gamma_j^{i_j}$, $j = 1, \dots, n$. Also we can assume that $k_j \geq \Delta$ if $1 \leq j \leq q_0$ and $1 \leq k_j < \Delta$ if $q_0 < j \leq q_1$. And we observe that there are $\delta_{(i)}$ such ψ in our basis. Thus W vanishes at z with order at least

$$\sum_{(i)} \left(\sum_{j=1}^{q_0} i_j (k_j - \Delta) \right) \delta_{(i)} = \sum_{(i)} i_j \delta_{(i)} \sum_{j=1}^{q_0} (k_j - \Delta) = \delta \sum_{j=1}^{q_0} (k_j - \Delta).$$

Therefore,

$$\sum_{j=1}^q N_f(r, Q_j) - \frac{1}{\delta}N_W(r, 0) \leq \sum_{j=1}^q N_f^\Delta(r, Q_j). \quad (3.8)$$

We now estimate on the left hand side of the inequality (3.7). Assume that N is divisible by d and $N \geq nd$. Then

$$\Delta = \binom{N+n}{n} = \frac{(N+1)\dots(N+n)}{n!}. \tag{3.9}$$

On the other hand, since the number of non-negative integer m -tuples with sum $\leq T$ is equal to the number of non-negative integer $(m+1)$ -tuples with sum exactly $T \in \mathbb{Z}$, which is $\binom{T+m}{m}$. It follows from Lemma 1 that,

$$\begin{aligned} \delta &\geq \sum_{(i)} i_j \delta_{(i)} = \sum_{\widehat{(i)}} i_j \delta_{(i)} = d^n \sum_{\widehat{(i)}} i_j = \frac{d^n}{n+1} \sum_{\widehat{(i)}} \sum_{j=1}^{n+1} i_j = \frac{d^n}{n+1} \sum_{\widehat{(i)}} (N/d - n) \\ &= \frac{d^n}{n+1} \binom{N/d}{n} (N/d - n) = \frac{N(N-d)\dots(N-nd)}{(n+1)!d}, \end{aligned} \tag{3.10}$$

where the sum $\sum_{(i)}$ is taken over the nonnegative integer $(n+1)$ -tuples with sum exactly N/d and $\sum_{\widehat{(i)}}$ is taken over the nonnegative integer $(n+1)$ -tuples with sum exactly $N/d - n$. So since (3.9) and (3.10) we have

$$\frac{\Delta N}{\delta} \leq (n+1)d \frac{(N+1)\dots(N+n)}{(N-d)\dots(N-nd)},$$

therefore

$$\left(qd - \frac{\Delta N}{\delta}\right) \geq d \left(q - (n+1) - \left(\frac{(N+1)\dots(N+n)}{(N-d)\dots(N-nd)} - 1 \right) (n+1) \right).$$

It follows, for every $\varepsilon > 0$,

$$\left(qd - \frac{\Delta N}{\delta}\right) T_f(r) \geq d(q - n - 1 - \varepsilon) T_f(r), \tag{3.11}$$

if we take N large enough such that

$$\frac{(N+1)\dots(N+n)}{(N-d)\dots(N-nd)} \leq 1 + \frac{\varepsilon}{(n+1)}. \tag{3.12}$$

Combining the formulas (3.7), (3.8), (3.11) and (3.12) together, for each $\varepsilon > 0$, and Δ in Main Theorem, we have

$$d(q - (n+1) - \varepsilon) T_f(r) \leq \sum_{j=1}^q N_f^\Delta(r, Q_j) + O(1),$$

so

$$\sum_{j=1}^q \left(1 - \frac{N_f^\Delta(r, Q_j)}{dT_f(r)} \right) \leq (n+1 + \varepsilon) + \frac{O(1)}{dT_f(r)}.$$

This is conclusion the proof of Main Theorem.

References

- [1] T. T. H. An, *A defect relation for non-Archimedean analytic curves in arbitrary projective varieties*, Proc. Amer. Math. Soc. **135** (2007), 1255-1261.
- [2] T. T. H. An and J. T-Y Wang, *An effective Schmidt's subspace theorem for non-linear forms over function fields*, Journal of Number Theory, Volume **125**(1) (2007), 210-228.
- [3] T. T. H. An, H.T. Phuong, *An explicit estimate on multiplicity truncation in the second main theorem for holomorphic curves encountering hypersurfaces in general position in projective space*, to appear in Houston Journal of Mathematics.
- [4] A. Boutabba, *Sur les courbes holomorphes p -adiques*, Ann. Fac. Sci. Toulouse Math. (6) **5**(1996), 29-52.
- [5] W. Cherry and Z.Ye, *Non-Archimedean Nevanlinna Theory in several variables and the Non-Archimedean Nevanlinna inverse problem*, Tran. Amer. Math. Soc., **349**(12)(1997), 5043-5071.
- [6] P. Corvaja and U. M. Zannier, *On a general Thue's equation*, Amer. J. Math. **126** (2004), 1033-1055.
- [7] P.C. Hu and C.C Yang, "Meromorphic Functions over Non-Archimedean Fields", Kluwer Academic Publishers, 2000.
- [8] H.H. Khoai and M.V. Tu, *P -adic Nevanlinna Cartan theorem*, Inter. J. Math., **6** (5) (1995), 719-731.
- [9] Y. Liu and M.Ru, *A defect relation for meromorphic maps on parabolic manifolds intersecting hypersurfaces*, Illinois J.Math. **49**, No 1 (2005), pp 237-257.
- [10] M. Ru, *A defect relation for holomorphic curves intersecting hypersurfaces* Amer. Journal of Math. **126** (2004), 215-226.
- [11] M. Ru, *A note on p -adic Nevanlinna theory*, Proc.Amer.Math.Soc. **129** (2001), 1263-1269.
- [12] B. L. Van Der Waerden, "Algebra", vol 2, 7th, ed., Springer-Verlag, New York, 1991.
- [13] P. Vojta, *On Cartan's theorem and Cartan's conjecture*, American Journal of Mathematics **119** (1997), 1-17.
- [14] Yan, Q.M and Chen, Z.H, *Weak Cartan-type Second Main Theorem for Holomorphic Curves*, to appear in Acta Mathematica Sinica.