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ON TRUNCATED DEFECT RELATION FOR NON-ARCHIMEDEAN ANALYTIC CURVES INTERSECTING HYPERSURFACES

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Abstract

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic, completed with respect to a non-Archimedean absolute value "$|\ |$". In this paper, we will estimate truncated defect relation for non-Archimedean analytic curves intersecting hypersurfaces in general position.

1 Introduction

We first introduce some standard notations in Nevanlinna theory. Let f be an entire function on $\mathbb{K}$, defined by a convergent series

$$f(z) = \sum_{n=m}^{\infty} a_n z^n, \quad (a_m \neq 0;\ m \geqslant 0).$$

For each real number $r \geqslant 0$, we define

$$\begin{aligned} |f|_r = \sup_n |a_n| r^n &= \sup\{|f(z)| : z \in \mathbb{K} \text{ with } |z| \leqslant r\} \\ &= \sup\{|f(z)| : z \in \mathbb{K} \text{ with } |z| = r\}. \end{aligned}$$

Let $f : \mathbb{K} \to \mathbb{P}^n(\mathbb{K})$ be a analytic map, $f = (f_0 : ... : f_n)$ be a reduced representative of f, where $f_0, ..., f_n$ are entire functions on $\mathbb{K}$ without common zeros,

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at least one of which is non-constant. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \log \|f\|_r,$$

where $\|f\|_r = \max\{|f_0|_r, ..., |f_n|_r\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{K})$ of degree d. Let G be the homogeneous polynomial in $n+1$ variables with coefficients in $\mathbb{K}$ of degree d defining D. The proximity function of f is defined by

$$m_f(r, D) = m_f(r, G) = \log \frac{\|f\|_r^d}{|G \circ f|_r}.$$

Note that up to a constant term, $m_f(r, D)$ is independent of the choice of defining form G. Let $n_f(r, G)$ be the number of zeros of $G \circ f$ in the disk $|z| < r$, counting multiplicity, and $n_f^{\Delta}(r, G)$ be the number of zeros of $G \circ f$ in the disk $|z| < r$, truncated multiplicity by a positive integer Δ. The counting function and truncated function are defined by

$$N_f(r, D) = N_f(r, G) = \int_0^r \frac{n_f(t, G) - n_f(0, G)}{t} dt + n_f(0, G) \log r;$$

$$N_f^{\Delta}(r, D) = N_f^{\Delta}(r, G) = \int_0^r \frac{n_f^{\Delta}(t, G) - n_f^{\Delta}(0, G)}{t} dt + n_f^{\Delta}(0, G) \log r.$$

It is clear that for any positive integer Δ, $N_f^{\Delta}(r, D) \leqslant N_f(r, D)$.

Let X be an n-dimensional (not necessarily smooth) projective subvariety of $\mathbb{P}^N(\mathbb{K})$. A collection of $q \geq n+1$ hypersurfaces $D_1, \ldots, D_q$ in $\mathbb{P}^N(\mathbb{K})$ is said to be *in general position with* X if for any subset $\{i_0, \ldots, i_n\}$ of $\{1, \ldots, q\}$ of cardinality $n+1$,

$$\{x \in X : G_{i_j}(x) = 0,\ j = 0, \ldots, n\} = \emptyset,$$

where $G_j, 1 \leqslant j \leqslant q$, be the homogeneous polynomials in $\mathbb{K}[x_0, ..., x_n]$ defining D_j.

For a hypersurface D, which is defined by homogeneous polynomial G, we define the defect

$$\delta_f(D) = \delta_f(G) = 1 - \limsup_{r \longrightarrow +\infty} \frac{N_f(r, G)}{(\deg G) T_f(r)},$$

and the truncated defect

$$\delta_f^{\Delta}(D) = \delta_f^{\Delta}(G) = 1 - \limsup_{r \longrightarrow +\infty} \frac{N_f^{\Delta}(r, G)}{(\deg G) T_f(r)},$$

where Δ be a positive integer. It is easy to see that

$$0 \leqslant \delta_f(D) \leqslant \delta_f^{\Delta}(D) \leqslant 1$$

for any positive integer Δ and hypersurface D.

In [1] (also [11] for the special case when $X = \mathbb{P}^N(\mathbb{K})$), the author showed that

Theorem A. *Let $X \subset \mathbb{P}^N(\mathbb{K})$ be a projective sub-variety of dimension $n \geq 1$ over $\mathbb{K}$. Let $D_1, .., D_q$ be hypersurfaces of degree $d_1, ..., d_q$ resp. in $\mathbb{P}^N(\mathbb{K})$ in general position with X. Let $f : \mathbb{K} \to X$ be a non-constant analytic map whose image is not completely contained in any of the hypersurfaces $D_1, \ldots, D_q$. Then*

$$\sum_{j=1}^{q} \delta_f(D_j) \leqslant n = \dim X. \tag{1.1}$$

Let $D_1, .., D_q$ be hypersurfaces of degree $d_1, ..., d_q$ resp. in $\mathbb{P}^n(\mathbb{K})$, for any $\varepsilon > 0$, we now define *bound of truncated level*, which is denoted by $\mathcal{B}_\varepsilon(D_1, ..., D_q)$, of the hypersurfaces $D_1, ..., D_q$ associating ε as follows: Let d be the least common multiple of d_j's and let N be the smallest natural number such that $N \geqslant nd$, divisible by d and $\dfrac{(N+1)...(N+n)}{(N-d)...(N-nd)} \leqslant 1 + \dfrac{\varepsilon}{n+1}$, then

$$\mathcal{B}_\varepsilon(D_1, ..., D_q) = \frac{(N+n)!}{N!n!}.$$

In this paper, we will estimate truncated defect relation for non-Archimedean analytic curves to projective space over $\mathbb{K}$. Our result is stated as follows.

Main Theorem. *Let $f : \mathbb{K} \to \mathbb{P}^n(\mathbb{K})$ be an algebraically non-degenerate analytic map, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{K})$ of degree d_j in general position. Then for every $\varepsilon > 0$, there exists a positive integer $\Delta = \mathcal{B}_\varepsilon(D_1, ..., D_q)$ such that*

$$\sum_{j=1}^{q} \delta_f^{\Delta}(D_j) \leqslant n + 1 + \varepsilon. \tag{1.2}$$

Theorem A as above gave a better bound than our result, but it seems impossible to get a truncated defect relation from the approach which is given in the paper [1]. The proof of our Main Theorem is based on the method which was first introduced by Corvaje and Zannier for number fields [6], then by Ru [10] and An-Phuong [3] for the complex field. By our method explicit bound truncation level of Δ in our result is the smallest as possible.

Unfortunately, Δ in Main Theorem depends on ε. It would be interesting if one can find a Δ term indenpends on $\varepsilon > 0$. It is very important, because we can cut term ε in the right side in (1.2) in that case.

2 Some Preparations

In this section, we show and recall some lemmas and theorem, which are necessary for proof of the our main result. (For detail, readers can find in [1], [5], [6], [7], [10], [12] and also [2]).

Throughout of this paper, we shall use the *lexicographic ordering* on the m-tuples $(i_1, ..., i_m) \in \mathbb{N}^m$ of natural numbers. Namely, $(j_1, ..., j_m) > (i_1, ..., i_m)$ if and only if for some $b \in \{1, ..., m\}$ we have $j_l = i_l$ for $l < b$ and $j_b > i_b$. With the n-tuples $(\mathbf{i}) = (i_1, ..., i_n)$ of non-negative integers, we denote $\sigma(\mathbf{i}) = \sum_{j=1}^n i_j$.

Let $g_1, ..., g_n \in \mathbb{K}[x_0, ..., x_n]$ be homogeneous polynomials of degree d, such that they define a subvariety of $\mathbb{P}^n(\mathbb{K})$ of dimension 0. For a fixed positive integer N, denote by V_N the space of homogeneous polynomials of degree N in $\mathbb{K}[x_0, ..., x_n]$. We define a filtration of V_N as follows. Arrange, by the lexicographic order, the n-tuples $(\mathbf{i}) = (i_1, ..., i_n)$ of non-negative integers such that $\sigma(\mathbf{i}) \leqslant N/d$. Define the spaces $W_{(\mathbf{i})} = W_{N,(\mathbf{i})}$ by

$$W_{(\mathbf{i})} = \sum_{(\mathbf{e}) \geqslant (\mathbf{i})} g_1^{e_1} ... g_n^{e_n} V_{N-d\sigma(\mathbf{e})}.$$

Clearly, $W_{(0,...,0)} = V_N$ and $W_{(\mathbf{i})} \supset W_{(\mathbf{i}')}$ if $(\mathbf{i}') > (\mathbf{i})$, so $W_{(\mathbf{i})}$ is a filtration of V_N.

For any pair $(\mathbf{i}')$ follows $(\mathbf{i})$ in ordering, as in Corvaja-Zannier's original proof we may choose a basis of quotient $\dfrac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}}$ from the set containing all equivalence classes of the form: $\gamma_1^{i_1} ... \gamma_n^{i_n} \eta$ modulo $W_{(\mathbf{i}')}$ with η being a monomial in $x_0, ..., x_n$ with total degree $N - d\sigma(\mathbf{i})$. Now we evaluate the dimensions, denoted by $\delta_{(\mathbf{i})}$, of the quotients of successive spaces in the filtration.

Lemma 1. *If $\sigma(\mathbf{i}) \leqslant N/d - n$ then*

$$\delta_{(\mathbf{i})} := \dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} = d^n.$$

The proof of the above lemma was give in [3].

Next, we will recall the following lemma which is well-know in p-adic Nevanlinna theory. The proof can be found, for example, in [8].

Lemma 2. *Let f be a nonconstant meromorphic function in $\mathbb{K}$, then*

$$m\left(r, \frac{f'}{f}\right) = O(1).$$

Let $f_1, ..., f_n$ be meromorphic functions over $\mathbb{K}$. Their Wronskian is

$$W(f) = W(f_1, ..., f_n) := \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

We denote

$$L = L(f) = L(f_1, ..., f_n) := \begin{vmatrix} 1 & \cdots & 1 \\ f_1'/f_1 & \cdots & f_n'/f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}/f_1 & \cdots & f_n^{(n-1)}/f_n \end{vmatrix}.$$

Lemma 3. *Let $f_1, ..., f_n$ be meromorphic functions over $\mathbb{K}$. Then for any $r > 0$ we have*

$$m_L(r) = \log^+ |L|_r = O(1).$$

Proof. Let g be a meromorphic function over $\mathbb{K}$, for any integer $k \geqslant 1$, we can write a logarithmic derivative of hight order as product

$$\frac{g^{(k)}}{g} = \frac{g^{(k)}}{g^{(k-1)}} \cdots \frac{g'}{g},$$

since Lemma 2, we have

$$m\left(r, \frac{g^{(k)}}{g}\right) = O(1). \tag{2.1}$$

Applies (2.1) to $f_s, s = 1, ..., n$, we have

$$m\left(r, \prod_{s=1}^{n} \frac{f_s^{(\mu(s))}}{f_s}\right) = O(1),$$

for any surjective $\mu : \{1, ..., n\} \longrightarrow \{0, ..., (n-1)\}$. Therefore

$$m_L(r) = O(1).$$

This finishes the proof. □

In [8], H.H. Khoai and M.V. Tu gave a form of inequality second main theorem type for an analytic curve intersecting hyperplanes, with ramification. For a convenience of readers, we will give here a simple proof of this result. The method in the proof of the following theorem bases on the method of Vojta, which is shown in [13], over $\mathbb{K}$.

Theorem 1. *Let $f = (f_0 : ... : f_n) : \mathbb{K} \to \mathbb{P}^n(\mathbb{K})$ be a analytic map whose image is not contained in any proper linear subspace. Let $H_1, ..., H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{K})$. Let $L_j, 1 \leqslant j \leqslant q$, be the linear forms defining $H_1, ..., H_q$. Denote by $W(f_0, ..., f_n)$ the Wronskian of $f_0, ..., f_n$. Then,*

$$\max_K \log \prod_{j\in K} \frac{\|f(z)\|_r}{|L_j(f)(z)|_r} + N_W(r,0) \leqslant (n+1)T_f(r) + O(1),$$

where the maximum is taken over all subsets K of $\{1, ..., q\}$ such that the linear forms $L_j, j \in K$, are linearly independent. $N_W(r,0)$ is the counting function for the zeros of Wronskian W of f.

Proof. Without loss of generality, we may assume (by adding more hyperplanes) that $H_1 \cap ... \cap H_q = \emptyset$. Then the subsets K can be further restricted to subsets having exactly $n+1$ elements.

Give such a subset K, write $K = \{s_0, ..., s_n\}$. Also write $\gamma_j = H_j \circ f$ for all $j \in K$. As in Cartan's original proof we have

$$\frac{W}{\gamma_{s_0} \dots \gamma_{s_n}} = C_K \begin{vmatrix} 1 & \dots & 1 \\ \dfrac{\gamma'_{s_0}}{\gamma_{s_0}} & \dots & \dfrac{\gamma'_{s_n}}{\gamma_{s_n}} \\ \vdots & \ddots & \vdots \\ \dfrac{\gamma^{(n)}_{s_0}}{\gamma_{s_0}} & \dots & \dfrac{\gamma^{(n)}_{s_n}}{\gamma_{s_n}} \end{vmatrix}, \tag{2.2}$$

where C_K is a constant. Let M_K denote the determinant on the right-hand side.

Obviously, W is a analytic function. Since Jensen's formula, we have

$$N_W(r,0) = \log |W|_r + O(1).$$

Hence

$$\begin{aligned} N_W(r,0) - \sum_{i=0}^{n} \log |L_{s_i}(f)(z)|_r &= \log \left| \frac{W}{\prod\limits_{i=0}^{n} L_{s_i}(f)(z)} \right|_r + O(1) \\ &= \log \left| \frac{W}{\gamma_{s_0} \dots \gamma_{s_n}} \right|_r + O(1), \end{aligned}$$

so

$$\sum_{i=0}^{n} \log \frac{\|f(z)\|_r}{|L_{s_i}(f)(z)|_r} - (n+1)T_f(r) + N_W(r,0) = \log \left| \frac{W}{\gamma_{s_0} \dots \gamma_{s_n}} \right|_r + O(1).$$

It then follows, by (2.2), that

$$\begin{aligned}
\max_K \sum_{j\in K} \log \frac{\|f(z)\|_r}{|L_j(f)(z)|_r} &- (n+1)T_f(r) + N_W(r,0)\\
&= \max_K \log \left|\frac{W}{\gamma_{s_0}\dots\gamma_{s_n}}\right|_r + O(1)\\
&= \max_K \log |M_K|_r + O(1)\\
&\leqslant \max_K \log^+ |M_K|_r + O(1).
\end{aligned}$$

Since Lemma 3, we have

$$\max_K \log^+ |M_K|_r = O(1)$$

which concludes the proof of the theorem. □

3 Proof of Main Theorem

We first recall the Nevanlinna first main theorem in non-Archimedean fields.
First Main Theorem. *Let $f : \mathbb{K} \longrightarrow \mathbb{P}^n(\mathbb{K})$ be an analytis curve and let Q be a homogeneous polynomial of degree d. If $Q(f) \not\equiv 0$, then every positive real number r,*

$$m_f(r,Q) + N_f(r,Q) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

Now let $f = (f_0 : ... : f_n) : \mathbb{K} \to \mathbb{P}^n(\mathbb{K})$ be an algebraically non-degenerate analytic map, and let $D_1, ..., D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{K})$ of degree d_j in general position. Let $G_j, 1 \leqslant j \leqslant q$, be the homogeneous polynomials in $\mathbb{K}[x_0, ..., x_n]$ of degree d_j defining D_j. Let d is the least common multiple of $d_j's$ and let $Q_j = G_j^{d/d_j}$ $\forall j = 1, ..., q$, then Q_j, $1 \leqslant j \leqslant q$ are homogeneous polynomials of the same degree of d.

Note that if $z \in \mathbb{K}$ is a zero of $Q_j \circ f = G_j^{d/d_j} \circ f$ with multiplicity α then z is a zero of $Q_j \circ f$ with multiplicity $\alpha \dfrac{d_j}{d}$. It implies that for every positive integer Δ and for all positive real number $r > 0$

$$N_f^{\Delta}(r, Q_j) = N_f^{\Delta}(r, G_j^{\frac{d}{d_j}}) = \frac{d}{d_j} N_f^{[\Delta \frac{d_j}{d}]}(r, G_j) = \frac{d}{d_j} N_f^{\Delta}(r, G_j),$$

so for all $j = 1, ..., q$

$$1 - \frac{N_f^{\Delta}(r, G_j)}{(\deg G_j)T_f(r)} = 1 - \frac{N_f^{\Delta}(r, Q_j)}{dT_f(r)}.$$

Hence

$$\sum_{j=1}^{q}\delta_f^{\Delta}(D_j)=\sum_{j=1}^{q}\delta_f^{\Delta}(G_j)=\sum_{j=1}^{q}\delta_f^{\Delta}(Q_j).$$

Given $z\in\mathbb{K}$, there exists a renumbering $\{i_1,...,i_q\}$ of the indices $\{1,...,q\}$ such that

$$|Q_{i_1}\circ f(z)|\leqslant|Q_{i_2}\circ f(z)|\leqslant...\leqslant|Q_{i_q}\circ f(z)|. \tag{3.1}$$

Since $Q_j, 1\leqslant j\leqslant n$ are in general position, by Hilbert's Nullstellensatz [12], for any integer $k, 0\leqslant k\leqslant n$, there is an integer $m_k\geqslant d$ such that

$$x_k^{m_k}=\sum_{j=1}^{n+1}\delta_{jk}(x_0,..,x_n)Q_{i_j}(x_0,...,x_n),$$

where $\delta_{jk}, 1\leqslant j\leqslant n+1, 0\leqslant k\leqslant n$, are the homogeneous forms with coefficients in $\mathbb{K}$ of degree m_k-d. So

$$|f_k(z)|^{m_k}\leqslant c_1\|f(z)\|^{m_k-d}\max\{|Q_{i_1}\circ f(z)|,...,|Q_{i_{n+1}}\circ f(z)|\}, \tag{3.2}$$

where $\|f(z)\|:=\max\{|f_0(z)|,...,|f_n(z)|\}$, c_1 is a positive constant depends only on the coefficients of $\delta_{jk}, 1\leqslant j\leqslant n+1, 0\leqslant k\leqslant n$, thus depends only on the coefficients of $Q_i, 1\leqslant i\leqslant q$. Note that, (3.2) holds for all $k=0,...,n$, so

$$\begin{aligned}\|f(z)\|^{m_k}&=\max_{k=0...n}\{|f_k(z)|^{m_k}\}\\&\leqslant c_1\|f(z)\|^{m_k-d}\max\{|Q_{i_1}\circ f(z)|,...,|Q_{i_{n+1}}\circ f(z)|\},\end{aligned}$$

therefore,

$$\|f(z)\|^{d}\leqslant c_1\max\{|Q_{i_1}\circ f(z)|,...,|Q_{i_{n+1}}\circ f(z)|\}. \tag{3.3}$$

By (3.1) and (3.3),

$$\prod_{j=1}^{q}\frac{\|f(z)\|^d}{|Q_j\circ f(z)|}=\Big(\prod_{k=1}^{n}\frac{\|f(z)\|^d}{|Q_{i_k}\circ f(z)|}\Big)\Big(\prod_{k=n+1}^{q}\frac{\|f(z)\|^d}{|Q_{i_k}\circ f(z)|}\Big)\leqslant c_1^{q-n}\prod_{k=1}^{n}\frac{\|f(z)\|^d}{|Q_{i_k}\circ f(z)|}.$$

Hence, by the definition,

$$\begin{aligned}\sum_{j=1}^{q}m_f(r,Q_j)&=\log\prod_{j=1}^{q}\frac{\|f(z)\|_r^d}{|Q_j\circ f(z)|_r}\\&\leqslant\max_{\{i_1,...,i_n\}}\log\prod_{k=1}^{n}\frac{\|f(z)\|_r^d}{|Q_{i_k}\circ f(z)|_r}+(q-n)\log c_1.\end{aligned} \tag{3.4}$$

Pick n distinct polynomials $g_1,...,g_n\in\{Q_1,...,Q_q\}$. By the general position assumption, they define a subvariety of dimension 0 in $\mathbb{P}^n(\mathbb{K})$. For a fixed large integer N, which will be chosen later, let V_N be the space of homogeneous polynomials of degree N in $\mathbb{K}[x_0, ..., x_n]$. We have constructed a filtration $W_{(\mathbf{i})}$ of V_N and

$$\delta_{(\mathbf{i})} := \dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} = d^n,$$

for any $(\mathbf{i}') > (\mathbf{i})$, which are consecutive n-tuples.

Set $\Delta := \dim V_N$. We now choose a suitable basis $\{\psi_1, ..., \psi_\Delta\}$ for V_N as the following way. We start with the last nonzero $W_{(\mathbf{i})}$ and pick any basis of it. Then we continue inductively as follows: suppose $(\mathbf{i}') > (\mathbf{i})$ are consecutive n-tuples such that $d\sigma(\mathbf{i}), d\sigma(\mathbf{i}') \leqslant N$ and assume that we have chosen a basis of $W_{(\mathbf{i}')}$. It follows directly from the definition that we may pick representatives in $W_{(\mathbf{i})}$ for the quotient space $W_{(\mathbf{i})}/W_{(\mathbf{i}')}$, of the form $g_1^{i_1}...g_n^{i_n}\eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. We extend the previously constructed basis in $W_{(\mathbf{i}')}$ by adding these representatives. In particular, we have obtained a basis for $W_{(\mathbf{i})}$ and our inductive procedure may go on unless $W_{(\mathbf{i})} = V_N$, in which case we stop. In this way, we have obtained a basis $\{\psi_1, ..., \psi_\Delta\}$ for V_N. Let $\phi_1, ..., \phi_\Delta$ be a fixed basis of V_N. Then $\{\psi_1, ..., \psi_\Delta\}$ can be written as linear forms $L_1, ..., L_\Delta$ in $\phi_1, ..., \phi_\Delta$ so that $\psi_t(f) = L_t(F)$, where $F = (\phi_1(f) : ... : \phi_\Delta(f)) : \mathbb{K} \to \mathbb{P}^{\Delta-1}(\mathbb{K})$. The linear forms $L_1, .., L_\Delta$ are linearly independent, and we know, from the assumption of algebraically non-degeneracy of f, that F is linearly non-degenerate.

For $z \in \mathbb{K}$, we now estimate $\log \prod_{t=1}^{\Delta} |L_t(F)(z)| = \log \prod_{t=1}^{\Delta} |\psi_t(f)(z)|$. Let ψ be an element of the basis, constructed with respect to $W_{(\mathbf{i})}/W_{(\mathbf{i}')}$, then we have $\psi = g_1^{i_1} \cdots g_n^{i_n}\eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. Then we have a bound

$$\begin{aligned}|\psi(f)(z)| &\leqslant |g_1(f)(z)|^{i_1} \cdots |g_n(f)(z)|^{i_n} |\eta(f)(z)| \\ &\leqslant c_2 |g_1(f)(z)|^{i_1} \cdots |g_n(f)(z)|^{i_n} \|f(z)\|^{N-d\sigma(\mathbf{i})},\end{aligned}$$

where c_2 is the positive constant depending only on ψ, not on f and z. Observe that there are precisely $\delta_{(\mathbf{i})}$ such functions ψ in our basis. Hence,

$$\begin{aligned}\log|\psi_t(f)(z)| &\leqslant i_1 \log|g_1(f)(z)| + \cdots + i_n \log|g_n(f)(z)| + (N - d\sigma(\mathbf{i}))\log\|f(z)\| + c_3 \\ &\leqslant i_1\Big(\log|g_1(f)(z)| - \log\|f(z)\|^d\Big) + \cdots \\ &\quad + i_n\Big(\log|g_n(f)(z)| - \log\|f(z)\|^d\Big) + N\log\|f(z)\| + c_3 \\ &\leqslant -i_1 \log\frac{\|f(z)\|^d}{|g_1(f)(z)|} - \cdots - i_n \log\frac{\|f(z)\|^d}{|g_n(f)(z)|} + N\log\|f(z)\| + c_3,\end{aligned}$$

where c_3 is the positive constant, which does not depend on f and r. Therefore,

$$\begin{aligned}\log\prod_{t=1}^{\Delta}|L_t(F)(z)| &= \log\prod_{t=1}^{\Delta}|\psi_t(f)(z)| \qquad (3.5)\\ &\leqslant -\sum_{(\mathbf{i})}\delta_{(\mathbf{i})}\left(i_1\log\frac{\|f(z)\|^d}{|g_1(f)(z)|}+\cdots+i_n\log\frac{\|f(z)\|^d}{|g_n(f)(z)|}\right)\\ &\quad+\Delta N\log\|f(z)\|+\Delta c_3\\ &= -\sum_{j=1}^{n}\log\frac{\|f(z)\|^d}{|g_j(f)(z)|}\Big(\sum_{(\mathbf{i})}\delta_{(\mathbf{i})}i_j\Big)+\Delta N\log\|f(z)\|+\Delta c_3,\end{aligned}$$

where the summations are taken over the n-tuples with $\sigma(\mathbf{i})\leqslant N/d$. Clearly that $\delta := \sum_{(\mathbf{i})}\delta_{(\mathbf{i})}i_j$ does not depend on j, $1\leqslant j\leqslant n$. Hence (3.5) becomes

$$\log\prod_{t=1}^{\Delta}|L_t(F)(z)|\leqslant -\delta\log\prod_{j=1}^{n}\frac{\|f(z)\|^d}{|g_j(f)(z)|}+\Delta N\log\|f(z)\|+\Delta c_3.$$

This implies

$$\begin{aligned}\log\prod_{j=1}^{n}\frac{\|f(z)\|^d}{|g_j(f)(z)|} &\leqslant \frac{1}{\delta}\log\prod_{t=1}^{\Delta}\frac{\|F(z)\|}{|L_t(F)(z)|}-\frac{\Delta}{\delta}\log\|F(z)\| \qquad (3.6)\\ &\quad+\frac{\Delta N}{\delta}\log\|f(z)\|+\frac{\Delta c_3}{\delta}.\end{aligned}$$

Since there are only finitely many choices $\{g_1,...,g_n\}\subset\{Q_1,...,Q_q\}$, we have a finite collection of linear forms $L_1,...,L_u$. From (3.6) we have

$$\begin{aligned}\max_{\{i_1,\dots,i_n\}}\log\prod_{k=1}^{n}\frac{\|f(z)\|_r^d}{|Q_{i_k}(f)(z)|_r} &\leqslant \frac{1}{\delta}\max_K\log\prod_{j\in K}\frac{\|F(z)\|_r}{|L_j(F)(z)|_r}-\frac{\Delta}{\delta}T_F(r)\\ &\quad+\frac{\Delta N}{\delta}T_f(r)+c_4,\end{aligned}$$

where $\max_K$ is taken over all subsets K of $\{1,...,u\}$ such that linear forms $L_j, j\in K$, are linearly independent, c_4 is positive constant independent of r. Applying Theorem 1 to analytic map $F:\mathbb{K}\to\mathbb{P}^{\Delta-1}(\mathbb{K})$ and linear forms $L_1,...,L_u$, and together with (3.4) we have

$$\begin{aligned}\sum_{j=1}^{q}m_f(r,Q_j) &\leqslant \max_{\{i_1,\dots,i_n\}}\log\prod_{k=1}^{n}\frac{\|f(z)\|_r^d}{|Q_{i_k}(f)(z)|_r}+(q-n)\log c_1\\ &\leqslant -\frac{1}{\delta}N_W(r,0)+\frac{\Delta N}{\delta}T_f(r)+O(1),\end{aligned}$$

where W is the Wronskian of $F_1, ..., F_\Delta$. By the First Main Theorem, we have

$$(qd - \frac{\Delta N}{\delta})T_f(r) \leqslant \sum_{j=1}^{q} N_f(r, Q_j) - \frac{1}{\delta} N_W(r, 0) + O(1). \tag{3.7}$$

We will estimate $\sum_{j=1}^{q} N_f(r, Q_j) - \frac{1}{\delta} N_W(r, 0)$ on the right hand side of the above inequality. For each $z \in \mathbb{K}$, without loss of generality, we may assume that $Q_j \circ f$ vanishes at z for $1 \leqslant j \leqslant q_1$ and $Q_j \circ f$ does not vanish at z for $j > q_1$. By the hypothesis Q_j are "in general position", we know $q_1 \leqslant n$. There are integers $k_j \geqslant 0$ and nowhere vanishing analytic functions γ_j in a neighborhood U of z such that

$$Q_j \circ f = (\zeta - z)^{k_j} \gamma_j, \text{ for } j = 1, ..., q,$$

where $k_j = 0$ if $q_1 < j \leqslant q$. For $\{Q_1, ..., Q_n\} \subset \{Q_1, ..., Q_q\}$, we can obtain a basis $\{\psi_1, ..., \psi_\Delta\}$ of V_N and linearly independent linear forms $L_1, ..., L_\Delta$ such that $\psi_t(f) = L_t(F)$. By the property of Wronskian,

$$\begin{aligned} W = W(F_1, ..., F_\Delta) &= CW(L_1(F), ..., L_\Delta(F)) \\ &= C \begin{vmatrix} \psi_1(f) & \dots & \psi_\Delta(f) \\ (\psi_1(f))' & \dots & (\psi_\Delta(f))' \\ \vdots & \ddots & \vdots \\ (\psi_1(f))^{(\Delta-1)} & \dots & (\psi_\Delta(f))^{(\Delta-1)} \end{vmatrix}. \end{aligned}$$

Let ψ be an element of basis, constructed with respect to $W_{(\mathbf{i})}/W_{(\mathbf{i}')}$, so we may write $\psi = Q_1^{i_1} ... Q_n^{i_n} \eta$, $\eta \in V_{N - d\sigma(\mathbf{i})}$. We have

$$\psi(f) = (Q_1(f))^{i_1} ... (Q_n(f))^{i_n} \eta(f),$$

where $(Q_j(f))^{i_j} = (\zeta - z)^{i_j . k_j} \gamma_j^{i_j}$, $j = 1, ..., n$. Also we can assume that $k_j \geqslant \Delta$ if $1 \leqslant j \leqslant q_0$ and $1 \leqslant k_j < \Delta$ if $q_0 < j \leqslant q_1$. And we observe that there are $\delta_{(\mathbf{i})}$ such ψ in our basis. Thus W vanishes at z with order at least

$$\sum_{(\mathbf{i})} \left(\sum_{j=1}^{q_0} i_j (k_j - \Delta) \right) \delta_{(\mathbf{i})} = \sum_{(\mathbf{i})} i_j \delta_{(\mathbf{i})} \sum_{j=1}^{q_0} (k_j - \Delta) = \delta \sum_{j=1}^{q_0} (k_j - \Delta).$$

Therefore,

$$\sum_{j=1}^{q} N_f(r, Q_j) - \frac{1}{\delta} N_W(r, 0) \leqslant \sum_{j=1}^{q} N_f^{\Delta}(r, Q_j). \tag{3.8}$$

We now estimate on the left hand side of the inequality (3.7). Assume that N is divisible by d and $N \geqslant nd$. Then

$$\Delta = \binom{N+n}{n} = \frac{(N+1)...(N+n)}{n!}. \tag{3.9}$$

On the other hand, since the number of non-negative integer m-tuples with sum $\leqslant T$ is equal to the number of non-negative integer $(m+1)$-tuples with sum exactly $T \in \mathbb{Z}$, which is $\binom{T+m}{m}$. It follows from Lemma 1 that,

$$\delta \geqslant \sum_{(\mathbf{i})} i_j \delta_{(\mathbf{i})} = \sum_{\widehat{(\mathbf{i})}} i_j \delta_{(\mathbf{i})} = d^n \sum_{\widehat{(\mathbf{i})}} i_j = \frac{d^n}{n+1} \sum_{\widehat{(\mathbf{i})}} \sum_{j=1}^{n+1} i_j = \frac{d^n}{n+1} \sum_{\widehat{(\mathbf{i})}} (N/d - n) \tag{3.10}$$

$$= \frac{d^n}{n+1} \binom{N/d}{n} (N/d - n) = \frac{N(N-d)...(N-nd)}{(n+1)!d},$$

where the sum $\sum_{(\mathbf{i})}$ is taken over the nonnegative integer $(n+1)$-tuples with sum exactly N/d and $\sum_{\widehat{(\mathbf{i})}}$ is taken over the nonnegative integer $(n+1)$-tuples with sum exactly $N/d - n$. So since (3.9) and (3.10) we have

$$\frac{\Delta N}{\delta} \leqslant (n+1)d \frac{(N+1)...(N+n)}{(N-d)...(N-nd)},$$

therefore

$$(qd - \frac{\Delta N}{\delta}) \geqslant d\Bigg(q - (n+1) - \Bigg(\frac{(N+1)...(N+d)}{(N-d)...(N-nd)} - 1\Bigg)(n+1)\Bigg).$$

It follows, for every $\varepsilon > 0$,

$$(qd - \frac{\Delta N}{\delta})T_f(r) \geq d(q - n - 1 - \varepsilon)T_f(r), \tag{3.11}$$

if we take N large enough such that

$$\frac{(N+1)...(N+n)}{(N-d)...(N-nd)} \leqslant 1 + \frac{\varepsilon}{(n+1)}. \tag{3.12}$$

Combining the formulas (3.7), (3.8), (3.11) and (3.12) together, for each $\varepsilon > 0$, and Δ in Main Theorem, we have

$$d(q - (n+1) - \varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} N_f^{\Delta}(r, Q_j) + O(1),$$

so

$$\sum_{j=1}^{q} \Bigg(1 - \frac{N_f^{\Delta}(r, Q_j)}{dT_f(r)}\Bigg) \leqslant (n + 1 + \varepsilon) + \frac{O(1)}{dT_f(r)}.$$

This is conclusion the proof of Main Theorem.

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