

ON QUASI-BAER RINGS OF ORE EXTENSIONS

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Abstract

Let R be a ring and $S = R[x; \sigma, \delta]$ its Ore extension. We prove under some conditions that R is a quasi-Baer ring if and only if the Ore extension $R[x; \sigma, \delta]$ is a quasi-Baer ring. Examples are provided to illustrate and delimit our results.

1 Introduction

Throughout this paper, R denotes an associative ring with unity. For a subset X of R , $r_R(X) = \{a \in R \mid Xa = 0\}$ and $\ell_R(X) = \{a \in R \mid aX = 0\}$ will stand for the right and the left annihilator of X in R respectively. By [9], a right annihilator of X is always a right ideal, and if X is a right ideal then $r_R(X)$ is a two-sided ideal. An Ore extension of a ring R is denoted by $R[x; \sigma, \delta]$, where σ is an endomorphism of R and δ is a σ -derivation, i.e., $\delta: R \rightarrow R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Recall that elements of $R[x; \sigma, \delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x; \sigma, \delta]$ is given by the multiplication in R and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$. We say that a subset X of R is (σ, δ) -stable if $\sigma(X) \subseteq X$ and $\delta(X) \subseteq X$. A ring R is *(quasi)-Baer* if the right annihilator of every nonempty subset (every right ideal) of R is generated by an idempotent. From [1], an idempotent $e \in R$ is left (resp. right) *semicentral* in R if $exe = xe$ (resp. $exe = ex$), for all $x \in R$. Equivalently, $e^2 = e \in R$ is left (resp. right)

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semicentral if eR (resp. Re) is an ideal of R . Since the right annihilator of a right ideal is an ideal, we see that the right annihilator of a right ideal is generated by a left semicentral in a quasi-Baer ring. We use $\mathcal{S}_\ell(R)$ and $\mathcal{S}_r(R)$ for the sets of all left and right semicentral idempotents, respectively. Also note $\mathcal{S}_\ell(R) \cap \mathcal{S}_r(R) = \mathcal{B}(R)$, where $\mathcal{B}(R)$ is the set of all central idempotents of R . If R is a semiprime ring then $\mathcal{S}_\ell(R) = \mathcal{S}_r(R) = \mathcal{B}(R)$. Recall that R is a *reduced* ring if it has no nonzero nilpotent elements. A ring R is *abelian* if every idempotent of R is central. We can easily observe that every reduced ring is abelian.

According to [10], an endomorphism σ of a ring R is said to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for all $a \in R$. We call a ring R σ -*rigid* if there exists a rigid endomorphism σ of R . Following Hashemi and Moussavi [4], a ring R is σ -*compatible* if for each $a, b \in R$, $a\sigma(b) = 0 \Leftrightarrow ab = 0$. Moreover, R is said to be δ -*compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both σ -compatible and δ -compatible, we say that R is (σ, δ) -*compatible*. A ring R is σ -*rigid* if and only if R is (σ, δ) -compatible and reduced [4, Lemma 2.2]. Also, if R is σ -rigid then $R[x; \sigma, \delta]$ is reduced [10, Theorem 3.3]. From [8], a ring R is said to be a σ -*skew Armendariz* ring if for $p = \sum_{i=0}^n a_i x^i$ and $q = \sum_{j=0}^m b_j x^j$ in $R[x; \sigma]$, $pq = 0$ implies $a_i \sigma^i(b_j) = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. From [5], a ring R is called an (σ, δ) -*skew Armendariz* ring if for $p = \sum_{i=0}^n a_i x^i$ and $q = \sum_{j=0}^m b_j x^j$ in $R[x; \sigma, \delta]$, $pq = 0$ implies $a_i x^i b_j x^j = 0$ for each i, j . Note that (σ, δ) -skew Armendariz rings are generalization of σ -skew Armendariz rings, σ -rigid rings and Armendariz rings, see [8], for more details. It was proved in [7, Corollary 12], that if R is a σ -rigid ring then $R[x; \sigma, \delta]$ is a quasi-Baer ring if and only if R is quasi-Baer. Also in [4, Corollary 2.8], it was shown that, if R is (σ, δ) -compatible, then $R[x; \sigma, \delta]$ is a quasi-Baer ring if and only if R is quasi-Baer.

The aim of this paper is to show that if R is an (σ, δ) -skew Armendariz ring with σ an automorphism such that Re is (σ, δ) -stable for all $e \in \mathcal{S}_\ell(R)$, then R is a quasi-Baer ring if and only if $R[x; \sigma, \delta]$ is a quasi-Baer ring. Many examples are provided to illustrate and delimit results and to show that they are not consequences of [4, Corollary 2.8]. Moreover, we obtain a partial generalization of [7, Corollary 12].

2 Preliminaries and Examples

For any $0 \leq i \leq j$ ($i, j \in \mathbb{N}$), $f_i^j \in \text{End}(R, +)$ will denote the map which is the sum of all possible words in σ, δ built with i letters σ and $j - i$ letters δ (e.g., $f_n^n = \sigma^n$ and $f_0^n = \delta^n$, $n \in \mathbb{N}$). The next lemma appears in [11, Lemma 4.1].

Lemma 2.1. *For any $n \in \mathbb{N}$ and $r \in R$ we have $x^n r = \sum_{i=0}^n f_i^n(r) x^i$ in the ring*

$R[x; \sigma, \delta]$.

Lemma 2.2. [5, Lemma 5]. *Let R be an (σ, δ) -skew Armendariz ring. If $e^2 = e \in R[x; \sigma, \delta]$ where $e = e_0 + e_1x + e_2x^2 + \cdots + e_nx^n$, then $e = e_0$.*

Lemma 2.3. *Let R be a ring, σ an endomorphism and δ be a σ -derivation of R . Then $\sigma(Re) \subseteq Re$ implies $\delta(Re) \subseteq Re$ for all $e \in \mathcal{B}(R)$.*

Proof. Let $e \in \mathcal{B}(R)$ and $r \in R$. Then $\delta(re) = \delta(ere) = \sigma(er)\delta(e) + \delta(er)e = \sigma(ere)\delta(e) + \delta(er)e = se\delta(e) + \delta(er)e$, for some $s \in R$, but $e \in \mathcal{B}(R)$, then $e\delta(e) = e\delta(e)e$, so $\delta(re) = (se\delta(e) + \delta(er))e$. Therefore $\delta(Re) \subseteq Re$. \square

Lemma 2.4. *Let R be a ring, σ an endomorphism of R and δ be a σ -derivation of R . If R is (σ, δ) -compatible. Then for $a, b \in R$, $ab = 0 \Rightarrow af_i^j(b) = 0$ for all $j \geq i \geq 0$.*

Proof. If $ab = 0$, then $a\sigma^i(b) = a\delta^j(b) = 0$ for all $i \geq 0$ and $j \geq 0$, because R is (σ, δ) -compatible. Then $af_i^j(b) = 0$ for all i, j . \square

Lemma 2.5. *Let R be a ring, σ an endomorphism of R and δ be a σ -derivation of R . If R is σ -rigid then R is (σ, δ) -skew Armendariz.*

Proof. If R is σ -rigid then R is (σ, δ) -compatible by [4, Lemma 2.2]. Let $f = \sum_{i=0}^n a_i x^i$, $g = \sum_{j=0}^m b_j x^j \in R[x; \sigma, \delta]$ such that $fg = 0$, then $a_i b_j = 0$ for all i, j , by [7, Proposition 6]. So $a_i f_\ell^j(b_j) = 0$, for all $0 \leq \ell \leq i \leq n$, $0 \leq j \leq m$, by Lemma 2.4. Hence $a_i x^i b_j x^j = \sum_{\ell=0}^i a_i f_\ell^j(b_j) x^{\ell+j} = 0$. Therefore R is (σ, δ) -skew Armendariz. \square

The next example illustrates that there exists a ring R and an automorphism σ of R such that Re is σ -stable for all $e \in \mathcal{S}_\ell(R)$, but R is not σ -rigid.

Example 2.6. [8, Example 1]. *Consider the ring*

$$R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, t \in \mathbb{Q} \right\},$$

where \mathbb{Z} and \mathbb{Q} are the set of all integers and all rational numbers, respectively. The ring R is commutative, let $\sigma: R \rightarrow R$ be an automorphism defined by

$$\sigma \left(\begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix}.$$

(1) R is not σ -rigid.

$$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \sigma \left(\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right) = 0, \text{ but } \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \neq 0, \text{ if } t \neq 0.$$

(2) $\sigma(Re) \subseteq Re$ for all $e \in \mathcal{S}_\ell(R)$. R has only two idempotents:

$e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, let $r = \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \in R$, we have $\sigma(re_0) \in Re_0$ and $\sigma(re_1) \in Re_1$.

Also we have an example of an endomorphism σ of a ring R such that Re is σ -stable for all $e \in \mathcal{S}_\ell(R)$ and R is not σ -compatible.

Example 2.7. Let \mathbb{K} be a field and $R = \mathbb{K}[t]$ a polynomial ring over \mathbb{K} with the endomorphism σ given by $\sigma(f(t)) = f(0)$ for all $f(t) \in R$.

- (1) R is not σ -compatible (so not σ -rigid). Take $f = a_0 + a_1t + a_2t^2 + \dots + a_nt^n$ and $g = b_1t + b_2t^2 + \dots + b_mt^m$, since $g(0) = 0$ so, $f\sigma(g) = 0$, but $fg \neq 0$.
 (2) R has only two idempotents 0 and 1 so Re is σ -stable for all $e \in \mathcal{S}_\ell(R)$.

There is an example of a ring R and an endomorphism σ of R such that R is σ -skew Armendariz and R is not σ -compatible.

Example 2.8. Consider a ring of polynomials over \mathbb{Z}_2 , $R = \mathbb{Z}_2[x]$. Let $\sigma: R \rightarrow R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then:

- (i) R is not σ -compatible. Let $f = \bar{1} + x$, $g = x \in R$, we have $fg = (\bar{1} + x)x \neq 0$, however $f\sigma(g) = (\bar{1} + x)\sigma(x) = 0$.
 (ii) R is σ -skew Armendariz [8, Example 5].

In the next example, $S = R/I$ is a ring and $\bar{\sigma}$ an endomorphism of S such that S is $\bar{\sigma}$ -compatible and not $\bar{\sigma}$ -skew Armendariz.

Example 2.9. Let \mathbb{Z} be the ring of integers and \mathbb{Z}_2 be the ring of integers modulo 4. Consider the ring

$$R = \left\{ \begin{pmatrix} a & \bar{b} \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, \bar{b} \in \mathbb{Z}_4 \right\}.$$

Let $\sigma: R \rightarrow R$ be an endomorphism defined by $\sigma \left(\begin{pmatrix} a & \bar{b} \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -\bar{b} \\ 0 & a \end{pmatrix}$.

Take the ideal $I = \left\{ \begin{pmatrix} a & \bar{0} \\ 0 & a \end{pmatrix} \mid a \in 4\mathbb{Z} \right\}$ of R . Consider the factor ring

$$R/I \cong \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{pmatrix} \mid \bar{a}, \bar{b} \in 4\mathbb{Z} \right\}.$$

(1) R/I is not $\bar{\sigma}$ -skew Armendariz. In fact, $\left(\begin{pmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{pmatrix} + \begin{pmatrix} \bar{2} & \bar{1} \\ 0 & \bar{2} \end{pmatrix} x \right)^2 = 0 \in$

$(R/I)[x; \bar{\sigma}]$, but $\begin{pmatrix} \bar{2} & \bar{1} \\ 0 & \bar{2} \end{pmatrix} \bar{\sigma} \begin{pmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{pmatrix} \neq 0$.

(2) R/I is $\bar{\sigma}$ -compatible. Let $A = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{pmatrix}$, $B = \begin{pmatrix} \bar{a}' & \bar{b}' \\ 0 & \bar{a}' \end{pmatrix} \in R/I$. If $AB = 0$ then $\overline{aa'} = 0$ and $\overline{ab'} = \overline{ba'} = 0$, so that $A\bar{\sigma}(B) = 0$. The same for the converse. Therefore R/I is $\bar{\sigma}$ -compatible.

3 Ore extensions over quasi-Baer rings

It was proved in [1, Theorem 1.2], that if R is a quasi-Baer ring and σ an automorphism of R then $R[x; \sigma]$ is a quasi-Baer ring. The following example shows that “ σ is an automorphism” is not a superfluous condition in Proposition 3.2.

Example 3.1. [6, Example 2.8]. *There is an example of a quasi-Baer ring R and an endomorphism σ of R such that $R[x; \sigma]$ is not a quasi-Baer ring. In fact, let $R = \mathbb{K}[t]$ be the polynomial ring over a field \mathbb{K} and σ be the endomorphism given by $\sigma(f(t)) = f(0)$. Then the ring $R[x; \sigma]$ is not a quasi-Baer ring.*

Proposition 3.2. *Let R be a ring, σ an automorphism and δ be a σ -derivation of R . Suppose that Re is (σ, δ) -stable for all $e \in \mathcal{S}_\ell(R)$. If R is quasi-Baer then the Ore extension $R[x; \sigma, \delta]$ is quasi-Baer.*

Proof. Let $S = R[x; \sigma, \delta]$ and I be an ideal of S . We claim that $r_S(I) = eS$, for some idempotent $e \in R$. We can suppose that $I \neq 0$, we set

$$I_0 = \{0\} \cup \{a \in R \mid \exists a_0, a_1, \dots, a_{n-1} \in R \text{ such that } ax^n + \sum_{i=0}^{n-1} a_i x^i \in I, n \in \mathbb{N}\}.$$

It is clear that I_0 is a nonzero left ideal of R . Given $a \in I_0$ and $r \in R$, there is an element in I of the form $ax^n + \sum_{i=0}^{n-1} a_i x^i$. Multiplying on the right

by $\sigma^{-n}(r)$ gives an element of the form $arx^n + \sum_{i=0}^{n-1} b_i x^i$, for some elements

$b_0, b_1, \dots, b_{n-1} \in R$, and so $ar \in I_0$, thus I_0 is a two-sided ideal. So there exists an idempotent $e \in R$ such that $r_R(I_0) = eR$. We have $eS \subseteq r_S(I)$. To

see this, let $0 \neq f(x) = \sum_{k=0}^n a_k x^k \in I$, then $f(x)e = \sum_{k=0}^n (\sum_{i=k}^n a_k f_k^i(e)) x^k$, where

f_k^i are sums of all possible words in σ, δ built with k letters σ and $i-k$ letters δ . Re is f_k^i -stable ($0 \leq k \leq i$), so there exists $u_k^i \in R$ such that $f_k^i(e) = u_k^i e$ ($0 \leq$

$k \leq i$). Therefore $f(x)e = \sum_{k=0}^n (\sum_{i=k}^n a_k u_k^i) e x^k$, if we set $\alpha_k = \sum_{i=k}^n a_k u_k^i e$, then

$f(x)e = \sum_{k=0}^n \alpha_k x^k$. If $\alpha_n \neq 0$, then $\alpha_n \in I_0$ and so, $\alpha_n e = \alpha_n = 0$ (because

$r_R(I_0) = eR$). Contradiction, hence $\alpha_n = 0$. Now suppose that $\alpha_j = 0$ for

$j = n, n-1, \dots, k+1$ with $k \in \mathbb{N}$. But $f(x)e = \alpha_k x^k + \sum_{\ell=0}^{k-1} \alpha_\ell x^\ell$, with the same

manner as above we have $\alpha_k = 0$. So we can get $\alpha_n = \alpha_{n-1} = \dots = \alpha_0 = 0$.

Consequently $eS \subseteq r_S(I)$.

Conversely, we can claim that $r_S(I) \subseteq eS$. Let $0 \neq f(x) = \sum_{k=0}^n a_k x^k \in I$

and $\lambda(x) = \sum_{j=0}^m b_j x^j \in S$, such that $f(x)\lambda(x) = 0$, we shall show that $\lambda(x) =$

$\sigma^{-n}(e)\lambda(x)$. If we set $\xi(x) = \lambda(x) - \sigma^{-n}(e)\lambda(x) = \sum_{j=0}^m (b_j - \sigma^{-n}(e)b_j)x^j$, we have

$$f(x)\xi(x) = \left(\sum_{i=0}^n a_i x^i\right) \left(\sum_{j=0}^m (b_j - \sigma^{-n}(e)b_j)x^j\right) = a_n \sigma^n(b_m - \sigma^{-n}(e)b_m)x^{n+m} + Q = 0,$$

where Q is a polynomial with $\deg(Q) < n+m$. Thus $a_n \sigma^n(b_m - \sigma^{-n}(e)b_m) = 0$, since $a_n \neq 0$, then $a_n \in I_0$. Hence $\sigma^n(b_m - \sigma^{-n}(e)b_m) \in r_R(I_0) = eR$. So $\sigma^n(b_m - \sigma^{-n}(e)b_m) = e\sigma^n(b_m - \sigma^{-n}(e)b_m)$, then $b_m - \sigma^{-n}(e)b_m = \sigma^{-n}(e)(b_m - \sigma^{-n}(e)b_m) = 0$ (because $\sigma^{-n}(e)$ is idempotent), hence $b_m - \sigma^{-n}(e)b_m = 0$. Now, suppose that $b_j - \sigma^{-n}(e)b_j = 0$ for $j = m, m-1, \dots, k+1$ with $k \in \mathbb{N}$ and showing that $b_k - \sigma^{-n}(e)b_k = 0$. Effectively, $f(x)\xi(x) = a_n \sigma^n(b_k - \sigma^{-n}(e)b_k)x^{n+k} + Q' = 0$, where Q' is a polynomial with $\deg(Q') < n+k$, then $a_n \sigma^n(b_k - \sigma^{-n}(e)b_k) = 0$, with the same manner as below, we obtain $b_k - \sigma^{-n}(e)b_k = 0$. Therefore $b_j - \sigma^{-n}(e)b_j = 0$ for all $0 \leq j \leq m$, then $\xi(x) = 0$. But $\lambda(x) = \sigma^n(e)\lambda(x)$ or $\sigma^n(e) = ue$ for some $u \in R$, but e is left semicentral then $\lambda(x) = eue\lambda(x)$. Hence $r_S(I) \subseteq eS$. So $R[x; \sigma, \delta]$ is a quasi-Baer ring. \square

In Example 2.7, Re is (σ, δ) -stable for all $e \in \mathcal{S}_\ell(R)$ but R is not (σ, δ) -compatible. Thus, Proposition 3.2 is not a consequence of [4, Corollary 2.8].

There is a quasi-Baer ring R , σ an automorphism of R and δ a σ -derivation of R such that Re is (σ, δ) -stable for all $e \in \mathcal{S}_\ell(R)$.

Example 3.3. Consider the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, where \mathbb{Z} is the set of all integers numbers. By [2, Example 1.3(ii)], R is a quasi-Baer ring. Define $\sigma: R \rightarrow R$ and $\delta: R \rightarrow R$ by

$$\sigma \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, \quad \delta \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix} \text{ for all } a, b, c \in \mathbb{Z}.$$

Clearly, σ is an automorphism of R and δ is a σ -derivation. The nonzero idempotents of R are of the form

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix},$$

where $t \in \mathbb{Z}$. e_2 is right semicentral not left semicentral and e_1 is left semicentral not right semicentral, so the only left semicentral nonzero idempotents of R are e_0 and e_1 . Re_0 is (σ, δ) -stable. Let $r = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$, since $\sigma(re_1) = \begin{pmatrix} x & -xt \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}$, then Re_1 is σ -stable, also Re_1 is δ -stable. Therefore Re is (σ, δ) -stable for all $e \in \mathcal{S}_\ell(R)$.

Example 3.4. Consider the ring $S = \begin{pmatrix} D & D \oplus D \\ 0 & D \end{pmatrix}$, where D is a simple domain which is not a division ring. By [3, Example 4.11], R is a quasi-Baer ring and has nonzero idempotents of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & (b, d) \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & (b, d) \\ 0 & 1 \end{pmatrix},$$

where $b, d \in D$, with σ and δ as in Example 3.3, Re is (σ, δ) -stable for all $e \in \mathcal{S}_\ell(R)$.

Corollary 3.5. Let R be an abelian or a semiprime ring, σ an automorphism and δ be a σ -derivation of R , such that $\sigma(Re) \subseteq Re$ for all $e \in \mathcal{B}(R)$. If R is quasi-Baer then $R[x; \sigma, \delta]$ is quasi-Baer.

Proof. By Lemma 2.3 and Proposition 3.2. □

In the remainder of this section we focus on the converse of Proposition 3.2. We begin with the next example which shows that there exists a ring R and a derivation δ of R such that $R[x; \delta]$ is quasi-Baer but R is not quasi-Baer.

Example 3.6. [1, Example 1.6]. There is a ring R and a derivation δ of R such that $R[x; \delta]$ is a Baer ring. But R is not quasi-Baer. Let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t}) = 1$ where $\bar{t} = t + (t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field \mathbb{Z}_2 of two elements. Consider the Ore extension $R[x; \delta]$. If we set $e_{11} = \bar{t}x$, $e_{12} = \bar{t}$, $e_{21} = \bar{t}x^2 + x$ and $e_{22} = 1 + \bar{t}x$ in $R[x; \delta]$, then they form a system of matrix units in $R[x; \delta]$. Now the centralizer of these matrix units in $R[x; \delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $R[x; \delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So the ring $R[x; \delta]$ is a Baer ring, but R is not quasi-Baer.

Proposition 3.7. Let R be an (σ, δ) -skew Armendariz ring. If $R[x; \sigma, \delta]$ is quasi-Baer then R is quasi-Baer.

Proof. Let I be an ideal of R and $S = R[x; \sigma, \delta]$, then since S is quasi-Baer, there exists an idempotent $e \in S$ such that $r_S(IS) = eS$ with $e = e_0 + e_1x + \dots + e_nx^n$ ($n \in \mathbb{N}$). By Lemma 2.2, we have $e_0 \in r_R(I)$. Thus $e_0R \subseteq r_R(I)$.

Conversely, let $a \in r_R(I)$ then $a \in r_S(IS) \cap R = e_0S \cap R$, so $a = e_0f$ for some $f = f_0 + f_1x + \dots + f_mx^m \in S$. Then $a = e_0f_0$ and so $a \in e_0R$. Therefore $r_R(I) \subseteq e_0R$. Consequently, R is a quasi-Baer ring. □

By Example 2.8, there is a ring R and σ an endomorphism of R such that R is σ -skew Armendariz and R is not σ -compatible. So that, Proposition 3.7 is not a consequence of [4, Corollary 2.8]. By the next result, we see that Proposition 3.7 is a partial generalization of [7, Corollary 12].

Corollary 3.8. *Let R be an σ -rigid ring. If $R[x; \sigma, \delta]$ is quasi-Baer then R is quasi-Baer.*

Proof. It follows from Lemma 2.5 and Proposition 3.7. □

One might expect the converse of Proposition 3.2 to hold when R is a (σ, δ) -skew Armendariz ring. However [8, Example 5] and [6, Example 2.8], shows that this converse does not hold in general.

Example 3.9. *We consider a commutative polynomial ring over \mathbb{Z}_2 . $R = \mathbb{Z}_2[x]$, let $\sigma: R \rightarrow R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. By [6, Example 2.8], $R[x; \sigma]$ is not quasi-Baer and R is quasi-Baer. But, by [8, Example 5], R is σ -skew Armendariz. Note that R has only two idempotents 0 and 1, so $\sigma(Re) \subseteq Re$ for all $e \in \mathcal{S}_\ell(R)$. Thus “ σ is an automorphism” is not a superfluous condition in the next theorem.*

Theorem 3.10. *Let R be a (σ, δ) -skew Armendariz ring with σ an automorphism such that Re is (σ, δ) -stable for all $e \in \mathcal{S}_\ell(R)$. Then R is a quasi-Baer ring if and only if $R[x; \sigma, \delta]$ is a quasi-Baer ring.*

Proof. It follows immediately from Proposition 3.2 and Proposition 3.7. □

Example 3.11. *Let $R = \mathbb{C}$ where \mathbb{C} is the field of complex numbers. Then R is a Baer (so quasi-Baer) reduced ring. Define $\sigma: R \rightarrow R$ and $\delta: R \rightarrow R$ by $\sigma(z) = \bar{z}$ and $\delta(z) = z - \bar{z}$, where \bar{z} is the conjugate of z . σ is an automorphism of R and δ is a σ -derivation. R has only two idempotents 0 and 1, so we have the stability indicated in Theorem 3.10.*

We claim that R is a (σ, δ) -skew Armendariz ring. Consider $R[x; \sigma, \delta]$. Let $p = a_0 + a_1x + \cdots + a_nx^n$ and $q = b_0 + b_1x + \cdots + b_mx^m \in R[x; \sigma, \delta]$. Assume that $pq = 0$. Since R is σ -rigid, we have $a_i b_j = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$, by [7, Proposition 6]. thus $a_i x^i b_j x^j = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$, because $R[x; \sigma, \delta]$ is reduced, by [10, Theorem 3.3].

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