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ON COMMUTATIVITY OF SEMIRINGS

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Abstract

We prove the following results. (1) If R is a semiring such that $(ab)^k = a^k b^k$ for all $a, b \in R$ and (i) fixed non negative integers $k = n,\ n+1,\ n+2$ or (ii) fixed positive integers $k = m,\ m+1,\ n,\ n+1$ where $(m, n) = 1$ then R is semicommutative. If R is also additively cancellative then R is commutative. Thus we generalize the results of [7] and [2]. (2) If R is a $(n+1)!$ – torsion free semiring such that $(ab)^n + b^n a^n = (ba)^n + a^n b^n$ is central for all $a, b \in R$ then R is semicommutative. (3) If R is a $n!$ – torsion free semiring such that $a^n b + b^n a = ba^n + ab^n$ for all $a, b \in R$ or $(ab)^n = a^n b^n$ for all $a, b \in R$ then R is semicommutative.

For the definition of semiring we refer [4]. All semirings in this paper are with an identity element. $\mathbb{Z}_0^+$ will denote the set of all non negative integers. A finite additively cancellative semiring (not necessarily with an identity element) is a ring. If R is an additively cancellative semiring (not necessarily with an identity element) such that $a^m = a$ for all $a \in R$ and fixed integer $m > 1$ then R is a ring: For, let $a \in R$. Then $(2a)^m = 2a$. This can be written as $(2^m - 2)a + 2a = 2a$. Thus $(2^m - 2)a = 0$. Moreover we prove the following.

Theorem 1. *Let R be an additively cancellative semiring such that $a^m = a^n$ for all $a \in R$ and fixed positive integers m, n $(m \neq n)$. Then R is a ring.*

Proof Let $a \in R$. We have $2^m a^m = (2a)^m = (2a)^n = 2^n a^n$. Assume that $m > n$. Therefore $(2^m - 2^n)a^n + 2^n a^n = 2^n a^n$. Hence $(2^m - 2^n)a^n = 0$.

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Replacing a by $a+1$, expanding by Binomial Theorem and multiplying by a^{n-1}, we get $(2^m - 2^n)a^{n-1} = 0$. Continuing in this way, we get $(2^m - 2^n)a = 0$.

Example 2. Let $R = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in (\mathbb{Z}_0^+, +, \cdot) \right\}$ Then R is an additively cancellative semiring without identity and $A^4 = A^3$ for all $A \in R$. But R is not a ring.

A semiring R will be called semicommutative if for each a, $b \in R$, there exists $x \in R$ such that $ab + x = ba + x$.

Example 3. Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in (\{0, 1\}, \max, \min) \right\}$. Then R is a semicommutative semiring but not commutative.

The following theorem generalizes the theorem of Ligh and Richoux [7] and a theorem of Bell [2].

Theorem 4. *Let R be a semiring such that*

$$(ab)^k = a^k b^k \quad \textit{for all} \quad a, b \in R \tag{$*$}$$

and (i) *fixed non negative integers $k = n,\ n+1,\ n+2$ or* (ii) *fixed positive integers $k = m,\ m+1,\ n,\ n+1$ where $(m, n) = 1$. Then R is semicommutative.*

Proof (i) Assume that $n > 0$. From first and second equations in $(*)$, we get

$$a^n b^n ab = a^{n+1} b^{n+1} \,. \tag{1}$$

Replacing a by $a+1$ in (1), expanding by Binomial Theorem, then multiplying by a^{n-1} from left and using (1), we get $a^{n-1}b^n ab + x_1 = a^n b^{n+1} + x_1$ for some $x_1 \in R$. Continuing this process, we have

$$b^n ab + x_n = ab^{n+1} + x_n \quad \text{for some } x_n \in R \,. \tag{2}$$

Similarly from second and third equations in $(*)$, we get

$$b^{n+1} ab + y_{n+1} = ab^{n+2} + y_{n+1} \quad \text{for some} \quad y_{n+1} \in R \,. \tag{3}$$

Adding bx_n in (3) , multiplying (2) by b from left and then using it, we get

$$bab^{n+1} + z = ab^{n+2} + z \quad \text{where} \quad z = bx_n + y_{n+1} \in R \,. \tag{4}$$

Applying the argument as above, we have $ba + u = ab + u$ for some $u \in R$. If $n = 0$ then the result follows easily.
(ii) There exist positive integers r and s such that $1 + rn = sm$. Applying the argument as in (i), we get

$$\begin{aligned} b^n ab + x_1 &= ab^{n+1} + x_1 & (1) \\ b^m ab + x_2 &= ab^{m+1} + x_2 & (2) \end{aligned}$$

for some $x_1, x_2 \in R$. Replacing b by b^r in (1) and b by b^s in (2), we get

$$\begin{aligned} b^{r\,n}ab^r + x_3 &= ab^{r\,n+r} + x_3 && (3)\\ b^{s\,m}ab^s + x_4 &= ab^{s\,m+s} + x_4 && (4)\end{aligned}$$

for some $x_3, x_4 \in R$. Now (4) can be written as

$$b^{1+r\,n}ab^s + x_4 = ab^{s\,m+s} + x_4\,. \tag{5}$$

If $r \geq s$ then multiplying (5) by b^{r-s} from right, we get

$$b^{1+r\,n}ab^r + x_5 = ab^{s\,m+r} + x_5 \quad \text{where } x_5 = x_4b^{r-s} \in R\,. \tag{6}$$

Adding bx_3 in (6) and using (3), we get $bab^{r\,(n+1)} + x_6 = ab^{r\,(n+1)+1} + x_6$ where $x_6 = bx_3 + x_5 \in R$. Hence $ba + x_7 = ab + x_7$ for some $x_7 \in R$. If $s \geq r$ then adding bx_3b^{s-r} in (5) and using (3), we get $bab^{r\,n+s} + x_8 = ab^{1+r\,n+s} + x_8$ where $x_8 = bx_3b^{s-r} + x_4 \in R$. Now $ba + x_9 = ab + x_9$ for some $x_9 \in R$.

Theorem 5. *Let R be a semiring such that $(ab)^{k-1} + ba^{k-1} = (ba)^{k-1} + a^{k-1}b$ is central for all $a, b \in R$ and fixed positive integers $k = m$, n where $(m, n) = 1$. Then R is semicommutative.*

Proof Let $(ab)^{k-1} + ba^{k-1} = (ba)^{k-1} + a^{k-1}b = z$ for some central element z of R. Then $a^kb + x = ba^k + x$ where $x = (ab)^{k-1}a \in R$. Similarly as above, R is semicommutative.

Definition 6. Let $n > 1$ be an integer. A semiring R will be called n-torsion free if for any $a, b, x \in R$, $na + x = nb + x$ implies that $a + y = b + y$ for some $y \in R$.

Theorem 7. *Let R be a $(n+1)!$-torsion free semiring such that $(ab)^n + b^na^n = (ba)^n + a^nb^n$ is central for all $a, b \in R$. Then R is semicommutative.*

Proof $(ab)^n + b^na^n = (ba)^n + a^nb^n = z$ for some central element z of R. Then $a^{n+1}b^n + x = b^na^{n+1} + x$ where $x = (ab)^na \in R$. Replacing a by $a+1$, expanding by Binomial Theorem and using above equation we get, $\alpha b^n + y = b^n\alpha + y$ for some $y = a^{n+1}b^n + x + b^n + u$ and $u \in R$ where $\alpha = \displaystyle\sum_{i=1}^{n}\binom{n+1}{i}a^{n+1-i}$. Again replacing a by a+l and repeating the process, after nth step we get $(n+1)!\,ab^n + v = (n+1)!\,b^na + v$ for some $v \in R$. Repeating the same technique we get $(n+1)!\,n!\,ab + w = (n+1)!\,n!\,ba + w$ for some $w \in R$.

Theorem 8. *Let R be a $n!$-torsion free semiring such that $a^nb + b^na = ba^n + ab^n$ for all $a, b \in R$ or $(ab)^n = a^nb^n$ for all $a, b \in R$. Then R is semicommutative.*

Proof (i) Replaying a by $a+1$, expanding by Binomial Theorem and using above equation we get $\left(\displaystyle\sum_{i=1}^{n-1}\binom{n}{i}a^{n-i}\right)b + x = b\left(\displaystyle\sum_{i=1}^{n-1}\binom{n}{i}a^{n-i}\right) + x$ for

some $x \in R$. Again repeating the process, after $(n-1)$th step we obtain $(n!)\,ab + y = (n!)ba + y$ for some $y \in R$.
(ii) Let $g(u, j; v, k)$ be a monomial in u and v where the sum of powers of the factors u is j and the sum of powers of the factors v is k. Let us take $u = ab$ and $v = b$. Replacing a by $(a+1)$, expanding by Binomial Theorem and using the equation we get $g(ab, n-1; b, 1) + g(ab, n-2; b, 2) + \ldots + g(ab, 1; b, n-1)$ $+ x_1 = \left(\sum_{i=1}^{n-1} \binom{n}{i} a^{n-i}\right) b^n + x_1$ for some $x_1 \in R$. Again replacing a by $(a+1)$ and repeating the process, we obtain $g(ab, n-2; b, 2) + g(ab, n-3; b, 3) + \ldots +$ $g(ab, 1; b, n-1) + x_2 = \left(\sum_{i=1}^{n-1} \binom{n}{i} \left(\sum_{j=1}^{n-i-1} \binom{n-i}{j} a^{n-i-j}\right)\right) b^n + x_2$ for some $x_2 \in R$. Repeating the process, after nth step, we obtain $g(ab, 1; b, n-1)$ $+ x_n = (n!)ab^n + x_n$ for some $x_n \in R$ where $g(ab, 1; b, n-1) = (n-1)!(ab)b^{n-1} +$ $(n-1)!b(ab)b^{n-2} + \ldots + (n-1)!b^{n-1}(ab)$. Repeating the same technique, we obtain $(n!)^2 ba + y_n = (n!)^2 ab + y_n$ for some $y_n \in R$.

Theorem 9. *Let R be $(n!)$-torsion free semiring such that $a^n b^m = b^m a^n$ for all $a, b \in R$ and fixed integers $n \geq m \geq 1$. Then R is semicommutative.*

Proof Similarly as above.

The existence of an identity element in the theorems 1, 4, 5, 7, 8 and 9 is essential (see Example 2).

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