East-West J. of Mathematics: Vol. 20, No 1 (2018) pp. 86-106

SOME FIXED POINT THEOREMS FOR GENARALIZED α-β-FG-CONTRACTIONS IN b-METRIC SPACES AND CERTAIN APPLICATIONS TO THE INTEGRAL EQUATIONS

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Abstract

The purpose of this paper is to introduce a generalized α-β-FG-contraction and prove some fixed point theorems in b-metric spaces. These theorems are generalizations of some results in literature. An example is given to illustrate the obtained results and to show that these results are proper extensions of the existing ones. Then we apply the obtained theorem to study the existence of solutions of the integral equation.

1 Introduction and preliminaries

In recent times, there have been many types of contractions to ensure the existence and uniqueness of the fixed point of mapings in various spaces. In 2012 Wardowski [11] introduced the notion of an F-contraction and proved fixed point results in metric spaces as generalization of the Banach contraction principle. After that, the notion of an F-contraction has been generalized and considered under the name F-weak contraction or F-generalized contraction. Moreover, the fixed point results for F-weak contraction in metric spaces were

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Key words: α-admissible mapping, FG-contraction, b-metric, fixed point.
2010 AMS Mathematics classification: Primary 47H10, 54H25; Secondary 47J25, 65J15

also established and proved by many authors in [9, 10, 12]. Later, F-weak contraction was continuously extended to generalized F-contraction in [3].

In 2016 Parvaneh *et al.* [7] introduced the notion of an α-β-FG-contraction and stated some fixed point theorems in b-metric spaces. The results extended those of Wardowski and several authors in b-metric spaces.

In this paper, we introduce the notion of a generalized α-β-FG-contraction and prove some fixed point results in b-metric spaces. An example is given to illustrate the obtained results and to show that these results are proper extensions of the existing ones. Then we also apply the obtained theorem to study the existence of solutions of the integral equation.

Now, we recall some notions and lemmas which will be useful in what follows.

Definition 1.1 ([11], Definition 2.1). Let $\mathcal{F}$ be the family of all functions $F : (0, \infty) \longrightarrow \mathbb{R}$ such that

(F1) F is strictly increasing;

(F2) For each sequence $\{\alpha_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} (\alpha^k F(\alpha)) = 0$.

Definition 1.2 ([3], Definition 7). Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. Then T is called a *generalized F-contraction* on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \le F\big(M(x, y)\big)$$

where

$$\begin{aligned} M(x,y) \quad = \quad & \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2} \\ & d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty, \frac{d(T^2x,x)+d(T^2x,Ty)}{2}\Big\}. \end{aligned}$$

Definition 1.3 ([2], Definition 2.7). Let X be a nonempty set, $s \ge 1$ be a real number and $D : X \times X \longrightarrow [0, \infty)$ satisfy the following properties.

1. $D(x, y) = 0$ if and only if $x = y$;
2. $D(x, y) = D(y, x)$ for all $x, y \in X$;
3. $D(x, z) \le s\big[D(x, y) + D(y, z)\big]$ for all $x, y, z \in X$.

Then (X, D, s) is called a *b-metric space.*

Definition 1.4 ([7], Definition 2). Let $\Delta\mathcal{F}$ be a family of all functions $F : (0,\infty) \longrightarrow \mathbb{R}$ such that

(F1) F is continuous and increasing;

(F2) For each sequence $\{\alpha_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty.$$

Let Δ be a family all of pairs (G,β), where $G : (0,\infty) \longrightarrow \mathbb{R}, \beta : [0,\infty) \longrightarrow [0,1)$, such that

(G1) For each sequence $\{t_n\} \subset (0,\infty)$, $\limsup_{n\to\infty} G(t_n) \geq 0$ if and only if $\limsup_{n\to\infty} t_n \geq 1$;

(G2) For each sequence $\{t_n\} \subset [0,\infty)$, $\limsup_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$;

(G3) For each sequence $\{t_n\} \subset (0,\infty)$, $\sum_{n=1}^{\infty} G\big(\beta(t_n)\big) = -\infty$.

Example 1.5 ([7], Example 2.1). Let $k \in (0,1), F(t) = G(t) = \ln t$ for all $t \in (0,\infty)$ and $\beta(t) = k \in (0,1)$ for all $t \in [0,\infty)$. Then $F \in \Delta_{\mathcal{F}}$ and $(G,\beta) \in \Delta$.

Definition 1.6 ([7], Definition 2.2). Let (X,D,s) be a b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0,\infty)$ be a function. Then T is called an *α-β-FG-contraction* if for all $x,y \in X$ with $\alpha(x,y) \geq 1$ and $D(Tx,Ty) > 0$, we have

$$F\big(sD(Tx,Ty)\big) \leq F\big(M_s(x,y)\big) + G\big(\beta(M_s(x,y))\big) \tag{1.1}$$

where $F \in \Delta\mathcal{F}$, $(G,\beta) \in \Delta$ and

$$M_s(x,y) = \max\Big\{D(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2s}\Big\}.$$

Definition 1.7 ([6], Definition 7). Let (X,D,s) be a b-metric space.

1. A sequence $\{x_n\}$ is called *convergent* to $x \in X$ if $\lim_{n\to\infty} D(x_n,x) = 0$.
2. A sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} D(x_n,x_m) = 0$.
3. (X,D,s) is called *complete* if every Cauchy sequence is a convergent sequence.

Lemma 1.8 ([1], Lemma 1). *Let (X, D, s) be a b-metric space and $\{x_n\}$, $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = v$. Then*

1. $\frac{1}{s^2}D(u,v) \le \liminf_{n\to\infty} D(x_n, y_n) \le \limsup_{n\to\infty} D(x_n, y_n) \le s^2 D(u,v)$.
2. $\frac{1}{s}D(u,y) \le \liminf_{n\to\infty} D(x_n, y) \le \limsup_{n\to\infty} D(x_n, y) \le sD(u,y)$ *for all* $y \in X$.

Definition 1.9 ([5], Definition 1). Let X be a nonempty set, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be a function. Then T is called *a triangular α-admissible mapping* if for all $x, y \in X$,

(T1) $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$;

(T2) $\alpha(x, z) \ge 1$ and $\alpha(z, y) \ge 1$ imply $\alpha(x, y) \ge 1$.

Definition 1.10 ([7], Definition 1.3). Let (X, D, s) be a b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be two functions. Then T is called an *α-continuous mapping* on (X, D, s) if for all $x \in X$,

$$\lim_{n\to\infty} x_n = x \text{ and } \alpha(x_n, x_{n+1}) \ge 1 \text{ for all } n \in \mathbb{N} \text{ imply } \lim_{n\to\infty} Tx_n = Tx.$$

Lemma 1.11 ([5], Lemma 7). *Let X be a nonempty set, $T : X \longrightarrow X$ be a triangular α-admissible mapping and $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(x_n, x_m) \ge 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

2 Main results

By adding four terms

$$D(T^2x, Tx), \frac{D(T^2x, y)}{s}, \frac{D(T^2x, Ty)}{s^2}, \frac{D(T^2x, x) + D(T^2x, Ty)}{2s^2}$$

to the notion of an α-β-FG-contraction, we introduce the notion of a generalized α-β-FG-contraction in b-metric spaces as follows.

Definition 2.1. Let (X, D, s) be a b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be two functions. Then T is called a *generalized α-β-FG-contraction* if there exist $F \in \Delta\mathcal{F}$ and $(G, \beta) \in \Delta$ such that for all $x, y \in X$ with $\alpha(x, y) \ge 1$ and $D(Tx, Ty) > 0$,

$$F\big(sD(Tx, Ty)\big) \le F\big(L_s(x, y)\big) + G\big(\beta(L_s(x, y))\big) \tag{2.1}$$

where

$$\begin{aligned} & L_s(x,y) \\ = \;& \max\Big\{D(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2s}, D(T^2x,Tx), \\ & \frac{D(T^2x,y)}{s}, \frac{D(T^2x,Ty)}{s^2}, \frac{D(T^2x,x)+D(T^2x,Ty)}{2s^2}\Big\}. \end{aligned} \tag{2.2}$$

The following example shows that there exists a generalized α-β-FG-contraction which is not an α-β-FG-contraction.

Example 2.2. Let $X = \{1, 2, 3, 4, 5\}$ and

$$D(x,y) = \begin{cases} 0 \text{ if } (x,y) \in \{(1,1); (2,2); (3,3); (4,4); (5,5)\} \\ 1 \text{ if } (x,y) \in \{(1,2); (1,3); (2,1); (3,1)\} \\ 2 \text{ if } (x,y) \in \{(2,3), (3,2)\} \\ 10 \text{ if } (x,y) \in \{(1,4); (1,5); (4,1); (5,1)\} \\ 4 \text{ otherwise.} \end{cases}$$

Define $T : X \to X, \alpha : X \times X \to [0,\infty), \beta : [0,\infty) \to [0,1)$ by

$$T1 = T2 = T3 = 1; T4 = 2, T5 = 3,$$

$$\alpha(x,y) = 2 \text{ for all } x, y \in X,$$

$$F(t) = G(t) = \ln t \text{ for all } t \in (0,\infty) \text{ and } \beta(t) = \frac{4}{5} \text{ for all } t \in [0,\infty).$$

Then

(1) (X, D, s) is a b-metric space with $s = 2$.

(2) T is a generalized α-β-FG-contraction but T is not an α-β-FG-contraction.

Proof. (1). It is easy to prove that (X, D, s) is a b-metric space with $s = 2$.

(2). Since $\alpha(x,y) = 2 > 1$ for all $x, y \in X$. Since $F(t) = G(t) = \ln t$ for all $t \in (0,\infty)$ and $\beta(t) = \frac{4}{5}$ for all $t \in [0,\infty)$, (2.1) becomes

$$sD(Tx,Ty) \le \frac{4}{5} L_s(x,y) \tag{2.3}$$

where $L_s(x,y)$ is defined by (2.2). For $D(Tx,Ty) > 0$, we obtain the following table.

x	y	$D(Tx,Ty)$	$sD(Tx,Ty)$	$\frac{4}{5}L_s(x,y)$ is equal or greater than
1	4	1	2	$D(1,4)=8$
1	5	1	2	$D(1,5)=8$
2	4	1	2	$D(2,4)=\frac{16}{5}$
2	5	1	2	$D(2,5)=\frac{16}{5}$
3	4	1	2	$D(3,4)=\frac{16}{5}$
3	5	1	2	$D(3,5)=\frac{16}{5}$
4	1	1	2	$D(4,1)=8$
4	2	1	2	$D(4,2)=\frac{16}{5}$
4	3	1	2	$D(4,3)=\frac{16}{5}$
4	5	2	4	$D(T^2 4,5)=8$
5	1	1	2	$D(5,1)=8$
5	2	1	2	$D(5,2)=\frac{16}{5}$
5	3	1	2	$D(5,3)=\frac{16}{5}$
5	4	2	4	$D(T^2 5,4)=8$

From the above table, the inequality (2.3) is satisfied for all $x, y \in X$ with $\alpha(x,y) \geq 1$ and $D(Tx,Ty) > 0$. Therefore, T is a generalized α-β-FG-contraction.

However, let $x = 4, y = 5$, we have $D(T4,T5) = D(2,3) = 2$ and

$$M_s(x,y) = \max\left\{D(4,4), D(4,T4), D(5,T5), \frac{D(4,T5)+D(5,T4)}{2s}\right\} = 4.$$

This implies that the inequality (1.1) in Definition 1.6 is not satisfied and hence T is not an α-β-FG-contraction. □

The following theorem states the existence and uniqueness of fixed points for generalized α-β-FG-contraction in b-metric spaces.

Theorem 2.3. *Let (X,D,s) be a complete b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0,\infty)$ be two functions, $F \in \Delta\mathcal{F}$ and $(G,\beta) \in \Delta$ such that*

(1) T is a triangular α-admissible mapping;

(2) T is a generalized α-β-FG-contraction;

(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(4) T is α-continuous.

Then

(1) T has a fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x_0 = x^*$.

(2) If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point, where $Fix(T) = \{x \in X | Tx = x\}$.

Proof. (1). First, we define the sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$. Since T is a triangular α-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, using Lemma 1.11 we conclude that the following statement for all $m, n \in \mathbb{N}$ with $n < m$,

$$\alpha(x_n, x_m) \geq 1. \tag{2.4}$$

This implies that

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{2.5}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and $\lim_{n\to\infty} T^n x_{n_0} = x_{n_0}$ by definition of the sequence $\{x_n\}$. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then $D(x_n, Tx_n) = D(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$. Since T is a generalized α-β-FG-contraction, we have

$$\begin{aligned} F\big(sD(x_n, x_{n+1})\big) &= F\big(sD(Tx_{n-1}, Tx_n)\big) \\ &\leq F\big(L_s(x_{n-1}, x_n)\big) + G\big(\beta(L_s(x_{n-1}, x_n))\big) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
L_s(x_{n-1},x_n) &= \max\Big\{D(x_{n-1},x_n), D(x_{n-1},Tx_{n-1}), D(x_n,Tx_n),\\
&\quad \frac{D(x_{n-1},Tx_n)+D(x_n,Tx_{n-1})}{2s},\\
&\quad D(T^2x_{n-1},Tx_{n-1}), \frac{D(T^2x_{n-1},x_n)}{s}, \frac{D(T^2x_{n-1},Tx_n)}{s^2},\\
&\quad \frac{D(T^2x_{n-1},x_{n-1})+D(T^2x_{n-1},Tx_n)}{2s^2}\Big\}\\
&= \max\Big\{D(x_{n-1},x_n), D(x_{n-1},x_n), D(x_n,x_{n+1}),\\
&\quad \frac{D(x_{n-1},x_{n+1})+D(x_n,x_n)}{2s},\\
&\quad D(x_{n+1},x_n), \frac{D(x_{n+1},x_n)}{s}, \frac{D(x_{n+1},x_{n+1})}{s^2},\\
&\quad \frac{D(x_{n+1},x_{n-1})+D(x_{n+1},x_{n+1})}{2s^2}\Big\}\\
&= \max\Big\{D(x_{n-1},x_n), D(x_n,x_{n+1}), \frac{D(x_{n-1},x_{n+1})}{2s}\Big\}\\
&= \max\big\{D(x_{n-1},x_n), D(x_n,x_{n+1})\big\}.
\end{aligned}$$

If there exists some $n \geq 1$ such that $D(x_n,x_{n+1}) \geq D(x_{n-1},x_n)$, then (2.6) becomes

$$F\big(sD(x_n,x_{n+1})\big) \leq F\big(D(x_n,x_{n+1})\big) + G\big(\beta(D(x_n,x_{n+1}))\big).$$

This implies that $G\big(\beta(D(x_n,x_{n+1}))\big) \geq 0$. By using the condition (G_1) of Δ, we get that $\beta(D(x_n,x_{n+1})) \geq 1$. It is a contradiction. Hence, for all $n \geq 1$, $D(x_n,x_{n+1}) \leq D(x_{n-1},x_n)$. Then (2.6) becomes

$$F\big(sD(x_n,x_{n+1})\big) \leq F\big(D(x_{n-1},x_n)\big) + G\big(\beta(D(x_{n-1},x_n))\big). \tag{2.7}$$

By using the increasing of F and (2.7), we obtain

$$\begin{aligned}
&F\big(D(x_n,x_{n+1})\big)\\
\leq\ & F\big(sD(x_n,x_{n+1})\big)\\
\leq\ & F\big(D(x_{n-1},x_n)\big) + G\big(\beta(D(x_{n-1},x_n))\big)\\
\leq\ & F\big(sD(x_{n-1},x_n)\big) + G\big(\beta(D(x_{n-1},x_n))\big)\\
\leq\ & F\big(D(x_{n-2},x_{n-1})\big) + G\big(\beta(D(x_{n-2},x_{n-1}))\big) + G\big(\beta(D(x_{n-1},x_n))\big)\\
&\ldots\\
\leq\ & F\big(D(x_0,x_1)\big) + \sum_{i=1}^{n} G\big(\beta(D(x_{i-1},x_i))\big).
\end{aligned}$$

Taking the limit in above inequality as $n \to \infty$ and using the condition (G_3) of Δ, we have

$$\lim_{n\to\infty} F\big(D(x_n, x_{n+1})\big) = -\infty.$$

By combining this with the condition $(F2)$ of Δ_F, we get

$$\lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{2.8}$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in X. On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence in X. Then there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which

$$n_k > m_k > k \geq 1 \text{ and } D(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{2.9}$$

This implies

$$D(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.10}$$

From (2.9), we get

$$\varepsilon \leq D(x_{m_k}, x_{n_k}) \leq sD(x_{m_k}, x_{m_k+1}) + sD(x_{m_k+1}, x_{n_k}). \tag{2.11}$$

Taking the upper limit as $k \to \infty$ in (2.11) and using (2.8), we obtain

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} D(x_{m_k+1}, x_{n_k}). \tag{2.12}$$

Furthermore, we have

$$\begin{aligned}
D(x_{m_k}, x_{n_k}) &\leq sD(x_{m_k}, x_{n_k-1}) + sD(x_{n_k-1}, x_{n_k}),\\
D(x_{m_k+1}, x_{n_k-1}) &\leq sD(x_{m_k+1}, x_{m_k}) + sD(x_{m_k}, x_{n_k-1}),\\
D(x_{m_k+2}, x_{n_k-1}) &\leq sD(x_{m_k+2}, x_{m_k}) + sD(x_{m_k}, x_{n_k-1})\\
&\leq s^2D(x_{m_k+2}, x_{m_k+1}) + s^2D(x_{m_k+1}, x_{m_k})\\
&\qquad +sD(x_{m_k}, x_{n_k-1}),\\
D(x_{m_k+2}, x_{n_k}) &\leq sD(x_{m_k+2}, x_{n_k-1}) + sD(x_{n_k-1}, x_{n_k}),\\
D(x_{m_k+2}, x_{m_k}) &\leq sD(x_{m_k+2}, x_{m_k+1}) + sD(x_{m_k+1}, x_{m_k}).
\end{aligned}$$

Taking the upper limit as $k \to \infty$ in the above inequalitys and using (2.8), (2.10), we get

$$\limsup_{k\to\infty} D(x_{m_k}, x_{n_k}) \leq \varepsilon s, \tag{2.13}$$

$$\limsup_{k\to\infty} D(x_{m_k+1}, x_{n_k-1}) \leq \varepsilon s, \tag{2.14}$$

$$\limsup_{k\to\infty} D(x_{m_k+2}, x_{n_k-1}) \leq \varepsilon s, \tag{2.15}$$

$$\limsup_{k\to\infty} D(x_{m_k+2}, x_{n_k}) \leq \varepsilon s^2, \tag{2.16}$$

$$\limsup_{k\to\infty} D(x_{m_k+2}, x_{m_k}) = 0. \tag{2.17}$$

Furthermore, we have

$$D(x_{m_k}, x_{n_k}) \le sD(x_{m_k}, x_{m_k+1}) + sD(x_{m_k+1}, x_{n_k}).$$

Taking the lower limit as $k \to \infty$ in the above inequality and using (2.8), (2.9), we conclude that $\liminf_{k\to\infty} D(x_{m_k+1}, x_{n_k}) \ge \frac{\varepsilon}{s} > 0$. Therefore, there exists $k_0 \in \mathbb{N}$ such that $D(x_{m_k+1}, x_{n_k}) > 0$ for all $k \ge k_0$. Combining this with (2.4) and using (2.1), we obtain the following for all $k \ge k_0$,

$$\begin{aligned} & F\big(sD(x_{m_k+1}, x_{n_k})\big) \\ = \; & F\big(sD(Tx_{m_k}, Tx_{n_k-1})\big) \\ \le \; & F\big(L_s(x_{m_k}, x_{n_k-1})\big) + G\big(\beta(L_s(x_{m_k}, x_{n_k-1}))\big) \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} L_s(x_{m_k}, x_{n_k-1}) \; = \; & \max\Big\{D(x_{m_k}, x_{n_k-1}), D(x_{m_k}, Tx_{m_k}), D(x_{n_k-1}, Tx_{n_k-1}), \\ & \frac{D(x_{m_k}, Tx_{n_k-1}) + D(x_{n_k-1}, Tx_{m_k})}{2s}, \\ & D(T^2x_{m_k}, Tx_{m_k}), \frac{D(T^2x_{m_k}, x_{n_k-1})}{s}, \frac{D(T^2x_{m_k}, Tx_{n_k-1})}{s^2}, \\ & \frac{D(T^2x_{m_k}, x_{m_k}) + D(T^2x_{m_k}, Tx_{n_k-1})}{2s^2}\Big\}. \\ = \; & \max\Big\{D(x_{m_k}, x_{n_k-1}), D(x_{m_k}, x_{m_k+1}), D(x_{n_k-1}, x_{n_k}), \\ & \frac{D(x_{m_k}, x_{n_k}) + D(x_{n_k-1}, x_{m_k+1})}{2s}, \\ & D(x_{m_k+2}, x_{m_k+1}), \frac{D(x_{m_k+2}, x_{n_k-1})}{s}, \frac{D(x_{m_k+2}, x_{n_k})}{s^2}, \\ & \frac{D(x_{m_k+2}, x_{m_k}) + D(x_{m_k+2}, x_{n_k})}{2s^2}\Big\}. \end{aligned}$$

Taking the upper limit of $L_s(x_{m_k}, x_{n_k-1})$ as $k \to \infty$ and using (2.9), (2.13) - (2.17), we get

$$0 < \frac{\varepsilon}{2s} \le \limsup_{k\to\infty} L_s(x_{m_k}, x_{n_k-1}) \le \varepsilon. \tag{2.19}$$

Now, taking the upper limit in (2.18) as $k \to \infty$, using (2.12), (2.19) and the properties of F, we obtain

$$\begin{aligned} F(s.\frac{\varepsilon}{s}) \; & \le \; F\big(s \limsup_{k\to\infty} D(x_{m_k+1}, x_{n_k})\big) \\ & \le \; \limsup_{k\to\infty} F\big(L_s(x_{m_k}, x_{n_k-1})\big) + \limsup_{k\to\infty} G\big(\beta(L_s(x_{m_k}, x_{n_k-1}))\big) \\ & \le \; F(\varepsilon) + \limsup_{k\to\infty} G\big(\beta(L_s(x_{m_k}, x_{n_k-1}))\big). \end{aligned}$$

Therefore,

$$\limsup_{k\to\infty} G\big(\beta(L_s(x_{m_k}, x_{n_k-1}))\big) \geq 0.$$

This implies that

$$\limsup_{k\to\infty} \beta(L_s(x_{m_k}, x_{n_k-1})) \geq 1.$$

Since $\beta(t) < 1$ for all $t \geq 0$, we have

$$\limsup_{k\to\infty} \beta(L_s(x_{m_k}, x_{n_k-1})) = 1.$$

By using the property of β, we obtain

$$\limsup_{k\to\infty} L_s(x_{m_k}, x_{n_k-1}) = 0,$$

which contradicts (2.19). Hence, $\{x_n\}$ is a Cauchy sequence in X. Since (X, D, s) is a complete b-metric space, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.20}$$

This implies that $\lim_{n\to\infty} T^n x_0 = x^*$ by definition of the sequence $\{x_n\}$.

Now, we will prove that x^* is a fixed point of T. By using (2.5), (2.20) and the α-continuous property of T, we obtain $\lim_{n\to\infty} Tx_n = Tx^*$. Therefore,

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

This implies that x^* is a fixed point of T and $\lim_{n\to\infty} T^n x_0 = x^*$.

(2). Let x, y be two fixed points of T. Suppose that $x \neq y$. Then, $Tx \neq Ty$. Note that, $\alpha(x, y) \geq 1$. Following (2.1), we have

$$F\big(sD(Tx, Ty)\big) \leq F\big(L_s(x, y)\big) + G\big(\beta(L_s(x, y))\big)$$

where

$$\begin{aligned} L_s(x, y) &= \max\Big\{D(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s}, \\ &\quad D(T^2x, Tx), \frac{D(T^2x, y)}{s}, \frac{D(T^2x, Ty)}{s^2}, \frac{D(T^2x, x) + D(T^2x, Ty)}{2s^2}\Big\} \\ &= D(x, y). \end{aligned}$$

This implies that $F\big(sD(Tx, Ty)\big) \leq F\big(D(x, y)\big) + G\big(\beta(D(x, y))\big)$. By using the increasing property of F, we get that $G\big(\beta(D(x, y))\big) \geq 0$ which yields that

$\beta(D(x,y)) \geq 1$. It is a contradiction. Therefore, $x = y$, that means T has a unique fixed point. □

In the following theorem, the assumption on α-continuous of T in Theorem 2.3 is replaced by another condition.

Theorem 2.4. *Let (X, D, s) be a complete b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be two functions, $F \in \Delta\mathcal{F}$ and $(G, \beta) \in \Delta$ such that*

(1) T is a triangular α-admissible mapping;

(2) T is a generalized α-β-FG-contraction;

(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(4) If $\{x_n\}$ is a sequence in X and $\lim\limits_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point x^* in X and $\lim\limits_{n\to\infty} T^n x_0 = x^*$.

(2) If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. (1). As in the proof of Theorem 2.3 we conclude that the sequence $\{x_n\}$ is defined by $x_n = T^n x_0 = Tx_{n-1}$ satisfying

$$\alpha(x_n, x_m) \geq 1, \tag{2.21}$$

$$\lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{2.22}$$

for all $n, m \in \mathbb{N}$ with $n > m$ and there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.23}$$

This implies that $\lim\limits_{n\to\infty} T^n x_0 = x^*$. Now, we will show that x^* is a fixed point of T. If for each $n \in \mathbb{N}$ there exists $i_n \in \mathbb{N}$ such that $x_{i_n+1} = Tx^*$ and $i_n > i_{n-1}, i_0 = 1$, we have $x^* = \lim\limits_{n\to\infty} x_{i_n+1} = \lim\limits_{n\to\infty} Tx_{i_n} = Tx^*$. Thus, x^* is a fixed point of T. If there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geq n_0$, we have $D(Tx_n, Tx^*) > 0$ for all $n \geq n_0$. Combining (2.21) and (2.23) with the assumption (4), we obtain $\alpha(x_n, x^*) \geq 1$. Then, by using (2.1) we conclude that for all $n \geq n_0$,

$$F\big(sD(Tx_n, Tx^*)\big) \leq F\big(L_s(x_n, x^*)\big) + G\big(\beta(L_s(x_n, x^*))\big). \tag{2.24}$$

where

$$\begin{aligned} L_s(x_n, x^*) &= \max\Big\{D(x_n, x^*), D(x_n, Tx_n), D(x^*, Tx^*), \\ &\quad \frac{D(x_n, Tx^*) + D(x^*, Tx_n)}{2s}, \\ &\quad D(T^2x_n, Tx_n), \frac{D(T^2x_n, x^*)}{s}, \frac{D(T^2x_n, Tx^*)}{s^2}, \\ &\quad \frac{D(T^2x_n, x_n) + D(T^2x_n, Tx^*)}{2s^2}\Big\} \\ &= \max\Big\{D(x_n, x^*), D(x_n, x_{n+1}), D(x^*, Tx^*), \\ &\quad \frac{D(x_n, Tx^*) + D(x^*, x_{n+1})}{2s}, \\ &\quad D(x_{n+2}, x_{n+1}), \frac{D(x_{n+2}, x^*)}{s}, \frac{D(x_{n+2}, Tx^*)}{s^2}, \\ &\quad \frac{D(x_{n+2}, x_n) + D(x_{n+2}, Tx^*)}{2s^2}\Big\}. \end{aligned}$$

Taking the upper limit in $L_s(x_n, x^*)$ as $n \to \infty$ and using (2.22) and Lemma 1.8, we obtain

$$\limsup_{n\to\infty} L_s(x_n, x^*) = D(x^*, Tx*). \tag{2.25}$$

Suppose that $D(x^*, Tx^*) > 0$, taking the upper limit in (2.24) as $n \to \infty$, using (2.25), the properties of F and Lemma 1.8, we have

$$F\big(D(x^*, Tx^*)\big) \le F\big(D(x^*, Tx^*)\big) + \limsup_{n\to\infty} G\big(\beta(L_s(x_n, x^*))\big).$$

Then $\limsup_{n\to\infty} G\big(\beta(L_s(x_n, x^*))\big) \ge 0$ which yields that $\limsup_{n\to\infty} \beta\big(L_s(x_n, x^*)\big) \ge 1$. Since $\beta(t) < 1$ for all $t \ge 0$, we have $\limsup_{n\to\infty} \beta\big(L_s(x_n, x^*)\big) = 1$ which implies

$$\lim_{n\to\infty} L_s(x_n, x^*) = 0.$$

By combining this with (2.25), we get $D(x^*, Tx^*) = 0$. It is a contradiction. Hence, x^* is a fixed point of T.

(2). As in the proof of Theorem 2.3. □

From the Theorem 2.3 and Theorem 2.4, we obtain the following results.

Corollary 2.5. *Let (X, D, s) be a complete b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be two functions, $F \in \Delta\mathcal{F}$ and $(G, \beta) \in \Delta$ such that*

(1) T is a triangular α-admissible mapping;

(2) There exist $F \in \Delta\mathcal{F}$ and $(G, \beta) \in \Delta$ such that for all $x, y \in X$, $D(Tx, Ty) > 0$,

$$\alpha(x, y)F\big(sD(Tx, Ty)\big) \leq F\big(L_s(x, y)\big) + G\big(\beta(L_s(x, y))\big)$$

where $L_s(x, y)$ is defined by (2.2);

(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(4) (i) Either T is α-continuous or

(ii) If $\{x_n\}$ is a sequence in X and $\lim_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x_0 = x^*$.

(2) If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

By choosing $F(t) = G(t) = \ln t$ for all $t \in (0, \infty)$ in Theorem 2.3 and Theorem 2.4, we get the following corollary which is analogous with [7, Corollary 2.5].

Corollary 2.6. *Let (X, D, s) be a complete b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be two functions such that*

(1) T is a triangular α-admissible mapping;

(2) For all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $D(Tx, Ty) > 0$,

$$sD(Tx, Ty) \leq \beta\big(L_s(x, y)\big)L_s(x, y)$$

where $(\ln, \beta) \in \Delta$, and $L_s(x, y)$ is defined by (2.2);

(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(4) (i) Either T is α-continuous or

(ii) If $\{x_n\}$ is a sequence in X and $\lim_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point x^* in X and $\lim_{n\to\infty} T^n x_0 = x^*$.

(2) If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Taking $G(t) = \ln t, t \in [0, \infty), \beta(t) = k \in (0, 1), t \in (0, \infty)$ and put $\tau = -\ln k$, from the Theorem 2.3 and Theorem 2.4, we obtain a generalization of the results in [3], [7] in the setting of b-metric spaces.

Corollary 2.7. *Let (X, D, s) be a complete b-metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be two functions, and $F \in \Delta\mathcal{F}$ such that*

(1) T is a triangular α-admissible mapping;

(2) There exists $\tau > 0$ such that for all $x, y \in X$ with $\alpha(x, y) > 1$ and $D(Tx, Ty) > 0$,

$$\tau + F\big(sD(Tx, Ty)\big) \leq F\big(L_s(x, y)\big)$$

where $L_s(x, y)$ is defined by (2.2);

(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(4) (i) Either T is α-continuous or

(ii) If $\{x_n\}$ is a sequence in X and $\lim\limits_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then

(1) T has a fixed point x^* and $\lim\limits_{n\to\infty} T^n x_0 = x^*$.

(2) If $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Next, by using Theorem 2.3, we study a Suzuki-Wardowski-type fixed point result in b-metric spaces.

Corollary 2.8. *Let (X, D, s) be a complete b-metric space, $T : X \longrightarrow X$ be a mapping, $F \in \Delta\mathcal{F}$ and $(G, \beta) \in \Delta$ such that*

(1) T is continuous;

(2) For all $x, y \in X$ and $x \neq y$, we have $D(x, Tx) \leq D(x, y)$;

(3) For all $x, y \in X$ and $D(Tx, Ty) > 0$, we have

$$F\big(sD(Tx, Ty)\big) \leq F\big(L_s(x, y)\big) + G\big(\beta(L_s(x, y))\big) \tag{2.26}$$

where $L_s(x, y)$ is defined by (2.2).

Then T has a unique fixed point x^* in X and $\lim\limits_{n\to\infty} T^n x_0 = x^*$.

Proof. Let $\alpha : X \times X \longrightarrow [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } D(x, Tx) \leq D(x, y), \\ 0 & \text{if otherwirse.} \end{cases}$$

Then $\alpha(x, y) \geq 1$ for all $x, y \in X, x \neq y$. From (2.26), we conclude that T is a generalized α-β-FG-contraction. Since T is continuous, T is α-continuous. Then the conclusions hold by Theorem 2.3. □

The following example shows that Theorem 2.3 is different from [7, Theorem 2.3, Theorem 2.4].

Example 2.9. Let b-metric space (X, D, s) and T, F, G, β be given as in Example 2.2. Define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ and for all $x, y \in X$. Then

(1) T is not an α-β-FG-contraction. It implies that [7, Theorem 2.3, Theorem 2.4] are not applicable to T, F, G, β, α.

(2) Theorem 2.3 is applicable to T, F, G, β, α.

Proof. (1). As in the proof of Example 2.2, T is not an α-β-FG-contraction. Then [7, Theorem 2.3, Theorem 2.4] are not applicable to T, F, G, β, α.

(2). Also, from Example 2.2, T is a generalized α-β-FG-contraction. Since $\alpha(x, y) > 1$ for all $x, y \in X$, T is a triangular α-admissible mapping. Furthermore, X is finite, then T is an α-continuous. Therefore, Theorem 2.3 is applicable to T, F, G, β, α. □

Next, we apply Corollary 2.6 to study the existence of solutions to a class of integral equation.

Example 2.10. Let $C[a, b]$ be the set of all continuous function on $[a, b]$, the b-metric D with $s = 2^{p-1}$ be defined by

$$D(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|^p$$

for all $x, y \in X$ and some $p > 1$. Consider the nonlinear integral equation

$$x(t) = g(t) + \int_a^b K(t, s, x(s)) ds \tag{2.27}$$

where $t \in [a, b]$, $g \in C[a, b]$ and $K : [a, b] \times [a, b] \times x[a, , b] \longrightarrow \mathbb{R}$ for each $x \in C[a, b]$. Suppose that the following statements hold:

(1) $K(t, s, x(s))$ is integrable with respect to s on $[a, b]$;

(2) $Tx \in C[a,b]$ for all $x \in C[a,b]$ where $Tx(t) = g(t) + \int_a^b K(t,s,x(s))ds$ for all $t \in [a,b]$;

(3) For all $x \in C[a,b]$ and $x(t) \geq 0$ for all $t \in [a,b]$, we have $Tx(t) \geq 0$ for all $t \in [a,b]$;

(4) For all $t,s \in [a,b]$ and $x,y \in C[a,b]$ such that $x(t),y(t) \in [0,\infty)$ and $x(t) \neq y(t)$ for all $t \in [a,b]$, we have

$$\begin{aligned} & |K(t,s,x(s)) - K(t,s,y(s))|^p \\ \leq\ & \varphi^p(t,s) \max\Big\{ |x(t)-y(t)|^p, |x(t)-Tx(t)|^p, \\ & |y(t)-Ty(t)|^p, \frac{|x(t)-Ty(t)|^p + |y(t)-Tx(t)|^p}{2^p}, \\ & |T^2x(t)-Tx(t)|^p, \frac{|T^2x(t)-y(t)|^p}{2^{p-1}}, \frac{|T^2x(t)-Ty(t)|^p}{2^{2p-2}}, \\ & \frac{|T^2x(t)-x(t)|^p + |T^2x(t)-Ty(t)|^p}{2^{2p-1}} \Big\}, \end{aligned}$$

where $\varphi : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ is continuous function satisfying

$$0 < \sup_{t \in [a,b]} \Big(\int_a^b \varphi^p(t,s)ds \Big) < \frac{1}{2^{p-1}(b-a)^{p-1}}.$$

Then the nonlinear integral equation (2.27) has a solution $x \in C[a,b]$.

Proof. Define a mapping $T : C[a,b] \longrightarrow C[a,b]$ by

$$Tx(t) = g(t) + \int_a^b K(t,s,x(s))ds$$

for all $x \in C[a,b]$ and for all $t \in [a,b]$. It follows from hypothesis (1) and hypothesis (2) that T is well-defined. Notice that the existence of a solution to (2.27) is equivalent to the existence of a fixed point of T. Now, we show that all the hypotheses of Corollary 2.6 are satisfied.

Define a mapping $\alpha : C[a,b] \times C[a,b] \longrightarrow \mathbb{R}$ by

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x(t), y(t) \in [0,\infty) \text{ for all } t \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

(1). We shall show that T is a triangular α-admissible mapping. Indeed, for $x,y \in C[a,b]$ such that $\alpha(x,y) \geq 1$, we have $x(t) \geq 0$ and $y(t) \geq 0$ for all $t \in [a,b]$. It follows from condition (3) that $Tx(t) \geq 0$ and $Ty(t) \geq 0$ for all $t \in [a,b]$. Therefore, $\alpha(Tx,Ty) \geq 1$. In addition, for $x,y,z \in C[a,b]$ such that $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$, we get $x(t), y(t), z(t) \geq 0$ for all $t \in [a,b]$. It implies that $\alpha(x,y) \geq 1$. Therefore, T is a triangular α-admissible mapping.

(2). We claim that assumption (2) in Corollary 2.6 holds. Indeed, let $q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For $x, y \in C[a,b]$ with $\alpha(x,y) \geq 1$ and $D(Tx,Ty) > 0$, we conclude that $x(t), y(t) \in [0,\infty)$ and $x(t) \neq y(t)$ for all $t \in [a,b]$. Combining this with condition (4), we have

$$\begin{aligned}
& 2^{p-1}\left|Tx(t) - Ty(t)\right|^p \\
\leq\; & 2^{p-1}\left|\int_a^b K(t,s,x(s))ds - \int_a^b K(t,s,y(s))ds\right|^p \\
=\; & 2^{p-1}\left|\int_a^b (K(t,s,x(s)) - K(t,s,y(s)))ds\right|^p \\
\leq\; & 2^{p-1}\left(\int_a^b \left|K(t,s,x(s)) - K(t,s,y(s))\right| ds\right)^p \\
\leq\; & 2^{p-1}\left[\left(\int_a^b ds\right)^{\frac{1}{q}}\left(\int_a^b |K(t,s,x(s)) - K(t,s,y(s))|^p ds\right)^{\frac{1}{p}}\right]^p \\
\leq\; & 2^{p-1}(b-a)^{p-1}\left(\int_a^b \varphi(t,s)^p ds\right)\max\Big\{|x(t)-y(t)|^p, |x(t)-Tx(t)|^p, \\
& |y(t)-Ty(t)|^p, \frac{|x(t)-Ty(t)|^p + |y(t)-Tx(t)|^p}{2^p}, \\
& |T^2x(t)-Tx(t)|^p, \frac{|T^2x(t)-y(t)|^p}{2^{p-1}}, \frac{|T^2x(t)-Ty(t)|^p}{2^{2p-2}}, \\
& \frac{|T^2x(t)-x(t)|^p + |T^2x(t)-Ty(t)|^p}{2^{2p-1}}\Big\} \\
=\; & 2^{p-1}(b-a)^{p-1}\Big(\int_a^b \varphi^p(t,s)ds\Big)L_s(x,y) \\
\leq\; & 2^{p-1}(b-a)^{p-1}\sup_{t\in[a,b]}\Big(\int_a^b \varphi^p(t,s)ds\Big)L_s(x,y) \\
=\; & \lambda L_s(x,y),
\end{aligned}$$

where $\lambda = 2^{p-1}(b-a)^{p-1}\sup_{t\in[a,b]}\Big(\int_a^b \varphi^p(t,s)ds\Big) \in (0,1)$. This implies that assumption (2) in Corollary 2.6 is satisfied with $\beta(t) = \lambda$ for all $t \geq 0$.

(3). From the definition of α and using assumption (3), we conclude that there exits $x_0 \in C[a,b]$ such that $\alpha(x_0, Tx_0) \geq 1$.

(4). Let $\{x_n\} \subset C[a,b]$ such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\lim_{n\to\infty} x_n = x \in$

$C[a,b]$. Then $x(t), x_n(t) \in [0,\infty)$ for all $t \in [a,b]$ and $n \geq 0$. Therefore, $\alpha(x_n, x) \geq 1$ for all $n \geq 1$.

By the above, we conclude that all the assumptions in Corollary 2.6 are satisfied. Thus, T has a fixed point $x \in C[a,b]$ and hence equation (2.27) has a solution $x \in C[a,b]$. $\square$

The following example guarantees the existence of the functions K and g satisfying all the assumption in Example 2.10.

Example 2.11. Let $C[0,1]$ be the set of all continuous functions on $[0,1]$, b-metric D with $s=2$ defined by

$$d(x,y) = \sup_{t\in[0,1]} |x(t)-y(t)|^2$$

for all $x, y \in C[0,1]$. Consider the integral equation

$$x(t) = -\frac{t^3}{\sqrt{5}} + 3t^2 + 1 + \int_0^1 \frac{(3s^2+2)t^3 x(s)}{2\sqrt{5}(1+x(s))} ds \qquad (2.28)$$

for all $t \in [0,1]$ and $x \in C[0,1]$. Put

$$g(t) = -\frac{t^3}{\sqrt{5}} + 3t^2 + 1, K(t,s,x(s)) = \frac{(3s^2+2)t^3 x(s)}{2\sqrt{5}(1+x(s))}$$

and $Tx(t) = -\frac{t^3}{\sqrt{5}} + 3t^2 + 1 + \int_0^1 \frac{(3s^2+2)t^3 x(s)}{2\sqrt{5}(1+x(s))} ds$ for all $t, s \in [0,1]$ and $x \in C[0,1]$. Then

(1). g is continuous on $[0,1]$. Since $x \in C[0,1]$, $K(t,s,x(s))$ is integral with respect to s on $[0,1]$.

(2). For all $t, s \in [0,1]$ and the sequence $t_n \in [0,1]$ with $\lim_{n\to\infty} t_n = t$. We have

$$\begin{aligned} |Tx(t_n) - Tx(t)| &\leq |g(t_n) - g(t)| + \frac{1}{2\sqrt{5}} \int_0^1 (3s^2+2)|t_n^3 - t^3| \left| \frac{x(s)}{1+x(s)} \right| ds \\ &\leq |g(t_n) - g(t)| + \frac{1}{2\sqrt{5}} \int_0^1 (3s^2+2)|t_n^3 - t^3| ds \\ &= |g(t_n) - g(t)| + \frac{3}{2\sqrt{5}} |t_n^3 - t^3|. \end{aligned}$$

This implies that $Tx \in C[0,1]$ for all $x \in C[0,1]$.

(3). It is easy to see that $g(t) \geq 0$ for $t \in [0,1]$. Furthermore, for $x \in C[0,1]$ such that $x(t) \geq 0$ for all $t \in [0,1]$, we get that $K(t,s,x(s)) \geq 0$ for all $t, s \in [0,1]$. It implies that $Tx(t) \geq 0$ for all $x \in C[0,1]$ and $t \in [0,1]$.

(4). For $x, y \in C[0,1]$ and $x(s) \neq y(s) \in [0,\infty)$ for all $s \in [0,1]$, we get

$$\begin{aligned}
|K(t,s,x(s)) - K(t,s,y(s))| &= \frac{(3s^2+2)t^3}{2\sqrt{5}}\Big|\frac{x(s)}{1+x(s)} - \frac{y(s)}{1+y(s)}\Big| \\
&= \frac{(3s^2+2)t^3}{2\sqrt{5}}\left|\frac{x(s)-y(s)}{[1+x(s)][1+y(s)]}\right| \\
&\leq \frac{(3s^2+2)t^3}{2\sqrt{5}}\big|x(s)-y(s)\big|.
\end{aligned}$$

By choosing $\varphi(t,s) = \dfrac{(3s^2+2)t^3}{2\sqrt{5}}$, it is easy to see that φ is continuous,

$$0 < \sup_{t\in[0,1]}\Big(\int_0^1 \varphi^2(t,s)ds\Big) < \frac{1}{2},$$

$$0 \leq |K(t,s,x(s)) - K(t,s,y(s))| \leq \varphi(t,s)\big|x(s)-y(s)\big|$$

and hence

$$\begin{aligned}
& |K(t,s,x(s)) - K(t,s,y(s))|^2 \\
\leq\ & \varphi^2(t,s)\max\Big\{|x(t)-y(t)|^2, |x(t)-Tx(t)|^2, \\
& |y(t)-Ty(t)|^2, \frac{|x(t)-Ty(t)|^2+|y(t)-Tx(t)|^2}{4}, \\
& |T^2x(t)-Tx(t)|^2, \frac{|T^2x(t)-y(t)|^2}{2}, \frac{|T^2x(t)-Ty(t)|^2}{4}, \\
& \frac{|T^2x(t)-x(t)|^2+|T^2x(t)-Ty(t)|^2}{8}\Big\}.
\end{aligned}$$

From the above, all the assumption to K and g in Example 2.10 are satisfied. Furthermore, it easy to check that $x(t) = 3t^2+1$ for all $t \in [0,1]$ is a solution of the integral equation (2.28).

Acknowledgements: The authors sincerely thank some anonymous referees for several helpful comments. The authors also thank members of The Dong Thap Group of Mathematical Analysis and its Applications for their discussions on the manuscript.

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