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SOME INTEGRAL INEQUALITIES AND THEIR APPLICATIONS VIA FRACTIONAL INTEGRALS

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Abstract

In this paper, we give some integral inequalities in finite measure spaces. By using these results, we establish some Jensen and Hermite-Hadamard type inequalities for convex functions via fractional integrals.

1 Introduction

Although arising from geometric, the convexity plays an important role in mathematical analysis; especially, in the study of mathematical inequalities it provides a very useful tool for estimating the integral mean value of a function defined on a closed interval. Some famous results for such estimations consist of Hermite-Hadamard, trapezoid, or Ostrowski inequalities, ect.

If $f : [a, b] \to \mathbb{R}$ is a integrable convex function on $[a, b]$, the Hermite-Hadamard inequality states (see, for example, [10]) that

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is a integrable concave function, the inequality (1.1) is reversed. For some

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generalizations, improvements and extensions of the inequality (1.1), please see [7, 10] and references therein.

The error in estimating the left hand-side of (1.1) is closely related to the Ostrowski inequality; it reads (see, for example, [8]) that if $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b) having the property $|f'(t)| \leq M$ for all $t \in (a, b)$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)M \tag{1.2}$$

for all $x \in [a, b]$. In 2015, the inequality (1.2) was extended to a functional generalization by Dragomir as follows.

Theorem 1.1 ([8]). *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is convex on $\mathbb{R}$, the following inequality holds:*

$$\Phi\left(f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right) \leq \frac{1}{b-a} \left[\int_a^x \Phi[(t-a)f'(t)]dt + \int_x^b \Phi[(t-b)f'(t)]dt \right] \tag{1.3}$$

for all $x \in [a, b]$.

Also, the error in estimating the right hand-side of (1.1) is closely related to the trapezoid inequality which states (see, for example, [10]) that if $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) with its derivative having the property $-\infty < \gamma \leq f'(x) \leq \Gamma < \infty$ for all $x \in (a, b)$, then we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{8}(\Gamma - \gamma). \tag{1.4}$$

This result was extended to the generalized trapezoid formula, namely, inequalities provide upper bounds for the quantity

$$\left| \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt \right|,$$

for any $x \in [a, b]$. In 2015, Dragomir [9] gave a functional generalization of generalized trapezoid inequalities as follows.

Theorem 1.2 ([9]). *Let $f : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a, b]$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is convex on $\mathbb{R}$, then we have the following inequality*

$$\begin{aligned} &\Phi\left(\frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt \right) \\ &\quad \leq \frac{x-a}{(b-a)^2} \int_a^b \Phi[f(a) - f(t)]dt + \frac{b-x}{(b-a)^2} \int_a^b \Phi[f(b) - f(t)]dt, \end{aligned} \tag{1.5}$$

for all $x \in [a, b]$.

To obtain inequalities (1.3) and (1.5), the author used the Montgomery identity (see [15] for details about this identity) and the property of convex functions.

A subtle combination between estimations of the left and right hand-side of (1.1) leads to the Simpson type inequalities. In [10], the authors investigated the extended Simpson-type inequalities. Namely, it provides upper bounds for the quantity

$$\left|\alpha\left[\frac{f(x)+f(y)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|,$$

where $0 \le \alpha \le 1, x \in [a, \frac{a+b}{2}]$ and $y \in [\frac{a+b}{2}, b]$, please see [10] for details.

Another well-known inequality related to the left hand-side of (1.1) is the Jensen integral inequality, see, for example, [1]. In recent years, the inequalities (1.1),(1.2), (1.4) as well as the Jensen inequality were improved and generalized to the framework of fractional integrals, see [1, 3, 4, 6, 7, 14, 16].

The main goal of the paper is to establish some Jensen, Hermite-Hadamard type inequalities and functional generalizations for Ostrowski and trapezoid type inequalities via fractional integrals. To obtain such results, we utilize a very simple and non-traditional method. More precisely, we propose some general inequalities in finite measure spaces, and then choose the suitable measure spaces to get the desired inequalities.

2 Integral inequalities for convex functions

The main purpose of this section is to propose some integral inequalities of generalized Hermite-Hadamard type in finite measure spaces.

Theorem 2.1. *Let $(X, \mathcal{F}, \mu)$ be a measure space with X compact. Let $f : X \to \mathbb{R}$ be a continuous function such that*

$$m := \min_{x\in X} f(x) < \max_{x\in X} f(x) =: M.$$

Let $I, J \in \mathcal{F}$ satisfy that $\mu(I) + \mu(J) = 1$. Then, for any $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$ and any continuously convex function $\Phi : \mathbb{R} \to \mathbb{R}$, the following inequalities hold:

$$\begin{aligned}
&\Phi\Big(\alpha\mu(I)+\beta\mu(J)-\frac{1}{\mu(X)}\int_X f(t)d\mu(t)\Big)\\
&\le \frac{1}{\mu(X)}\int_X \Phi\Big(\alpha\mu(I)+\beta\mu(J)-f(t)\Big)d\mu(t)\\
&\le \mu(I).\frac{1}{\mu(X)}\int_X \Phi(\alpha-f(t))d\mu(t)+\mu(J).\frac{1}{\mu(X)}\int_X \Phi(\beta-f(t))d\mu(t)\\
&\le \mu(I)\mathcal{M}(\alpha)+\mu(J)\mathcal{M}(\beta),
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\mathcal{M}(\alpha) = &\frac{M\Phi(\alpha-m)-m\Phi(\alpha-M)}{M-m}- \\ &-\frac{\Phi(\alpha-m)-\Phi(\alpha-M)}{M-m}.\frac{1}{\mu(X)}\int_X f(t)d\mu(t).\end{aligned} \tag{2.2}$$

Proof. Since $\mu(X) < +\infty$, we can write

$$\alpha\mu(I)+\beta\mu(J)-\frac{1}{\mu(X)}\int_X f(t)d\mu(t) = \frac{1}{\mu(X)}\int_X \left[\alpha\mu(I)+\beta\mu(J)-f(t)\right]d\mu(t).$$

It follows from Jensen's integral inequality (see, for example, [1]) that

$$\Phi\Big(\alpha\mu(I)+\beta\mu(J)-\frac{1}{\mu(X)}\int_X f(t)d\mu(t)\Big) \le \frac{1}{\mu(X)}\int_X \Phi\Big(\alpha\mu(I)+\beta\mu(J)-f(t)\Big)d\mu(t). \tag{2.3}$$

On the other hand, by the convexity of Φ and the hypothesis $\mu(I)+\mu(J)=1$, we infer that

$$\begin{aligned}&\int_X \Phi(\alpha\mu(I)+\beta\mu(J)-f(t))d\mu(t) \\ &=\int_X \Phi\big([\alpha-f(t)]\mu(I)+[\beta-f(t)]\mu(J)\big)d\mu(t) \\ &\le \mu(I)\int_X \Phi(\alpha-f(t))d\mu(t)+\mu(J)\int_X \Phi(\beta-f(t))d\mu(t).\end{aligned} \tag{2.4}$$

A combination of (2.3) and (2.4) yields the first two inequalities in (2.1).

To obtain the other inequality in (2.1), we write

$$\alpha-f(t)=\Big(1-\frac{M-f(t)}{M-m}\Big)(\alpha-M)+\frac{M-f(t)}{M-m}(\alpha-m).$$

Then,

$$\Phi(\alpha-f(t)) \le \Big(1-\frac{M-f(t)}{M-m}\Big)\Phi(\alpha-M)+\frac{M-f(t)}{M-m}\Phi(\alpha-m)$$

by the convexity of Φ. Integrating this inequality on X, we have

$$\begin{aligned}\frac{1}{\mu(X)}\int_X \Phi(\alpha-f(t))d\mu(t) \le &\frac{M\Phi(\alpha-m)-m\Phi(\alpha-M)}{M-m}- \\ &-\frac{\Phi(\alpha-m)-\Phi(\alpha-M)}{M-m}.\frac{1}{\mu(X)}\int_X f(t)d\mu(t).\end{aligned} \tag{2.5}$$

Similarly, we also obtain

$$\frac{1}{\mu(X)}\int_X \Phi(\beta - f(t))d\mu(t) \leq \frac{M\Phi(\beta-m) - m\Phi(\beta-M)}{M-m} - \frac{\Phi(\beta-m)-\Phi(\beta-M)}{M-m}.\frac{1}{\mu(X)}\int_X f(t)d\mu(t). \tag{2.6}$$

From (2.5) and (2.6), we deduce

$$\mu(I).\frac{1}{\mu(X)}\int_X \Phi(\alpha - f(t))d\mu(t) + \mu(J).\frac{1}{\mu(X)}\int_X \Phi(\beta - f(t))d\mu(t) \leq \mu(I)\mathcal{M}(\alpha) + \mu(J)\mathcal{M}(\beta),$$

where $\mathcal{M}$ is as in (2.2). This completes the proof of Theorem 2.1. □

Remark 2.2. Let $X = [a,b] \subset \mathbb{R}$ be a nontrivial interval, $\alpha = f(a), \beta = f(b)$, $I = [\frac{a}{b-a}, \frac{x}{b-a}], J = [\frac{x}{b-a}, \frac{b}{b-a}]$ for $x \in [a,b]$ and $d\mu(x) = dx$. Then, from the inequalities in (2.7) we get inequality (1.5).

Let $\alpha = \beta = 0$, in the following, we have an immediate consequence of Theorem 2.1.

Corollary 2.3. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space with X compact. If $f : X \to \mathbb{R}$ is a continuous function which is not a constant and $\Phi : \mathbb{R} \to \mathbb{R}$ is convex, then*

$$\Phi\left(\frac{1}{\mu(X)}\int_X f(t)d\mu(t)\right) \leq \frac{1}{\mu(X)}\int_X \Phi(f(t))d\mu(t) \leq \frac{M\Phi(m) - m\Phi(M)}{M-m} + \frac{\Phi(M)-\Phi(m)}{M-m}\cdot\frac{1}{\mu(X)}\int_X f(t)d\mu(t), \tag{2.7}$$

where

$$m = \min_{t\in X} f(t) \quad \textit{and} \quad M = \max_{t \in X} f(t).$$

Remark 2.4. Let $X = [a,b] \subset \mathbb{R}$ be a nontrivial interval, $f(x) = x$ for all $x \in [a,b]$ and $d\mu(x) = dx$. Then the inequalities in (2.7) become the well-known Hermite-Hadamard inequality for the real-valued convex function Φ defined on $[a,b]$ as follows:

$$\Phi\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b \Phi(x)dx \leq \frac{\Phi(a)+\Phi(b)}{2}.$$

3 Applications to fractional integrals

This section is devoted to present some applications of the results in the Corollary 2.3 to fractional integrals. To this end, we first recall some necessary notions on fractional integrals in the following definitions.

Definition 3.1 (see [7]). Let $\alpha \in (n, n+1]$ with $n \in \mathbb{N}$ and $x \in (a,b)$. The left- and right-side Riemann-Liouville fractional integrals of order α of a function f are given by

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad \text{and} \quad J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt,$$

respectively, where $\Gamma(\cdot)$ is the Euler's gamma function defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

Definition 3.2 (see [7]). Let $\alpha \in (n, n+1]$ with $n \in \mathbb{N}$ and $x \in (a,b)$. The left- and right-side Hadamard fractional integrals of order α of a function f are given by

$$H^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt$$

and

$$H^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt.$$

A generalization of the two kinds of fractional integrals above is the Katugampola fractional integral stated as follows.

Definition 3.3 ([11]). Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha > 0$ of $f \in X^p_c(a,b)$ are defined by

$${}^{\rho}I^{\alpha}_{a^+} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} f(t)dt$$

and

$${}^{\rho}I^{\alpha}_{b^-} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^{\rho} - x^{\rho})^{1-\alpha}} f(t)dt$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Note that the integrals in Definition 3.3 exist whenever f belongs to the space $\mathcal{X}^p_c(a,b)$. Here, the space $\mathcal{X}^p_c(a,b)$ consist of complex-valued Lebesgue measurable functions f defined on $[a,b]$ for which $\|f\|_{\mathcal{X}^p_c} < \infty$, where $c \in \mathbb{R}$, $1 \leq p \leq \infty$ and

$$\|f\|_{\mathcal{X}^p_c} = \begin{cases} \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{a \leq t \leq b} |t^c f(t)| & \text{if } p = \infty. \end{cases}$$

A close relationship between the Katugampola fractional integrals with Riemann-Liouville and Hadamard fractional integrals is given in the following theorem.

Theorem 3.4 ([11]). *Let $\alpha > 0$ and $\rho > 0$. Then, we have the following assertions, for $x > a$,*

(i) $\lim_{\rho\to 1} {}^{\rho}I^{\alpha}_{a^+} f(x) = J^{\alpha}_{a^+} f(x)$,

(ii) $\lim_{\rho\to 0^+} {}^{\rho}I^{\alpha}_{a^+} f(x) = H^{\alpha}_{a^+} f(x)$.

Similar results also hold for the right-side operators.

Finally, we recall the definition of conformable fractional integrals introduced by Khalil et al.[12] and then developed by T. Abdeljawad [2].

Definition 3.5 ([2]). Let $\alpha \in (n, n+1]$ and $\beta = \alpha - n$. The left (right) conformable fractional integrals of order α starting at a (respectively, b) are defined by

$$I^{\alpha}_{a} f(x) = J^{n+1}_{a^+}((x-a)^{\beta-1} f) = \frac{1}{n!}\int_a^x (x-t)^n (t-a)^{\beta-1} f(t)dt, \tag{3.1}$$

and by

$$_{b}I^{\alpha} f(x) = J^{n+1}_{b^-}((b-x)^{\beta-1} f) = \frac{1}{n!}\int_x^b (t-x)^n (b-t)^{\beta-1} f(t)dt. \tag{3.2}$$

3.1 Jensen type inequalities via fractional integrals

Theorem 3.6. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuously convex function. Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a continuous function satisfying $f \in \mathcal{X}^p_c(a,b)$. Then, the following Jensen-type inequalities via Katugampola fractional integrals hold:*

[(i)]

(i) If $x \in (a,b]$ and $\alpha > 0$, then

$$\Phi\left(\frac{\rho^{\alpha}\Gamma(\alpha+1)}{(x^{\rho}-a^{\rho})^{\alpha}} {}^{\rho}I^{\alpha}_{a^+} f(x)\right) \le \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(x^{\rho}-a^{\rho})^{\alpha}} {}^{\rho}I^{\alpha}_{a^+} \Phi(f(x)).$$

(ii) If $x \in [a,b)$ and $\alpha > 0$, then

$$\Phi\left(\frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-x^{\rho})^{\alpha}} {}^{\rho}I^{\alpha}_{b^-} f(x)\right) \le \frac{\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-x^{\rho})^{\alpha}} {}^{\rho}I^{\alpha}_{b^-} \Phi(f(x)).$$

Proof. By taking $X = [a,x]$ with $0 \le a < x \le b$ and

$$d\mu(t) = \frac{\rho^{1-\alpha} t^{\rho-1}}{\Gamma(\alpha)(t^{\rho}-a^{\rho})^{1-\alpha}} dt$$

for any $\alpha > 0$, it is easy to calculate that

$$\mu(X) = \int_a^x d\mu(t) = \int_a^x \frac{\rho^{1-\alpha} t^{\rho-1}}{\Gamma(\alpha)(t^\rho - a^\rho)^{1-\alpha}} dt$$
$$= \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^x \rho t^{\rho-1} (t^\rho - a^\rho)^{\alpha-1} dt = \frac{(x^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)}.$$

Substitute these quantities for the left inequality in Corollary 2.3, we obtain

$$\Phi\left(\frac{\rho^\alpha \Gamma(\alpha+1)}{(x^\rho - a^\rho)^\alpha} {}^\rho I_{a+}^\alpha f(x)\right) \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{(x^\rho - a^\rho)^\alpha} {}^\rho I_{a+}^\alpha \Phi(f(x)).$$

Similarly, by taking $X = [x, b]$ with $0 \leq a \leq x < b$ and

$$d\mu(t) = \frac{\rho^{1-\alpha} t^{\rho-1}}{\Gamma(\alpha)(b^\rho - t^\rho)^{1-\alpha}} dt$$

for any $\alpha > 0$, it is easy to calculate that

$$\mu(X) = \int_x^b d\mu(t) = \int_x^b \frac{\rho^{1-\alpha} t^{\rho-1}}{\Gamma(\alpha)(b^\rho - x^\rho)^{1-\alpha}} dt$$
$$= \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_x^b \rho t^{\rho-1} (b^\rho - t^\rho)^{\alpha-1} dt = \frac{(b^\rho - x^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)}.$$

Replace these quantities in the left inequality of Corollary 2.3, we get

$$\Phi\left(\frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - x^\rho)^\alpha} {}^\rho I_{b-}^\alpha f(x)\right) \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - x^\rho)^\alpha} {}^\rho I_{b-}^\alpha \Phi(f(x)),$$

which finishes the proof. □

An immediate consequence of Theorem 3.6 is as follows.

Corollary 3.7. *Under the hypotheses of Theorem 3.6, we have*

$$\Phi\left(\frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} [{}^\rho I_{b-}^\alpha f(a) + {}^\rho I_{a+}^\alpha f(b)]\right) \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} [{}^\rho I_{b-}^\alpha \Phi(f(a)) + {}^\rho I_{b-}^\alpha \Phi(f(b))].$$

Remark 3.8. (i) Letting $\rho \to 1$ in Theorem 3.6 and Corollary 3.7 and using Theorem 3.4, we obtain [1, Theorems 3.1 and 3.3] for Riemann-Liouville fractional integrals of order $\alpha > 0$.

(ii) Taking the limit $\rho \to 0^+$ in Theorem 3.6 and Corollary 3.7, we obtain the Jensen type inequalities for Hadamard fractional integrals of order $\alpha > 0$ of the continuous function f defined on $[a, b] \subset (0, \infty)$ and the convex continuous functions $\Phi : \mathbb{R} \to \mathbb{R}$ as follows.

- If $x \in (a, b] \subset (0, \infty)$ and $\alpha > 0$, then

$$\Phi\left(\frac{\Gamma(\alpha+1)}{(\ln x - \ln a)^\alpha} H_{a+}^\alpha f(x)\right) \leq \frac{\Gamma(\alpha+1)}{(\ln x - \ln a)^\alpha} H_{a+}^\alpha \Phi(f(x)).$$

- If $x \in [a,b) \subset (0,\infty)$ and $\alpha > 0$, then

$$\Phi\left(\frac{\Gamma(\alpha+1)}{(\ln b - \ln x)^{\alpha}} H^{\alpha}_{b-} f(x)\right) \leq \frac{\Gamma(\alpha+1)}{(\ln b - \ln x)^{\alpha}} H^{\alpha}_{b-} \Phi(f(x)).$$

- If $(a,b) \subset (0,\infty)$ and $\alpha > 0$, then

$$\Phi\left(\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}} [H^{\alpha}_{b-} f(a) + H^{\alpha}_{a+} f(b)]\right) \leq$$

$$\leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}} [H^{\alpha}_{b-} \Phi(f(a)) + H^{\alpha}_{a+} \Phi(f(b))].$$

Theorem 3.9. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuously convex function and $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a continuous function. Let $\alpha \in (n, n+1]$ with $n \in \mathbb{N}$ and $\beta = n - \alpha$. Then, the following Jensen-type inequalities for conformable fractional integrals of order α of f hold:*

(i) If $x \in (a,b]$, then

$$\Phi\Big(\frac{\Gamma(\alpha+1)}{\Gamma(\beta)(x-a)^{\alpha}} I^{\alpha}_{a} f(x)\Big) \leq \frac{\Gamma(\alpha+1)}{\Gamma(\beta)(x-a)^{\alpha}} I^{\alpha}_{a} \Phi(f(x)). \tag{3.3}$$

(ii) If $x \in [a,b)$, then

$$\Phi\Big(\frac{\Gamma(\alpha+1)}{\Gamma(\beta)(b-x)^{\alpha}} {}_{b}I^{\alpha} f(x)\Big) \leq \frac{\Gamma(\alpha+1)}{\Gamma(\beta)(b-x)^{\alpha}} {}_{b}I^{\alpha} \Phi(f(x)). \tag{3.4}$$

Proof. First, we take the space $X = [a,x]$ with $a < x \leq b$ and the measure

$$d\mu(t) = \frac{(x-t)^n (t-a)^{\beta-1}}{n!} dt.$$

Then, we have

$$\begin{aligned}
\mu(X) &= \frac{1}{n!} \int_a^x (x-t)^n (t-a)^{\beta-1} dt \\
&= \frac{1}{n!} \int_a^x [(x-a)-(t-a)]^n (t-a)^{\beta-1} dt \\
&= \frac{(x-a)^{n+\beta}}{n!} \int_a^x \Big(1 - \frac{t-a}{x-a}\Big)^n \Big(\frac{t-a}{x-a}\Big)^{\beta-1} \frac{dt}{x-a}.
\end{aligned}$$

Thus, the variable substitution $\lambda = \frac{t-a}{x-a}$ yields

$$\begin{aligned}\mu(X) &= \frac{(x-a)^{n+\beta}}{n!}\int_0^1 \lambda^{\beta-1}(1-\lambda)^n d\lambda \\ &= \frac{(x-a)^{n+\beta}}{n!}B(\beta, n+1) \\ &= \frac{(x-a)^{n+\beta}}{n!}\frac{\Gamma(\beta)\Gamma(n+1)}{\Gamma(n+\beta+1)} \\ &= \frac{(x-a)^{\alpha}\Gamma(\beta)}{\Gamma(\alpha+1)},\end{aligned}$$

where $B(\cdot,\cdot)$ is the beta function. Replace these quantities in the left inequality of Corollary 2.3, we get

$$\begin{aligned}&\Phi\Big(\frac{\Gamma(\alpha+1)}{\Gamma(\beta)(x-a)^{\alpha}}\frac{1}{n!}\int_a^x (x-t)^n(t-a)^{\beta-1}f(t)dt\Big) \\ &\leq \frac{\Gamma(\alpha+1)}{\Gamma(\beta)(x-a)^{\alpha}}\frac{1}{n!}\int_a^x (x-t)^n(t-a)^{\beta-1}\Phi(f(t))dt,\end{aligned}$$

or equivalently,

$$\Phi\Big(\frac{\Gamma(\alpha+1)}{\Gamma(\beta)(x-a)^{\alpha}}I_a^{\alpha}f(x)\Big) \leq \frac{\Gamma(\alpha+1)}{\Gamma(\beta)(x-a)^{\alpha}}I_a^{\alpha}\Phi(f(x)).$$

To prove the other inequality, we choose the space $X = [x, b]$ with $a \leq x < b$ and the measure

$$d\mu(t) = \frac{(t-x)^n(b-a)^{\beta-1}}{n!}dt.$$

The result is directly deduced from some simple calculations which are similar to the previous proof of the theorem. Hence, we have finished the proof. □

Corollary 3.10. *Under the hypotheses of Theorem 3.9, we have*

$$\Phi\Big(\frac{\Gamma(\alpha+1)}{2\Gamma(\beta)(b-a)^{\alpha}}[I_a^{\alpha}f(b)+{}_bI^{\alpha}f(a)]\Big) \leq \frac{\Gamma(\alpha+1)}{2\Gamma(\beta)(b-a)^{\alpha}}[I_a^{\alpha}\Phi(f(b))+{}_bI^{\alpha}\Phi(f(a))]. \tag{3.5}$$

Remark 3.11. When $\alpha \in (0, 1]$, that is, we take $n = 0$ in Theorem 3.9, then the following assertions hold for all continuously convex functions $\Phi : \mathbb{R} \to \mathbb{R}$ and all continuous functions $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$.

(i) If $x \in (a, b]$, then

$$\Phi\Big(\frac{\alpha}{(x-a)^{\alpha}}I_a^{\alpha}f(x)\Big) \leq \frac{\alpha}{(x-a)^{\alpha}}I_a^{\alpha}\Phi(f(x)).$$

(ii) If $x \in [a, b)$, then

$$\Phi\Big(\frac{\alpha}{(b-x)^{\alpha}}{}_bI^{\alpha}f(x)\Big) \leq \frac{\alpha}{(b-x)^{\alpha}}{}_bI^{\alpha}\Phi(f(x)).$$

3.2 Hermite-Hadamard type inequalities via fractional integrals

The following theorem was established in [7, Theorem 2.1]. However, we again propose it here by the different way.

Theorem 3.12. *If $\Phi : [a^\rho, b^\rho] \subset \mathbb{R} \to \mathbb{R}$ is a continuously convex function such that $\Phi \in \mathcal{X}_c^p(a^\rho, b^\rho)$, the Hermite-Hadamard type inequalities for the Katugampola fractional integrals of order $\alpha > 0$ of Φ holds*

$$\Phi\Big(\frac{a^\rho + b^\rho}{2}\Big) \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}[{}^\rho I_{a+}^\alpha \Phi(b^\rho) +{}^\rho I_{b-}^\alpha \Phi(a^\rho)] \leq \frac{\Phi(a^\rho) + \Phi(b^\rho)}{2}.$$

Proof. By taking $X = [a, b]$ with $0 \leq a < b$, $f(t) = t^\rho$ for all $t \in [a, b]$ and the measure

$$d\mu(t) = \frac{\rho^{1-\alpha} t^{\rho-1}}{\Gamma(\alpha)} \left(\frac{1}{(b^\rho - t^\rho)^{1-\alpha}} + \frac{1}{(t^\rho - a^\rho)^{1-\alpha}} \right) dt,$$

it is easy to see that

$$a^\rho = \min_{t\in[a,b]} f(t), \quad b^\rho = \max_{t\in[a,b]} f(t)$$

and

$$\mu(X) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^b t^{\rho-1} \left(\frac{1}{(b^\rho - t^\rho)^{1-\alpha}} + \frac{1}{(t^\rho - a^\rho)^{1-\alpha}} \right) dt = \frac{2(b^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)},$$

$$\begin{aligned}\int_X f(t) d\mu(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^b t^{2\rho-1} \left(\frac{1}{(b^\rho - t^\rho)^{1-\alpha}} + \frac{1}{(t^\rho - a^\rho)^{1-\alpha}} \right) dt \\ &= \frac{(a^\rho + b^\rho)(b^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)}.\end{aligned}$$

By replacing these quantities in Corollary 2.3, we obtain the desired result, the details are omitted. □

The following result on the Hermite-Hadamard type inequality for conformable fractional integrals was proposed by Anderson [5] (see also [13]), but under the hypothesis which the conformable fractional derivative of the function increases. We relax this hypothesis in the following.

Theorem 3.13. *If $\Phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a convex function, the Hermite-Hadamard type inequality for conformable fractional integrals of order $\alpha > 0$ of Φ holds*

$$\begin{aligned}\Phi\Big(\frac{a+b}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\beta)}[I_a^\alpha \Phi(b) + {}_bI^\alpha \Phi(a)] \\ &\leq \frac{\Phi(a) + \Phi(b)}{2}.\end{aligned}$$

Proof. First, we take $X = [a, b]$ a nontrivial interval of $\mathbb{R}$, the function $f(t) = t$ for all $t \in [a, b]$ and the measure

$$d\mu(t) = \frac{1}{n!}(b-t)^n(t-a)^{\beta-1}dt.$$

From the proof of Theorem 3.9, it is easy to see that

$$\mu(X) = \frac{(b-a)^\alpha \Gamma(\beta)}{\Gamma(\alpha+1)}. \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}
\int_X (t-a)d\mu(t) &= \frac{1}{n!}\int_a^b [(b-a)-(t-a)]^n (t-a)^\beta dt \\
&= \frac{(b-a)^{n+\beta+1}}{n!}\int_a^b \Big(1-\frac{t-a}{b-a}\Big)^n\Big(\frac{t-a}{b-a}\Big)^\beta \frac{dt}{b-a} \\
&= \frac{(b-a)^{\alpha+1}}{n!}\int_0^1 (1-\lambda)^n\lambda^\beta d\lambda \\
&= \frac{(b-a)^{\alpha+1}}{n!}B(n+1,\beta+1) \\
&= \frac{(b-a)^{\alpha+1}}{n!}\frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+2)} \\
&= \frac{(b-a)^{\alpha+1}\Gamma(\beta+1)}{\Gamma(\alpha+2)},
\end{aligned} \tag{3.7}$$

where we have used the change of variable $\lambda = \frac{t-a}{b-a}$. A combination of (3.6) and (3.7) gives

$$\begin{aligned}
\int_X t d\mu(t) &= \int_X (t-a)d\mu(t) + a\int_X d\mu(t) \\
&= \frac{(b-a)^{\alpha+1}\Gamma(\beta+1)}{\Gamma(\alpha+2)} + a\frac{(b-a)^\alpha\Gamma(\beta)}{\Gamma(\alpha+1)} \\
&= \frac{(b-a)^\alpha\Gamma(\beta)}{\Gamma(\alpha+1)}\Big(\frac{\beta(b-a)}{\alpha+1}+a\Big) \\
&= \frac{(b-a)^\alpha\Gamma(\beta)}{\Gamma(\alpha+1)}\frac{\beta b+(n+1)a}{\alpha+1}.
\end{aligned}$$

This, together with (3.6), leads to

$$\frac{1}{\mu(X)}\int_X t d\mu(t) = \frac{\beta b+(n+1)a}{\alpha+1}. \tag{3.8}$$

Replacing $d\mu(t)$ and (3.8) in (2.7), we get

$$\begin{aligned}\Phi\Big(\frac{\beta b+(n+1)a}{\alpha+1}\Big) &\le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\beta)} I_a^{\alpha}\Phi(b)\\ &\le \frac{b\Phi(a)-a\Phi(b)}{b-a}+\frac{\Phi(b)-\Phi(a)}{b-a}\frac{\beta b+(n+1)a}{\alpha+1}.\end{aligned} \tag{3.9}$$

An argument exactly like the previous one for the measure

$$d\mu(t)=\frac{1}{n!}(t-a)^n(b-t)^{\beta-1}dt$$

gives us the following estimates

$$\begin{aligned}\Phi\Big(\frac{(n+1)b+\beta a}{\alpha+1}\Big) &\le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\beta)} {}_bI^{\alpha}\Phi(a)\\ &\le \frac{b\Phi(a)-a\Phi(b)}{b-a}+\frac{\Phi(b)-\Phi(a)}{b-a}\frac{(n+1)b+\beta a}{\alpha+1}.\end{aligned} \tag{3.10}$$

From (3.9), (3.10) and the convexity of Φ, we infer

$$\begin{aligned}\Phi\Big(\frac{a+b}{2}\Big) &= \Phi\Big(\frac{1}{2}\frac{\beta b+(n+1)a}{\alpha+1}+\frac{1}{2}\frac{(n+1)b+\beta a}{\alpha+1}\Big)\\ &\le \frac{1}{2}\Phi\Big(\frac{\beta b+(n+1)a}{\alpha+1}\Big)+\frac{1}{2}\Phi\Big(\frac{(n+1)b+\beta a}{\alpha+1}\Big)\\ &\le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\beta)}[I_a^{\alpha}\Phi(b)+{}_bI^{\alpha}\Phi(a)]\\ &\le \frac{b\Phi(a)-a\Phi(b)}{b-a}+\frac{\Phi(b)-\Phi(a)}{b-a}\frac{a+b}{2}\\ &= \frac{\Phi(a)+\Phi(b)}{2},\end{aligned}$$

which are the desired results. □

Remark 3.14. Under the hypotheses of Theorem 3.13 and $\alpha\in(0,1]$, we have

$$\begin{aligned}\Phi\Big(\frac{a+b}{2}\Big) &\le \frac{\alpha}{2(b-a)^{\alpha}}[I_a^{\alpha}\Phi(b)+{}_bI^{\alpha}\Phi(a)]\\ &\le \frac{\Phi(a)+\Phi(b)}{2}.\end{aligned}$$

References

[1] S. Abbaszadeh, A. Ebadian, M. Jaddi, *Jensen type inequalities and their applications via fractional integrals,* Rocky Mountain J. Math. **48** (8) (2018), 2459-2488.

[2] T. Abdeljawad, *On conformable fractional calculus,* J. Comput. Appl. Math. **279** (2015), 57-66.

[3] A. A. Aljinovic, *Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral*, J. Math. 2014, Art. ID 503195, 6 pp.

[4] A. A. Aljinovic, M. Krnić, J. Pečarić, *Weighted Montgomery identity for the fractional integral of a function with respect to another function*, Georgian Math. J. **21**(1) (2014), 1-10.

[5] D. Anderson, *Taylor's formula and integral inequalities for conformable fractional derivatives*, Contributions in mathematics and engineering, 25-43, Springer, 2016.

[6] S. Asliyüce, A. F. Güvenilir, *Fractional Jensen's inequality,* Palest. J. Math., **7**(2) (2018), 554-558.

[7] H. Chen, U. N. Katugampola, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals,* J. Math. Anal. Appl., **446**(2) (2017), 1274-1291.

[8] S. S. Dragomir, *A functional generalization of Ostrowski inequality via Montgomery identity,* Acta Math. Univ. Comenian, **84**(1) (2015), 63-78.

[9] S. S. Dragomir, *A functional generalization of trapezoid inequality,* Vietnam J. Math. **43**(4) (2015), 663-675.

[10] K.-C. Hsu, S.-R. Hwang, K.-L. Tseng, *Some extended Simpson-type inequalities and applications*, Bull. Iranian Math. Soc., **43**(2) (2017), 409-425.

[11] U. N. Katugampola, *New approach to generalized fractional derivatives*, Bull. Math. Anal. Appl., **6**(4) (2014), 1-15.

[12] R. Khalil, M. A. Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative,* J. Comput. Appl. Math., **264** (2014), 65-70.

[13] M. A. Khan, T. Ali, S. S. Dragomir, M. Z. Sarikaya, *Hermite-Hadamard type inequalities for conformable fractional integrals*, Rev. R. Acad. Cienc. Exactas Fís, Nat. Ser. A Mat. RACSAM, **112**(4) (2018), 1033-1048.

[14] W. Liu, *Ostrowski type fractional integral inequalities for MT-convex functions*, Miskolc Math. Notes, **16**(1) (2015), 249-256.

[15] J. E. Pečarić, *On the Čebyšev inequality*, Bul. Stiint. Tehn. Inst. Politehn. "Traian Vuia" Timisoara 25, **39**(1) (1980), 5-9.

[16] M. Z. Sarikaya, H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Amer. Math. Soc., **145**(4)(2017), 1527-1538.