East-West J. of Mathematics: Vol. 8, No 1 (2006) pp. 61-76

M-SOLID STRONG QUASIVARIETIES OF PARTIAL ALGEBRAS

Saofee Busaman and **Klaus Denecke**⋆

Department of Mathematics and Computer Science
Prince of Songkla University
Pattani 94000, Thailand
e-mail: kdenecke@rz.uni-potsdam.de

⋆ *Institute of Mathematics*
University of Potsdam
Am Neuen Palais, 14415 Potsdam, Germany
e-mail: bsaofee@bunga.pn.psu.ac.th

Dedicated to H.-J. Hoehnke

Abstract

In this paper the hyperequational theory for strong and strong regular varieties of partial algebras will be extended to strong and strong regular quasivarieties of partial algebras. Two kinds of strong quasi-identities are defined and the corresponding model classes are characterized.

1 Introduction

Let $P^n(A) := \{f : A^n \multimap\!\!\rightarrow A\}$ be the set of all $n-ary$ partial operations defined on the set A and let $P(A) := \bigcup_{n=1}^{\infty} P^n(A)$ be the set of all partial operations on A. A *partial algebra* $\mathcal{A} = (A; (f_i^A)_{i\in I})$ of type $\tau = (n_i)_{i\in I}$ is a pair consisting of a set A and an indexed set $(f_i^A)_{i\in I}$ of partial operations where f_i^A is $n_i - ary$. Let $PAlg(\tau)$ be the class of all partial algebras of type τ. If $f \in P^n(A)$ is a partial operation, then $domf$ denotes the domain of f. To define the language corresponding to partial algebras of type τ we need an indexed set $(f_i)_{i\in I}$ of operation symbols, where f_i is n_i-ary and a finite or a countably infinite alphabet $X_n = \{x_1, x_2, \ldots, x_n\}$ $(X = \{x_1, x_2, \ldots\})$ of variables. Then *n-ary terms* of type τ are defined inductively as follows:

Key words: Partial algebra, strong quasivariety, strong M-hyperquasi-identity, M-solid strong quasivariety.
2000 AMS Mathematics Subject Classification: 08A55, 08C15, 08A15, 08B15.

(i) The variables $x_1, \ldots, x_n$ are n-ary terms.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and if f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

We denote by $W_\tau(X_n)$ the set of all n-ary terms of type τ and let $W_\tau(X) := \bigcup\limits_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ. From given terms by superposition one can produce new terms. For each m and n in $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$, the superposition operation S^n_m maps one n-ary term and n m-ary terms to an m-ary term, so that

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m).$$

The operation S^n_m is defined inductively, by setting
$S^n_m(x_j, t_1, \ldots, t_n) = t_j$ for any variable $x_j \in X_n$, and
$S^n_m(f_r(s_1, \ldots, s_{n_r}), t_1, \ldots, t_n) = f_r(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_r}, t_1, \ldots, t_n))$.

To each term $t \in W_\tau(X_n)$ and to each partial algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$ of type τ we obtain a partial operation $t^{\mathcal{A}}$, called term operation induced by t as follows:

(i) If $t = x_i \in X_n$ then $t^{\mathcal{A}} = x_i^{\mathcal{A}} := e_i^{n,A}$, where $e_i^{n,A}$ is the n-ary total projection on the i-th component.

(ii) Now assume that $t = f_i(t_1, \ldots, t_{n_i})$ where f_i is an n_i-ary operation symbol, and assume also that $t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}}$ are the term operations induced by the terms $t_1, \ldots, t_{n_i}$, and that the $t_j^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined, with values $t_j^{\mathcal{A}}(a_1, \ldots, a_n) = b_j$, for $1 \leq j \leq n_i$. If $f_i^A(b_1, \ldots, b_{n_i})$ is defined, then $t^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and $t^{\mathcal{A}}(a_1, \ldots, a_n) = f_i^A(t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_{n_i}^{\mathcal{A}}(a_1, \ldots, a_n))$.

Let $W_\tau(X_n)^{\mathcal{A}}$ be the set of all n-ary term operations of type τ. On the right hand side we formed the superposition of partial operations. This superposition can also be described by a (total) superposition operation (on partial operations) as follows

$$S^{n,A}_m(f^A, g_1^A, \ldots, g_n^A)(a_1, \ldots, a_m) := f^A(g_1^A(a_1, \ldots, a_m), \ldots, g_n^A(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m)$ for which $g_1^A, \ldots, g_n^A$ are defined and for which the values $b_1 = g_1^A(a_1, \ldots, a_m), \ldots, b_n = g_n^A(a_1, \ldots, a_m)$ form an n-tuple $(b_1, \ldots, b_n)$ belonging to the domain of f^A. In general the set $W_\tau(X)^{\mathcal{A}}$ is different from the set of all partial operations which is generated by $\{f_i^A \mid i \in I\}$ using the operation $S^{n,A}_m$ (see e.g. [9]).

In this paper we are interested in the following kind of identities in partial algebras.

Definition 1.1. ([8]) A pair $t_1 \approx t_2 \in W_\tau(X)^2$ is called a *strong identity* in a partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{s}{\models} t_1 \approx t_2$) iff the right hand side is defined whenever the left hand side is defined and both are equal, i.e. when both sides are defined, then the induced partial term operations $t_1^{\mathcal{A}}$ and $t_2^{\mathcal{A}}$ are equal.

Definition 1.2. A *quasi-equation* of type τ is a first order formula of the form

$$e : \forall x_1, \ldots, x_s(s_1 \approx t_1 \wedge s_2 \approx t_2 \wedge \ldots \wedge s_n \approx t_n \Rightarrow u \approx v)$$

where $s_1, \ldots, s_n, t_1, \ldots, t_n, u, v \in W_\tau(X)$ and where $\wedge, \Rightarrow$ are the binary propositional connectives conjunction and implication.
For abbreviation with $e' : s_1 \approx t_1 \wedge s_2 \approx t_2 \wedge \ldots \wedge s_n \approx t_n$ and $e'' : u \approx v$ we write

$$e : \forall x_1, \ldots, x_s(e' \Rightarrow e'').$$

Then the quasi-equation e is satisfied in the partial algebra $\mathcal{A}$ as a strong quasi-identity if from $s_1^{\mathcal{A}} = t_1^{\mathcal{A}} \wedge \ldots \wedge s_n^{\mathcal{A}} = t_n^{\mathcal{A}}$ it follows $u^{\mathcal{A}} = v^{\mathcal{A}}$. In this case we write $\mathcal{A} \underset{sq}{\models} e$.

We notice that in [2] strong quasi-identities are denoted as QE-equations.

Using the relation $\underset{sq}{\models}$ for every class K of partial algebras of type τ and for every set Σ of quasi-equations (i.e. implications of the form $e' \Rightarrow e''$) we form the sets

$$\begin{array}{lcll} QId^sK & := & \{e \in \Sigma \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{sq}{\models} e)\} & \text{and} \\ QMod^s\Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall\ e \in \Sigma \ (\mathcal{A} \underset{sq}{\models} e)\}. & \end{array}$$

Definition 1.3. Let $QV \subseteq PAlg(\tau)$ be a class of partial algebras. The class QV is called a *strong quasivariety* of partial algebras if $QV = QMod^sQId^sQV$.

We have that strong quasivarieties different from $\{\underline{\emptyset}\}$ (i.e. the empty partial algebra) are closed under formation of closed subalgebras, isomorphic images, and filtered products ([2]). Conversely, a class QV of partial algebras of type τ which is closed under formation of closed subalgebras, isomorphic images, and filtered products is a strong quasivariety of partial algebras. For more background on theories for partial algebras see [6].

2 Strong Quasi-identities

Since in general the set $W_\tau(X_n)^{\mathcal{A}}$ is different from the set of all partial operations generated by $\{f_i^{\mathcal{A}} \mid i \in I\}$ we need a new definition of terms over partial algebras of type τ.

Let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ, where each f_i has arity n_i and $X \cap \{f_i \mid i \in I\} = \emptyset$. We need additional symbols $\varepsilon_j^k \notin X$, for every $k \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ and $1 \leq j \leq k$. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element alphabet and let X be an arbitrary countable alphabet. The set of all n-ary terms of type τ over X_n is defined inductively as follows (see [1]):

(i) Every $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $w_1, \ldots, w_k$ are n-ary terms of type τ, then $\varepsilon_j^k(w_1, \ldots, w_k)$ is an n-ary term of type τ for all $1 \leq j \leq k$ and all $k \in \mathbb{N}^+$.

(iii) If $w_1, \ldots, w_n$ are n-ary terms of type τ and if f_i is an n_i-ary operation symbol, then $f_i(w_1, \ldots, w_{n_i})$ is an n-ary term of type τ.

Let $W_\tau^C(X_n)$ be the set of all n-ary terms of type τ defined in this way. Then $W_\tau^C(X) := \bigcup\limits_{n=1}^{\infty} W_\tau^C(X_n)$ denotes the set of all terms of this type.

Remark 2.1. $W_\tau(X) \subseteq W_\tau^C(X)$.

Every n-ary term $w \in W_\tau^C(X_n)$ induces an n-ary term operation $w^{\mathcal{A}}$ on any partial algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ of type τ. For $a_1, \ldots, a_n \in A$, the value $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined in the following inductive way (see [1]):

(i) If $w = x_i$ then $w^{\mathcal{A}} = x_i^{\mathcal{A}} = e_i^{n,A}$, where $e_i^{n,A}$ is as usual the n-ary total projection on the i-th component.

(ii) If $w = \varepsilon_j^k(w_1, \ldots, w_k)$ and we assume that $w_1^{\mathcal{A}}, \ldots, w_k^{\mathcal{A}}$ are the term operations induced by the terms $w_1, \ldots, w_k$ and that the $w_i^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined for $1 \leq i \leq k$, then $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and $w^{\mathcal{A}}(a_1, \ldots, a_n) = w_j^{\mathcal{A}}(a_1, \ldots, a_n)$.

(iii) Now assume that $w = f_i(w_1, \ldots, w_{n_i})$ where f_i is an n_i-ary operation symbol, and assume that the $w_j^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined, with values $w_j^{\mathcal{A}}(a_1, \ldots, a_n) = b_j$ for $1 \leq j \leq n_i$. If $f_i^A(b_1, \ldots, b_{n_i})$ is defined, then $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and $w^{\mathcal{A}}(a_1, \ldots, a_n) = f_i^A(w_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, w_{n_i}^{\mathcal{A}}(a_1, \ldots, a_n))$.

On the sets $W_\tau^C(X_n)$ we may introduce the following superposition operations. Let $w_1, \ldots, w_m$ be n-ary terms and let t be an m-ary term. Then we define an n-ary term $\overline{S}_n^m(t, w_1, \ldots, w_m)$ inductively by the following steps:

(i) For $t = x_j$, $1 \leq j \leq m$ (m-ary variable), we define
$\overline{S}_n^m(x_j, w_1, \ldots, w_m) = w_j$.

(ii) For $t = \varepsilon_j^k(s_1, \ldots, s_k)$ we set
$\overline{S}_n^m(t, w_1, \ldots, w_m) = \varepsilon_j^k(\overline{S}_n^m(s_1, w_1, \ldots, w_m), \ldots, \overline{S}_n^m(s_k, w_1, \ldots, w_m))$,
where $s_1, \ldots, s_k$ are m-ary, for all $k \in \mathbb{N}^+$ and $1 \leq j \leq k$.

(iii) For $t = f_i(s_1, \ldots, s_{n_i})$ we set
$\overline{S}^m_n(t, w_1, \ldots, w_m) = f_i(\overline{S}^m_n(s_1, w_1, \ldots, w_m), \ldots, \overline{S}^m_n(s_{n_i}, w_1, \ldots, w_m))$,
where $s_1, \ldots, s_{n_i}$ are m-ary.

This defines operations

$$\overline{S}^m_n : W^C_\tau(X_m) \times (W^C_\tau(X_n))^m \longrightarrow W^C_\tau(X_n),$$

which describe the superposition of terms.

The *term clone* of type τ is the heterogeneous algebra

$$clone\tau^C := ((W^C_\tau(X_n)); \overline{S}^m_n, e^k_j)_{n,m,k\in\mathbb{N}^+, 1\leq j\leq k},$$

where $e^k_j := x_j \in X_k$, $1 \leq j \leq k$.

Definition 2.2. A pair $s \approx t \in W^C_\tau(X)^2$ is called a *strong c-identity* in a partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{s}{\models} s \approx t$) iff the right hand side is defined whenever the left hand side is defined and both are equal, i.e. when both sides are defined, then the induced partial term operations $s^{\mathcal{A}}$ and $t^{\mathcal{A}}$ are equal.

Definition 2.3. A quasi-equation ce of type τ is satisfied in the partial algebra $\mathcal{A}$ as a strong *c-quasi-identity* if from $s^{\mathcal{A}}_1 = t^{\mathcal{A}}_1 \wedge \ldots \wedge s^{\mathcal{A}}_n = t^{\mathcal{A}}_n$ it follows $u^{\mathcal{A}} = v^{\mathcal{A}}$. In this case we write $\mathcal{A} \underset{sq}{\models} ce$.

Let $c\Sigma$ be a set of c-quasi-equations (i.e. implications of the form $ce' \Rightarrow ce''$). Let $cQ\tau$ denote the set of all c-quasi-equations of type τ and let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ. Consider the connection between $PAlg(\tau)$ and $cQ\tau$ given by the following two operators:

$$QId^s : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(cQ\tau) \text{ and}$$

$$QMod^s : \mathcal{P}(cQ\tau) \to \mathcal{P}(PAlg(\tau)) \text{ with}$$

$$\begin{array}{lcl} cQId^sK & := & \{ce \in c\Sigma \mid \forall \mathcal{A} \in K\ (\mathcal{A} \underset{sq}{\models} ce)\} \quad \text{and} \\ QMod^sc\Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall\ ce \in c\Sigma\ (\mathcal{A} \underset{sq}{\models} ce)\}. \end{array}$$

Clearly, the pair $(QMod^s, cQId^s)$ is a Galois connection between $PAlg(\tau)$ and $cQ\tau$, i.e it satisfies the following properties:

$$K_1 \subseteq K_2 \Rightarrow cQId^sK_2 \subseteq cQId^sK_1, c\Sigma_1 \subseteq c\Sigma_2 \Rightarrow QMod^sc\Sigma_2 \subseteq QMod^sc\Sigma_1$$

and

$$K \subseteq QMod^scQId^sK, c\Sigma \subseteq cQId^sQMod^sc\Sigma.$$

The products $QMod^scQId^s$ and $cQId^sQMod^s$ are closure operators and their fixed points form complete lattices.

Definition 2.4. Let $QV \subseteq PAlg(\tau)$ be a class of partial algebras. The class QV is called a *strong c-quasivariety* of partial algebras if $QV = QMod^scQId^sQV$.

3 Strong Hyperquasi-identities

In [3] (see also [5]) hyperquasi-identities for total algebras were introduced. We want to generalize this approach to partial algebras but instead of terms from $W_\tau(X)$ as in [3] we will use terms from $W_\tau^C(X)$. Since from now on we want to consider only terms from $W_\tau^C(X)$ instead of c-terms we will simply speak of terms.

To define strong hyperquasi-identities we need the concept of a hypersubstitution.

Definition 3.1. ([9]) Let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ and $W_\tau^C(X)$ be the set of all terms of this type. A mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau^C(X)$ which maps each n_i-ary fundamental operation f_i to a term of arity n_i is called a *hypersubstitution* of type τ.

Any hypersubstitution σ of type τ can be extended to a map $\widehat{\sigma} : W_\tau^C(X) \longrightarrow W_\tau^C(X)$ defined for all terms, in the following way:

(i) $\widehat{\sigma}[x_i] = x_i$ for every $x_i \in X_n$,

(ii) $\widehat{\sigma}[\varepsilon_j^k(s_1, \ldots, s_k)] = \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_k])$, where $s_1, \ldots, s_k \in W_\tau^C(X_n)$,

(iii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \overline{S}_n^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, where $t_1, \ldots, t_{n_i} \in W_\tau^C(X_n)$.

Let $Var(t)$ be the set of all variables occurring in the term t.

Definition 3.2. ([7]) The hypersubstitution σ is called *regular* if $Var(\sigma(f_i)) = \{x_1, \ldots, x_{n_i}\}$, for all $i \in I$.

Let $Hyp_R^C(\tau)$ be the set of all regular hypersubstitutions of type τ and let σ_R denote some member of $Hyp_R^C(\tau)$. Now we define a product of regular hypersubstitutions in the usual way, by $\sigma_{R_1} \circ_h \sigma_{R_2} := \widehat{\sigma}_{R_1} \circ \sigma_{R_2}$ where $\circ$ is the usual composition of functions and obtain:

Theorem 3.3. ([9]) *The algebra* $(Hyp_R^C(\tau); \circ_h, \sigma_{id})$ *with* $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ *is a monoid.*

Let $\mathcal{M} \subseteq \mathcal{H}yp_R^C(\tau)$ be a submonoid of the monoid of all regular hypersubstitutions. If $\mathcal{A} = (A; (f_i^A)_{i \in I})$ is a partial algebra of type τ, then we say that the equation $s \approx t \in W_\tau^C(X)^2$, is an *M-hyperidentity* in $\mathcal{A}$ if $\mathcal{A} \underset{s}{\models} \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]$ for every $\sigma_R \in M$. In this case we write $\mathcal{A} \underset{sMh}{\models} s \approx t$.

The relation $\underset{sMh}{\models}$ defines a Galois-connection $(H_M Mod^s, H_M Id^s)$ where these operators for classes $K \subseteq PAlg(\tau)$ of partial algebras of type τ and for sets $\Sigma \subseteq W_\tau^C(X)^2$ of equations of type τ are defined by

$$H_M Mod^s\Sigma \quad := \quad \{\mathcal{A}\in PAlg(\tau) \mid \forall\, s\approx t\in\Sigma\ (\mathcal{A} \underset{sMh}{\models} s\approx t)\},$$
$$H_M Id^s K \quad := \quad \{s\approx t\in W_\tau^C(X)^2 \mid \forall\mathcal{A}\in K\ (\mathcal{A} \underset{sMh}{\models} s\approx t)\}.$$

The products $H_M Mod^s H_M Id^s$ and $H_M Id^s H_M Mod^s$ are closure operators and their fixed points form two complete lattices which are dually isomorphic. The application of a regular hypersubstitution to a partial algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ of type $\tau = (n_i)_{i\in I}$ leads us to the concept of a *derived algebra* $\sigma_R(\mathcal{A}) = (A; (\sigma_R(f_i)^{\mathcal{A}})_{i\in I})$, where $\sigma_R(f_i)^{\mathcal{A}}$ is the term operation induced by the term $\sigma_R(f_i)$ on the algebra $\mathcal{A}$.
For a class K of partial algebras of type τ and for a set Σ of equations of regular terms of type τ (i.e. the both sides of the equation contain the same variables), we may form

$$\chi_M^E[\Sigma] := \bigcup_{\sigma_R\in M}\ \bigcup_{s\approx t\in\Sigma} \widehat{\sigma}_R[s]\approx\widehat{\sigma}_R[t] \quad\text{and}\quad \chi_M^A[K] := \bigcup_{\sigma_R\in M}\ \bigcup_{\mathcal{A}\in K}\sigma_R(\mathcal{A}).$$

In ([9]), was shown that χ_M^E and χ_M^A are closure operators which are completely additive by their definition and this means that they satisfy the condition

$$(\star) \qquad \mathcal{A} \underset{s}{\models} \chi_M^E[s\approx t] \Leftrightarrow \chi_M^A[\mathcal{A}] \underset{s}{\models} s\approx t.$$

The property $(\star)$ is called the conjugate property and (χ_M^E, χ_M^A) is called a conjugate pair of additive closure operators. A strong variety V of partial algebras of type τ is called *M-solid* if $\chi_M^A[V] = V$ (see [9]).

Theorem 3.4. ([9]) *Let V be a strong variety of partial algebras of type τ and let $\mathcal{M}$ be a monoid of regular hypersubstitutions of type τ. Then the following conditions are equivalent:*

(i) *There is a set $\Sigma\subseteq W_\tau^C(X)^2$ such that $V = H_M Mod^s\Sigma$.*

(ii) *V is M-solid, i.e. $\chi_M^A[V] = V$.*

(iii) *$Id^sV = H_M Id^s V$, i.e. every strong identity in V is an M-hyperidentity in V.*

(iv) *$\chi_M^E[Id^sV] = Id^sV$.*

Every set $\Sigma\subseteq W_\tau^C(X)^2$ for which there is a strong variety V of partial algebras of type τ with $\Sigma = H_M Id^s V$ is called an *M-hyperquational theory.* If $M = Hyp_R^C(\tau)$ we speak of a hyperequational theory and if $M = \{\sigma_{id}\}$, we have the usual case of an equational theory. M-hyperequational theories can be characterized by the following equivalent conditions:

Theorem 3.5. ([9]) *Let Σ be an equational theory of type τ and let M be a monoid of regular hypersubstitutions of type τ. Then the following conditions are equivalent:*

(i) *There is a class V of partial algebras of type τ such that $\Sigma = H_M Id^s V$.*

(ii) $\chi_M^E[\Sigma] = \Sigma$.

(iii) $Mod^s\Sigma = H_M Mod^s\Sigma$.

(iv) $\chi_M^A[Mod^s\Sigma] = Mod^s\Sigma$.

Definition 3.6. Let $\mathcal{A}$ be a partial algebra of type τ and let M be a submonoid of the monoid $Hyp_R^C(\tau)$. Then the quasi-equation

$$ce := (s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n \Rightarrow u \approx v)$$

of type τ in $\mathcal{A}$ is a *strong M-hyperquasi-identity* in $\mathcal{A}$ if for every regular hypersubstitution $\sigma_R \in M$, the formulas

$$\widehat{\sigma}_R[ce] := (\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \wedge \ldots \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n] \Rightarrow \widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v])$$

are strong quasi-identities in $\mathcal{A}$. For $M = Hyp_R^C(\tau)$, we speak simply of a *strong hyperquasi-identity* in $\mathcal{A}$.
A strong quasivariety V of type τ is called *M-solid* if $\chi_M^A[V] = V$. If *ce* is a strong M-hyperquasi-identity in $\mathcal{A}$ or in V, we will write $\mathcal{A} \underset{sMhq}{\models} ce$ or $V \underset{sMhq}{\models} ce$, respectively.

Example 3.7. Consider the strong regular quasivariety V of type $\tau = (2)$ defined by the following strong quasi-identities:

(S1) $x(yz) \approx (xy)z$,

(S2) $x^2 \approx x$,

(S3) $xyx \approx \varepsilon_1^2(x, y)$,

(S4) $xy \approx yx \Rightarrow \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y)$.

Because of (S1), (S2), (S3) we have to consider exactly the following binary terms over V:

$$t_1(x, y) = \varepsilon_1^2(x, y), t_2(x, y) = \varepsilon_2^2(x, y), t_3(x, y) = xy, t_4(x, y) = yx$$

and the regular hypersubstitutions $\sigma_{t_i}, i = 1, \ldots, 4$ which map the binary operation symbol f to the terms $t_i, i = 1, \ldots, 4$. It is easy to see that the application of each of these regular hypersubstitutions to (S1), (S2), (S3), (S4) gives a strong identity or a strong quasi-identity which is satisfied in V. This is enough to show that V is a solid strong quasivariety.

As usual, the relation $\underset{sMhq}{\models}$ induces a Galois-connection. For any set $c\Sigma$ of quasi-equations of type τ and for any class K of partial algebras of type τ we define:

$$\begin{array}{lcl} H_M QMod^s c\Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in c\Sigma(\mathcal{A} \underset{sMhq}{\models} ce)\}, \\ H_M QId^s QK & := & \{ce \in c\Sigma \mid \forall \mathcal{A} \in QK(\mathcal{A} \underset{sMhq}{\models} ce)\}. \end{array}$$

The products $H_M QMod^s H_M QId^s$ and $H_M QId^s H_M QMod^s$ are closure operators. The fixed points with respect to these closure operators form two complete lattices. For a quasi-equation ce, we define $\chi_M^{QE}[ce] := \{\widehat{\sigma}_R[ce] \mid \sigma_R \in M\}$, and for a set $c\Sigma$ of quasi-equations we set $\chi_M^{QE}[c\Sigma] := \bigcup\limits_{ce \in c\Sigma} \chi_M^{QE}[ce]$. Then the following lemma is very easy to prove.

Lemma 3.8. *Let M be a submonoid of $Hyp_R^C(\tau)$. Then the pair (χ_M^A, χ_M^{QE}) is a pair of additive closure operators having the property $\chi_M^A[\mathcal{A}] \underset{sq}{\models} ce \Leftrightarrow \mathcal{A} \underset{sq}{\models} \chi_M^{QE}[ce]$ for any quasi-equation ce (a conjugate pair).*

Proof By definition χ_M^A and χ_M^{QE} are additive closure operators. We will use that for every term t of type τ, for every regular hypersubstitution σ_R and for every partial algebra $\mathcal{A}$, we have $t^{\sigma_R(\mathcal{A})} = \widehat{\sigma}_R[t]^{\mathcal{A}}$ ([9]). Further we have $\chi_M^A[\mathcal{A}] \underset{sq}{\models} ce$

$\Leftrightarrow \chi_M^A[\mathcal{A}] \underset{sq}{\models} (s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n \Rightarrow u \approx v)$

$\Leftrightarrow \forall \sigma_R \in M\ (\sigma_R(\mathcal{A}) \underset{sq}{\models} (s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n \Rightarrow u \approx v))$

$\Leftrightarrow \forall \sigma_R \in M\ (s_1^{\sigma_R(\mathcal{A})} = t_1^{\sigma_R(\mathcal{A})} \wedge \ldots \wedge s_n^{\sigma_R(\mathcal{A})} = t_n^{\sigma_R(\mathcal{A})} \Rightarrow u^{\sigma_R(\mathcal{A})} = v^{\sigma_R(\mathcal{A})})$

$\Leftrightarrow \forall \sigma_R \in M\ (\widehat{\sigma}_R[s_1]^{\mathcal{A}} = \widehat{\sigma}_R[t_1]^{\mathcal{A}} \wedge \ldots \wedge \widehat{\sigma}_R[s_n]^{\mathcal{A}} = \widehat{\sigma}_R[t_n]^{\mathcal{A}} \Rightarrow \widehat{\sigma}_R[u]^{\mathcal{A}} = \widehat{\sigma}_R[v]^{\mathcal{A}})$

$\Leftrightarrow \forall \sigma_R \in M\ (\mathcal{A} \underset{sq}{\models} (\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \wedge \ldots \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n] \Rightarrow \widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v]))$

$\Leftrightarrow \forall \sigma_R \in M(\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[ce])$

$\Leftrightarrow \mathcal{A} \underset{sq}{\models} \chi_M^{QE}[ce].$ $\square$

If $c\Sigma$ is a set of quasi-equations of type τ, then classes of the form $H_M QMod^s c\Sigma$ are called *strong M-hyperquasi-equational* classes and the fixed points under $H_M QId^s H_M QMod^s$ are called *strong M-hyperquasi-equational theories*. Therefore we can characterize M-solid strong quasivarieties by the following conditions:

Theorem 3.9. *Let M be a submonoid of $Hyp_R^C(\tau)$. Then for every strong quasivariety $QV \subseteq PAlg(\tau)$ the following conditions are equivalent:*

(i) *QV is a strong M-hyperquasi-equational class.*

(ii) *QV is M-solid, i.e. $\chi_M^A[QV] = QV$.*

(iii) *$QId^sQV = H_M QId^sQV$, i.e. every strong quasi-identity in QV is strong M-hyperidentity in QV.*

(iv) *$\chi_M^{QE}[QId^sQV] = QId^sQV$,i.e. QId^sQV is closed under the operator χ_M^{QE}.*

Proof (i)$\Rightarrow$ (ii): Since χ_M^A is a closure operator, the inclusion $QV \subseteq \chi_M^A[QV]$ is clear and we have only to show the opposite inclusion. Assume that $\mathcal{B} \in \chi_M^A[QV]$. Then there is a regular hypersubstitution $\sigma_R \in M$ and a partial algebra $\mathcal{A} \in QV$ such that $\mathcal{B} = \sigma_R(\mathcal{A})$. Since QV is a strong M-hyperquasi-equational class, there is a set $c\Sigma$ of quasi-equations such that $QV = H_M QMod^s c\Sigma$ and $\mathcal{A} \in QV$ means that for all regular hypersubstitution $\sigma_R \in M$ and for all $ce \in c\Sigma$, we have $\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[ce]$. By the conjugate property from Lemma 3.8 we have that $\sigma_R(\mathcal{A}) \underset{sq}{\models} ce$ and therefore $\sigma_R(\mathcal{A}) \in QMod^s c\Sigma = QV$ since QV is a strong quasivariety.

(ii)$\Rightarrow$ (iii): From $\chi_M^A[QV] = QV$ implies that $QId^s\chi_M^A[QV] = QId^sQV$. Because of

$$\begin{aligned} QId^s\chi_M^A[QV] &= \{ce \mid \forall \sigma_R \in M, \forall \mathcal{A} \in QV(\sigma_R(\mathcal{A}) \underset{sq}{\models} ce)\} \\ &= \{ce \mid \forall \sigma_R \in M, \forall \mathcal{A} \in QV(\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[ce])\} \\ &= H_M QId^sQV \end{aligned}$$

we have $H_M QId^sQV = QId^sQV$.

(iii)$\Rightarrow$ (iv): The inclusion $QId^sQV \subseteq \chi_M^{QE}[QId^sQV]$ follows from the property of χ_M^{QE}. We only have to show the opposite inclusion. Let $\sigma_R \in M$ and $ce \in QId^sQV$. Then $\widehat{\sigma}_R[ce] \in QId^sQV$ since $QId^sQV = H_M QId^sQV$.

(iv)$\Rightarrow$ (i): From $\chi_M^{QE}[QId^sQV] = QId^sQV$ by applying the operator $QMod^s$ on both sides we obtain the equation

$$QV = QMod^sQId^sQV = QMod^s(\chi_M^{QE}[QId^sQV]).$$

Considering the right hand side, we get

$$\begin{aligned} QMod^s(\chi_M^{QE}[QId^sQV]) &= \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in QId^sQV, \forall \sigma_R \in M \\ &\qquad (\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[ce])\} \\ &= H_M QMod^sQId^sQV \end{aligned}$$

and therefore with $c\Sigma = QId^sQV$ and we have shown that QV is a strong M-hyperquasi-equational class. $\square$

The following Theorem is a consequence of the general theory of conjugate pairs of additive closure operators (see[7]).

Theorem 3.10. *Let M be a submonoid of $Hyp_R^C(\tau)$. Then for every strong quasi-equational theory $c\Sigma$, the following conditions are equivalent:*

(i) *$c\Sigma$ is a strong M-hyperquasi-equational theory, i.e. there is a class QV of partial algebras of type τ such that $c\Sigma = H_M QId^sQV$.*

(ii) *$\chi_M^{QE}[c\Sigma] = c\Sigma$.*

(iii) *$QMod^s c\Sigma = H_M QMod^s c\Sigma$.*

(iv) *$\chi_M^A[QMod^s c\Sigma] = QMod^s c\Sigma$.*

Proof The proof can be given in a similar way as in ([3]). $\square$

4 Weakly M-solid strong quasivarieties

Now we define a different concept of M-hypersatisfaction of a quasi-equation. This leads us to weakly M-solid strong quasivarieties. We will use the operator χ_M^E introduced in Section 3.

Definition 4.1. Let $\mathcal{A}$ be a partial algebra of type τ, let $\mathcal{M}$ be a monoid of regular hypersubstitutions, and let $ce := (s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n \Rightarrow u \approx v)$ be a quasi-equation of type τ. Then ce is called a *weakly strong M-hyperquasi-identity* in $\mathcal{A}$ if the implication:

$$\chi_M^E[\{s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n\}] \Rightarrow \chi_M^E[u \approx v]$$

is satisfied in $\mathcal{A}$. In this case we write $\mathcal{A} \underset{wsMhq}{\models} ce$. If every partial algebra $\mathcal{A}$ of a class QV has this property, we write $QV \underset{wsMhq}{\models} ce$.

Proposition 4.2. *If ce is a strong M-hyperquasi-identity in the class QV of partial algebras of type τ, then ce is a weakly strong M-hyperquasi-identity in QV but not conversely.*

Proof If ce is a strong M-hyperquasi-identity in QV then for every $\sigma_R \in M$ we have $\widehat{\sigma}_R[ce] \in QId^sQV$. Therefore we have

$$\forall \sigma_R \in M((\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \wedge \ldots \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n] \Rightarrow \widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v]) \in QId^sQV). (*)$$

Using the rules of the predicate calculus from $(*)$ we get,

$$(\forall\sigma_R \in M(\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \quad \wedge \ldots \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n]) \Rightarrow \\ \Rightarrow \forall\sigma_R \in M(\widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v])) \subseteq QId^sQV$$

and this means

$$(\chi_M^E[s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n] \Rightarrow \chi_M^E[u \approx v]) \subseteq QId^sQV(**)$$

and therefore ce is satisfied as a weakly strong M-hyperquasi-identity in QV. The converse is not true since it could be possible to find a regular hypersubstitution $\sigma_{R_1} \in M$ with

$$\widehat{\sigma}_{R_1}[s_1] \approx \widehat{\sigma}_{R_1}[t_1] \wedge \ldots \wedge \widehat{\sigma}_{R_1}[s_n] \approx \widehat{\sigma}_{R_1}[t_n] \not\Rightarrow \widehat{\sigma}_{R_1}[u] \approx \widehat{\sigma}_{R_1}[v] \notin QId^sQV$$

even if $(**)$ is satisfied. $\square$

Using this new concept we define:

Definition 4.3. A strong quasivariety QV of partial algebras of type τ is *weakly M-solid* if every $ce \in QId^sQV$ is a weakly strong M-hyperquasi-identity in QV.

Our next aim is to characterize weakly M-solid strong quasivarieties.
In the usual way the relation $\underset{wsMhq}{\models}$ induces a Galois connection if we define:

$$\begin{aligned} WH_MQMod^sc\Sigma &:= \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in c\Sigma(\mathcal{A} \underset{wsMhq}{\models} ce)\}, \\ WH_MQId^sQK &:= \{ce \in Q\tau \mid \forall\mathcal{A} \in QK(\mathcal{A} \underset{wsMhq}{\models} ce)\}. \end{aligned}$$

For sets $c\Sigma \subseteq Q\tau$ of quasi-equations and $QV \subseteq PAlg(\tau)$ of partial algebras of type τ. Then the pair (WH_MQMod^s, WH_MQId^s) is a Galois-connection between the power sets $\mathcal{P}(PAlg(\tau))$ and $\mathcal{P}(Q\tau)$ and the fixed points of the closure operators $WH_MQMod^sWH_MQId^s$ and $WH_MId^sWH_MQMod^s$ form two complete lattices which are dually isomorphic.
We are going to show that strong quasivarieties which are fixed points with respect to $WH_MQMod^sWH_MQId^s$ are weakly M-solid.

Proposition 4.4. *If QV is a strong quasivariety of partial algebras of type τ and $WH_MQMod^sWH_MQId^sQV = QV$ then QV is weakly M-solid.*

Proof ¿From the definition we get

$$QV = WH_MQMod^sWH_MQId^sQV = \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in QId^sQV(\mathcal{A} \underset{wsMhq}{\models} ce)\}$$

and this means that every strong quasi-identity in QV is a weak strong M-hyperquasi-identity in QV. $\square$

If we compare M-solid and weakly M-solid strong quasivarieties, we obtain:

Proposition 4.5. *Every M-solid strong quasivariety of type τ is also weakly M-solid.*

Proof If QV is M-solid, then by definition we have $\chi_M^A[QV] = QV$. Application of Theorem 3.9 gives $QId^sQV = H_M QId^sQV \subseteq WH_M QId^sQV$ by Proposition 4.2. But this means by Definition 4.3 that QV is weakly M-solid. $\square$

The fixed points with respect to the closure operato $WH_M QMod^s WH_M QId^s$ form also a complete lattice and Proposition 4.5 shows that this complete lattice contains the complete lattice of all M-solid strong quasivarieties of partial algebras of type τ. This does not yet mean that the complete lattice of M-solid strong quasivarieties is a complete sublattice of the complete lattice of weakly M-solid strong quasivarieties. We want to show that the lattice of all weakly M-solid strong quasivarieties is a complete sublattice of the complete lattice of all strong quasivarieties. A way to characterize complete sublattices of a complete lattice is via Galois-closed subrelations. We want to mention only the basic facts on Galois-closed subrelations and refer to [4] for more details.

Definition 4.6. ([4]) Let R and R' be relations between sets A and B, and (μ, ι) and (μ', ι') be the Galois-connections between A and B induced by R and R', respectively. The relation R' is called a *Galois-closed subrelation* of R if

1) $R' \subseteq R$, and
2) $\forall T \subseteq A, \forall S \subseteq B$ $(\mu'(T) = S$ and $\iota'(S) = T \Rightarrow \mu(T) = S$ and $\iota(S) = T)$.

Then the following Theorem is satisfied

Theorem 4.7. ([4]) *Let $R \subseteq A \times B$ be a relation between sets A and B with induced Galois connection (μ, ι). Let $\mathcal{H}_{\iota,\mu}$ be the corresponding lattice of closed subsets of A.*

(i) *If $R' \subseteq A \times B$ is a Galois-closed subrelation of R, then the class $\mathcal{U}_{R'} := \mathcal{H}_{\iota'\mu'}$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$.*

(ii) *If $\mathcal{U}$ is a complete sublattice of $\mathcal{H}$, then the relation*

$$R_{\mathcal{U}} := \bigcup \{T \times \mu(T) \mid T \in U\}$$

is a Galois-closed subrelation R' of R.

(iii) *For any Galois-closed subrelation R' of R and any complete sublattice $\mathcal{U}$ of $\mathcal{H}_{\iota\mu}$ we have $\mathcal{U}_{R_{\mathcal{U}}} = \mathcal{U}$ and $R_{\mathcal{U}_{R'}} = R'$.*

We want to apply Theorem 4.7 and prove at first.

Lemma 4.8. *$\underset{wsMhq}{\models}$ is a Galois closed subrelation of $\underset{sq}{\models}$.*

Proof Let $\mathcal{A}$ be a partial algebra of type τ and let ce be a quasi-equation of type τ such that $(\mathcal{A}, ce) \in \underset{wsMhq}{\models}$. Then $\mathcal{A} \underset{wsMhq}{\models} ce$ and by Definition 4.1 we have $\mathcal{A} \underset{sq}{\models} ce$. Therefore $\underset{wsMhq}{\models} \subseteq \underset{sq}{\models}$.
Assume that $K = WH_M QMod^s c\Sigma$ and $c\Sigma = WH_M QId^s K$ where $K \subseteq PAlg(\tau)$. If $\mathcal{A} \in K$, then $\mathcal{A} \underset{wsMhq}{\models} c\Sigma$, i.e. for all $ce \in c\Sigma$ we have $\mathcal{A} \underset{wsMhq}{\models} ce$. But then also $\mathcal{A} \underset{sq}{\models} ce$ by Definition 4.1, therefore $\mathcal{A} \in QMod^s c\Sigma$ and $K \subseteq QMod^s c\Sigma$.
Conversely, if $\mathcal{A} \in QMod^s c\Sigma$, then for every $ce \in c\Sigma$ we have $\mathcal{A} \underset{sq}{\models} ce$ and because of $c\Sigma = WH_M QId^s K$ also $\mathcal{A} \underset{wsMhq}{\models} ce$ for every $ce \in c\Sigma$ and this means $\mathcal{A} \in WH_M QId^s K = K$. Altogether we have $K = QMod^s c\Sigma$.
¿From $ce \in c\Sigma = WH_M QId^s K$ it follows $\mathcal{A} \underset{wsMhq}{\models} ce$ for all $\mathcal{A} \in K$. But then by Definition 4.1, $\mathcal{A} \underset{sq}{\models} ce$ and this means $ce \in QId^s K$ and thus $c\Sigma \subseteq QId^s K$. If $ce \in QId^s K$, then for all $\mathcal{A} \in K = WH_M QMod^s c\Sigma$ we have $\mathcal{A} \underset{sq}{\models} ce$, therefore $\mathcal{A} \underset{wsMhq}{\models} ce$ and $ce \in WH_M QId^s K = c\Sigma$. This shows that $QId^s K \subseteq c\Sigma$ and altogether $c\Sigma = QId^s K$. □

As a consequence we have

Corollary 4.9. *For every monoid $\mathcal{M}$ of regular hypersubstitutions the lattice of all weakly M-solid strong quasivarieties is a complete sublattice of the complete lattice of all strong quasivarieties of type τ.*

Proof This follows with Lemma 4.8 from Theorem 4.7. □

The next, step is to define the following operator χ_M^{wQE} on sets of quasi-equations. Let $ce : ce' \Rightarrow ce''$ be a quasi-equation. Then

$$\chi_M^{wQE}[ce] := \chi_M^{QE}[ce'] \Rightarrow \chi_M^{QE}[ce''].$$

For sets $c\Sigma$ of quasi-equations we define: $\chi_M^{wEQ}[c\Sigma] = \bigcup_{ce \in c\Sigma} \chi_M^{wQE}[ce]$.

This operator has the following properties:

Proposition 4.10. *The operator $\chi_M^{wQE} : \mathcal{P}(Q\tau) \to \mathcal{P}(Q\tau)$ is monotone and idempotent, but in general not extensive.*

Proof By definition the operator χ_M^{wQE} is additive and therefore monotone. We show the idempotency. Let $c\Sigma \subseteq Q\tau$ and $ce \in c\Sigma$. Then $\chi_M^{wQE}[ce] = \chi_M^{QE}[ce'] \Rightarrow \chi_M^{QE}[ce'']$ if ce is the implication $ce' \Rightarrow ce''$. Then

$$\begin{array}{rcl} \chi_M^{wQE}[\chi_M^{wQE}[ce]] & = & \chi_M^{QE}[\chi_M^{QE}[ce']] \Rightarrow \chi_M^{QE}[\chi_M^{QE}[ce'']] \\ & = & \chi_M^{QE}[ce'] \Rightarrow \chi_M^{QE}[ce''] \\ & = & \chi_M^{wQE}[ce] \end{array}$$

for every $ce \in c\Sigma$ since the operator χ_M^{QE} is idempotent. Since χ_M^{wQE} is additive, we obtain the idempotency. □

Finally we want to give an example showing that a strong quasivariety can satisfy an implication as a weakly strong M-hyperquasi-identity, but not as a strong M-hyperquasi-identity.
We consider the strong regular quasivariety V of type $\tau = (2)$ defined by

(i) $x(yz) \approx (xy)z$,

(ii) $x^2 \approx x$,

(iii) $xyuv \approx xuyv$,

(iv) $xy \approx yx \Rightarrow \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)$.

There are exactly the following binary terms over $QV : \varepsilon_1^2(x,y), \varepsilon_2^2(x,y), xy, yx, xyx, yxy$. We prove that (iv) is a weakly strong hyperquasi-identity in QV. That means, for every partial algebra $\mathcal{A} \in QV$ we have to prove

$$(\mathcal{A} \underset{shq}{\models} xy \approx yx) \Rightarrow (\mathcal{A} \underset{shq}{\models} \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)).$$

This becomes clear because of $\mathcal{A} \underset{shq}{\models} xy \approx yx \Leftrightarrow \forall\sigma_R(\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[xy] \approx \widehat{\sigma}_R[yx] \Leftrightarrow \mathcal{A} \underset{sq}{\models} \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y) \wedge \mathcal{A} \underset{sq}{\models} \varepsilon_2^2(x,y) \approx \varepsilon_1^2(x,y) \wedge \mathcal{A} \underset{sq}{\models} xy \approx yx \wedge \mathcal{A} \underset{sq}{\models} yx \approx xy \wedge \mathcal{A} \underset{sq}{\models} xyx \approx yxy \wedge \mathcal{A} \underset{sq}{\models} yxy \approx xyx)$. The implication $xy \approx yx \Rightarrow \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)$ is satisfied as a weakly strong hyperquasi-identity also in the case if $\mathcal{A} \underset{shq}{\models} xy \approx yx$ is wrong, for instance, if $\mathcal{A} \underset{shq}{\models} \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)$ is not satisfied and if $\mathcal{A} \underset{shq}{\models} \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)$ is satisfied. In this case $\mathcal{A}$ has more than one element and is commutative. But then $xy \approx yx \Rightarrow \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)$ is not a strong quasi-identity in $\mathcal{A}$ and $xy \approx yx \Rightarrow \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)$ is not satisfied as a strong hyperquasi-identity.

References

[1] F. Börner, *Varieties of Partial Algebras*, Beiträge zur Algebra und Geometrie, Vol. 37 (1996), No. 2, 259-287.

[2] P. Burmeister, *Lecture Notes on Universal Algebra - Many-Sorted Partial Algebras*, 2002.

[3] Ch. Chompoonut, K. Denecke, *M-solid Quasivarieties*, East-West J. of Mathematics: Vol. 4, No 2 (2002).

[4] K. Denecke, S. L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science* , CRC/Chapman and Hall 2002.

[5] E. Graczyńska *M-solid Quasivarieties*, Habilitationsschrift, Potsdam 2007.

[6] H.-J. Hoehnke, J. Schreckenberger, *Theories for Partial Algebras*, Shaker-Verlag, Aachen 2007.

[7] J. Koppitz, K. Denecke, *M-solid Varieties of Algebras*, Springer-Verlag, Series: Advances in Mathematic, Vol. 10, March 2006.

[8] B. Staruch, B. Staruch, *Strong Regular Varieties of Partial Algebras*, Algebra Universalis, 31 (1994), 157-176.

[9] D. Welke, *Hyperidentitäten Partieller Algebren*, Ph.D.Thesis, Universität Potsdam, 1996.