East-West J. of Mathematics: Vol. 9, No 2 (2007) pp. 183-190

ON STRICTLY GENERALIZED P-QUASI-BAER RINGS

Banh Duc Dung*, **Truong Cong Quynh**†
and
Le Duc Thoang‡

* *Department of Mathematics, University of Transport in HoChiMinh City*
2-D3, Van Thanh Bac, Ward. 25, Binh Thanh Dist., HoChiMinh City, Vietnam
e-mail: banhducdung@yahoo.com

† *Department of Mathematics, DaNang University*
459 Ton Duc Thang, DaNang, Vietnam
e-mail: tc_quynh@ud.edu.vn

‡ *Department of Basic Science, Phu Yen University,*
18A Tran Phu, Ward. 7, TuyHoa City, Vietnam
e-mail: leducthoang@yahoo.com.vn

Abstract

A ring R is called *strictly generalized right p-quasi-Baer* if for any nonzero element $x \in R$, there exists a positive integer n such that $x^n \neq 0$ and the right annihilator of x^nR is generated by an idempotent. The class of strictly right generalized right p-quasi-Baer rings is a new class of generalized right p-quasi-Baer rings and contains right principally quasi-Baer rings. In this paper, many properties of these rings and relations to another kinds of rings are studied, the closeness of this class of rings and some relative classes under direct products or direct sums is investigated.

1 Introduction

Throughout this paper, R is an associative ring with identity $1 \neq 0$ and all modules are unitary modules. We write M_R (resp. ${}_RM$) to indicate that M is a right (resp. left) R-module. The notation $A \leq M$ ($A < M$) means A is a (proper) submodule of M. The right (resp. left) annihilator of a subset S of a

Key words: PP ring, (strictly)generalized PP ring, p-quasi-Baer ring, (strictly) Gp-quasi-Baer ring, von Neumann regular ring.
2000 AMS Mathematics Subject Classification: 16D50, 16D70, 16D80.

ring R are denoted by $r(S)$ (resp. $l(S)$). If $S = \{x\}$, we usually abbreviate it to $r(x)$ (resp. $l(x)$). Z_r, Z_l will stand for the right singular ideal and the left singular ideal of R, respectively.

A ring R is called a *Baer ring* (resp. *quasi-Baer ring*) if the right annihilator of every non-empty subset (resp. right ideal) in R is generated by an idempotent. It is well-known that, Baer rings and quasi-Baer rings are left-right symmetry. In 2001, Birkenmeier, Kim and Park (in [2]) defined a ring R to be a *right principally quasi-Baer ring*, or simply *right p-quasi-Baer ring*, if the right annihilator of every principal right ideal in R is generated by an idempotent. Similarly, *left p-quasi-Baer rings* can be defined. It is proved in ([2], Proposition 1.7) that, if the right annihilator of every finitely generated (right) ideal of R is generated (as a right ideal) by an idempotent then R is also a right p-quasi-Baer ring. The concept of Gp-quasi-Baer rings is a generalization of p-quasi-Baer rings and was defined by Kwak [13]. A ring R is called a *generalized right p-quasi-Baer ring*, briefly *right Gp-quasi-Baer*, if for any $x \in R$ there exists a positive integer n (depending on x) such that the right annihilator x^nR is generated by an idempotent.

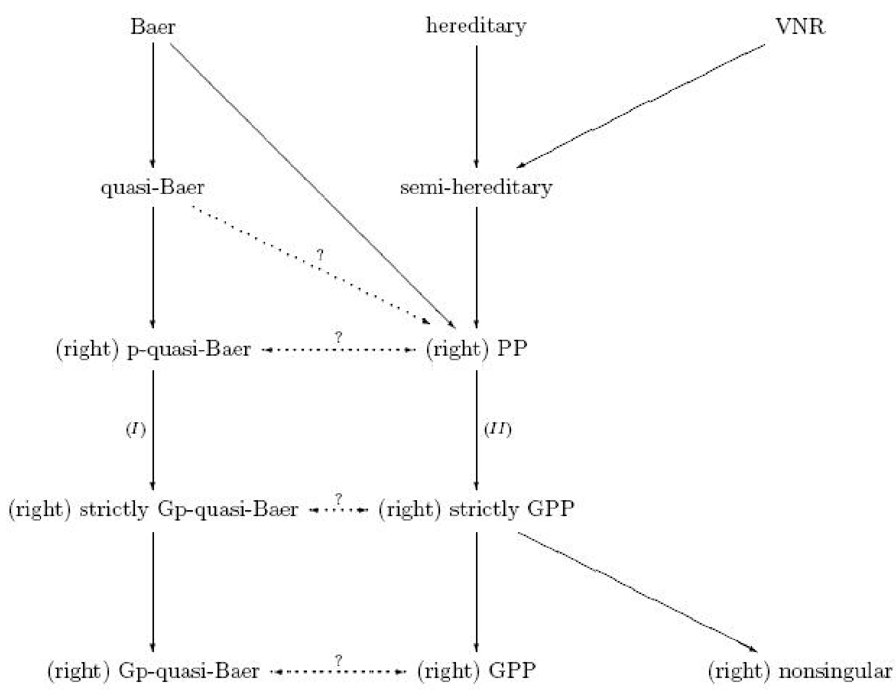
If for every element a of a ring R, there exists an element $b \in R$ such that $a = aba$ then R is called a *von Neumann regular* (=VNR) ring. A ring R is a (*semi*$-$) *hereditary ring* if every (finitely generated) right ideal of R is projective. A ring R is a VNR ring if and only if every (finitely generated) right or left ideal is generated by an idempotent ([12], Theorem 1.1). So, a VNR ring is left and right semi-hereditary. A ring R is called a *right* (resp. $left$) PP *ring* if all principal *right* (resp. $left$) ideals are projective. The ring is both left and right PP is called PP (see [16], [4], [14]). It is well-known that a ring R is a right PP ring if and only if for each element $a \in R$, the homomorphism $\varphi : R \to aR$ defined by $\varphi(r) = ar$ splits (i.e., Kerφ is a direct summand of R), if and only if the right annihilator of each element of R is generated by an idempotent (see Wisbauer [16]). The class of generalized right PP rings was defined by Hirano [9], and Dung and Thuyet in [6] defined the class of strictly generalized right PP rings. A ring R is called (resp. *strictly*) *generalized right PP*, briefly *right* (resp. *strictly*) GPP, if for any (resp. nonzero) $x \in R$ there exists a positive integer n (depending on x) such that (resp. $x^n \neq 0$ and) the right ideal x^nR is projective, or equivalently, if for any (resp. nonzero) $x \in R$ the right annihilator of (resp. non-zero) element x^n is generated by an idempotent for some positive integer n, depending on x. Clearly, the class of (one-sided) PP rings is a generalization of Baer rings and (one-sided) semi-hereditary rings (also VNR rings and hereditary rings). Right PP rings are obviously right (strictly) GPP. For basic concepts and results that are not defined here we refer to the texts of Anderson and Fuller [1], Dung, Huynh, Smith and Wisbauer [5], Faith [8], Lam [14], Wisbauer [16].

2 Strictly Gp-quasi-Baer rings

Note that for a ring R to be right Gp-quasi-Baer, we only need to find a positive integer n such that the right annihilator of $x^n R$ is generated by an idempotent for each non-nilpotent element $x \in R$. We now consider the following class of rings.

Definition 2.1. A ring R is called *strictly generalized right p-quasi-Baer*, briefly *right strictly Gp-quasi-Baer*, if for any nonzero $x \in R$ there exists a positive integer n (depending on x) such that $x^n \neq 0$ and the right annihilator of $x^n R$ is generated by an idempotent. *Strictly generalized left p-quasi-Baer rings* are defined similarly. A ring which is both left and right strictly Gp-quasi-Baer is called a *strictly Gp-quasi-Baer ring.*

Clearly, (right) p-quasi-Baer rings are (right) strictly Gp-quasi-Baer and (right) strictly Gp-quasi-Baer rings are (right) Gp-quasi-Baer. We have the following implications:



We have now some properties of this class of rings but we need first some notions. Recall that a ring R is called *reduced* if it has no nonzero nilpotent element, and *abelian* (or *normal*) if all idempotents are in its center. A ring R is called *semicommutative* if for every $a \in R$, $r(a)$ is an ideal of R. Clearly, reduced rings are semicommutative and semicommutative rings are abelian. An idempotent $e \in R$ is called *left* (resp., *right*) *semicentral* if $xe = exe$ (resp., $ex = exe$) for all $x \in R$. If all idempotents of a ring R is left (resp. right) semicentral then R is also called *left* (resp., *right*) *semicentral*. The set of left (resp., right) semicentral idempotents of R is denoted by $\mathcal{S}_l(R)$ (resp., $\mathcal{S}_r(R)$). Note that $\mathcal{S}_l(R) \cap \mathcal{S}_r(R) = \mathcal{B}(R)$ is the set of all central idempotents of R.

For a reduced ring R, we have $l(x) = r(x) = l(x^n) = r(x^n) = l(Rx) = r(xR) = l(Rx^n) = r(x^nR)$, for every $x \in R$ and every positive integer n. Therefore, the following result is immediate:

Proposition 2.2. *For a reduced ring R, the following conditions are equivalent:*

(1) *R is (right) PP;*

(2) *R is strictly generalized (right) PP;*

(3) *R is generalized (right) PP;*

(4) *R is (right) p-quasi-Baer;*

(5) *R is strictly generalized (right) p-quasi-Baer;*

(6) *R is generalized (right) p-quasi-Baer;* □

Lemma 2.3. *If R is a semicommutative ring then, $r(a^n) = r(a^nR)$ for any $a \in R$ and positive integer n.*

Proof By a direct calculation. □

Proposition 2.4. *Let R be a ring, $a \in R$ and n a positive integer. If $r(a^n) = eR$ for some left semicentral idempotent $e \in R$ then $r(a^nR) = r(a^{n+1}R) = eR$.*

Proof First, we prove that in this case, $r(a^nR) = r(a^n)$. The fact that $r(a^nR) \leq r(a^n)$ is obvious. Let $x \in r(a^n)$. Since $x \in eR$, we have $x = ex$. It implies that $a^nRx = a^nRex = a^neRex = 0$ from the fact that e is left semicentral, i.e. $x \in r(a^nR)$. Now, applying [15, Lemma 3], $eR = r(x^nR) = r(x^n) = r(x^{n+1}) = r(x^{n+1}R)$, it completes the proof. □

Corollary 2.5. *Let R be a semicommutative ring, $a \in R$ and n a positive integer. If $r(a^nR) = eR$ for some idempotent $e \in R$ then $r(a^nR) = r(a^{n+1}R)$.* □

Corollary 2.6. *Let R be a left semicentral ring. If R is a right PP* (*resp. GPP, strictly GPP*) *then R is right p-quasi-Baer* (*resp. generalized right p-quasi-Baer, strictly generalized right p-quasi-Baer*). □

If R is a semicommutative ring, then R is a generalized right PP-ring if and only if R is a generalized right p-quasi-Baer ring [13, Proposition 3.3]. Moreover, we have

Proposition 2.7. *If R be a semicommutative ring, then the following conditions are equivalent:*

(1) *R is (right) PP;*

(2) *R is (right) p-quasi-Baer;*

(3) *R is strictly generalized (right) PP;*

(4) *R is strictly generalized (right) p-quasi-Baer;*

Proof It follows immediately from Lemma 2.3, [7, Proposition 2], [6, Proposition 2.7] and [2, Proposition 1.14]. □

Example 2.8. (1) The notions right PP and right p-quasi-Baer are distinct by [2, Examples 1.3 and 1.5]. Especially, there is a regular ring (hence PP) that is neither right nor left p-quasi-Baer ([2, Example 1.6]), and there is a quasi-Baer (hence right p-quasi-Baer) ring that is not right PP [2, Examples 1.3].

(2) Let D be a domain and let R be the trivial extension of D by D. Then R is semicommutative (but not reduced) and R is a generalized right PP ring, but it is not a right (strictly) PP ring. Thus R is a generalized right p-quasi-Baer ring by [13, Proposition 3.3], but it is not (strictly) right p-quasi-Baer by [2, Proposition 1.14].

(3) The semicommutativity of the ring R in Proposition 2.7 can not reduce to abelian property because, there is an abelian p-quasi-Baer ring (so also an abelian strictly Gp-quasi-Baer ring) that is neither left nor right PP (see [2, Example 1.16]). □

Question 2.9. Does the implication (I) or (II) in the diagram above has inverse?

Proposition 2.10. *If R is a strictly generalized right p-quasi-Baer ring, then the center $C(R)$ of R is a PP-ring (also a $p-quasi-Baer$ ring).*

Proof By the same argument of the proof of [2, Proposition 1.2], we have that, the center of a strictly generalized right p-quasi-Baer ring is also strictly generalized right p-quasi-Baer. But the center of a ring is always commutative, so the center of a strictly generalized right p-quasi-Baer ring must be a PP ring, also a p-quasi-Baer ring by Proposition 2.7. □

Proposition 2.11. *Let R be a ring. The following conditions are equivalent:*

(1) *R is (resp. strictly) generalized right p-quasi-Baer;*

(2) *For any (resp. nonzero) element $a \in R$, there exists a positive integer n, depending on a, such that the right annihilator of (resp. $non-zero$) ideal Ra^nR is generated by an idempotent.*

(3) *For any (resp. nonzero) element $a \in R$, there exists a positive integer n, depending on a, and an idempotent $e \in S_r(R)$ such that (resp. $0 \neq$) $Ra^nR \leq Re$ and $r(Ra^nR) \cap Re = (1-e)Re$.*

Proof We need only prove for the strictly Gp-quasi-Baer case.

(1) $\Leftrightarrow$ (2). Follows from $r(I) = r(RI)$, where I is any right ideal of R.

(1) $\Rightarrow$ (3). Let R is a strictly generalized right p.p.-Baer ring and a a nonzero element in R. There exists a positive integer n such that $a^nR \neq 0$ and $r(a^nR) = r(Ra^nR) = fR$ with $f \in \mathcal{S}_l(R)$. So $Ra^nR \subseteq lr(Ra^nR) = R(1-f)$. Let $e = 1-f$, then $e \in \mathcal{S}_r(R)$ and $r(Ra^nR) \cap Re = (1-e)R \cap Re = (1-e)Re$.

(3) $\Rightarrow$ (1). Assume (3) holds. We show that $r(a^nR)(= r(Ra^nR)) = (1-e)R$. The fact that $(1-e)R \subseteq r(Ra^nR)$ is obvious. Now, let $x \in r(Ra^nR)$, then $xe = exe + (1-e)xe \in r(Ra^nR) \cap Re = (1-e)Re$. So $ex = exe = 0$ and hence, $a = (1-e)a \in (1-e)R$. Thus, $r(Ra^nR) = (1-e)R$ and R is strictly generalized right p-quasi-Baer. □

Remark. The implications (1) $\Leftrightarrow$ (3) in Proposition 2.11 seemed to be stated by Kwak in [13, Proposition 3.7] (from the thought of [2, Proposition 1.9]), but it was incorrect. We correct it and extend for the strictly Gp-quasi-Baer case.

For a given positive integer n, a ring R is called *n-generalized right p-quasi-Baer* if for every $a \in R$ there exist a/an (left semicentral) idempotent $e \in R$ such that $r(a^nR) = eR$. Clearly, 1-generalized right p-quasi-Baer rings are right p-quasi-Baer and, n-generalized right p-quasi-Baer rings are generalized right p-quasi-Baer rings for every positive integer n. The class of (right principally) quasi-Baer rings is closed under direct products by [2, Proposition 2.1]. Similarly, we have the following result:

Proposition 2.12. *Let $R = \prod_{i\in I} R_i$ and n a positive integer. Then, R is n-generalized right p-quasi-Baer if and only if R_i is a n-generalized right p-quasi-Baer for each $i \in I$.* □

The class of (generalized, n-generalized, strictly generalized) right p-quasi-Baer rings may not be closed under infinitely direct sums. For example, the infinite direct sum of domains is not a (generalized, strictly generalized) right p-quasi-Baer ring. However, we have a relative result as follows:

Proposition 2.13. *Let $R = \prod_{i\in I} R_i$ and $S = \langle \bigoplus_{i\in I} R_i, 1\rangle$ the subring of R generated by $\bigoplus_{i\in I} R_i$ and 1_R. If R_i are (resp. strictly) generalized right p-quasi-Baer and semicommutative, then S is also a (resp. strictly) generalized right p-quasi-Baer ring.*

Proof Assume that R_i are strictly Gp-quasi-Baer and semicommutative and $0 \neq x = (x_i) \in S$. Consider the following cases:

Case 1: $x \in \bigoplus_{i\in I} R_i$. Since x has only finite nonzero x_i, we may assume that $x_{i_j} \neq 0$ with $j = 1, 2, .., k$ and $x_i = 0$ with $i \neq i_j$. Then, for each i_j there exists a positive integer n_{i_j} and an idempotent $e_{i_j} \in R_{i_j}$ such that $x_{i_j}^{n_{i_j}} \neq 0$ and $r(x_{i_j}^{n_{i_j}} R) = e_{i_j} R$. Take $n = max\{n_{i_j}\}, j = 1, .., k$. Since R_{i_j} are semicommutative, applying Proposition 2.4 we have $r(x_{i_j}^n R) = e_{i_j} R, j = 1, .., k$. Put $e = (e_i) \in S$ such that $e_i = e_{i_j}$ with $i = i_j$ and $e_i = 1_{R_i}$ with $i \neq i_j$. Then, it easy to see that $e^2 = e$ and $r(x^n S) = eS$.

Case 2: $x \notin \bigoplus_{i\in I} R_i$. In this case, $x = (x_i)$ has the form as follows: There is a positive integer k such that, almost x_i are of the form $k1_{R_i} = 1_{R_i} + .. + 1_{R_i}$ (k times) except finite terms, say $x_{i_j},\ j = 1, .., t$. By the same argument of Case 1, for each $x_{i_j} \in R_{i_j}$, there corresponds an idempotent $e_{i_j} \in R_{i_j}$ (if $x_{i_j} = 0$ then $e_{i_j} = 1_{R_{i_j}}$), and we may choose a positive integer n such that $r(x_{i_j}^n R) = e_{i_j} R, j = 1, .., t.$. Put $e = (e_i) \in S$ such that $e_i = e_{i_j}$ with $i = i_j$ and $e_i = 0_{R_i}$ with $i \neq i_j$. It can be checked that $e^2 = e$ and $r(x^n S) = eS$, as desired. □

Corollary 2.14. *Let $R = \prod_{i\in I} R_i$ and $S = \langle \bigoplus_{i\in I} R_i, 1\rangle$ the subring of R generated by $\bigoplus_{i\in I} R_i$ and 1_R. If R_i are (resp. $n-$generalized) right p-quasi-Baer, then S is also a (resp. $n-$ generalized) right p-quasi-Baer ring.* □

Corollary 2.15. *The direct sum of finite left semicentral (resp. strictly) generalized right p-quasi-Baer rings is also a (resp. strictly) generalized right p-quasi-Baer rings.* □

Proposition 2.16. *Let R be a strictly generalized right p-quasi-Baer ring.*

(1) *R is semiprime if and only if $\mathcal{S}_l = \mathbf{B}(R)$*

(2) *If every essential right ideal is an essential extension of an ideal of R, then R is right nonsingular.*

Proof (1). For $e \in \mathcal{S}_l$, then $eR(1-e)$ is an ideal and $(1-e)Re = 0$. We have $eR(1-e)^2 = 0$. If R is semiprime, then $eR(1-e) = 0$. Thus e is central. Conversely, assume that $\mathcal{S}_l = \mathbf{B}(R)$. Suppose that I is an nonzero ideal of R with $I^2 = 0$. Let $0 \neq x \in I$ (note that $x^2 = 0$). Since R is strictly generalized right p-quasi-Baer, there exists an idempotent $e \in \mathbf{B}(R)$ such that $r(xR) = eR$. We have $(xR)^2 \subseteq I^2 = 0$ and so $xR \leq r(xR)$. It implies that $x = ex$ and hence $xR = exR = xRe = 0$. This is a contradiction.

(2). Assume that $Z_r \neq 0$. Let $0 \neq x \in Z_r$. Since R is strictly generalized right p-quasi-Baer, there exist a positive integer n and $e \in \mathcal{S}_l$ such that $r(x^n R) = eR$. On the other hand, $r(x^n) \leq^e R_R$. By hypothesis, there exists an ideal I such that $I \leq r(x^n)$ and $I \leq^e R_R$. Then $x^n RI = 0$ and so $I \leq r(x^n R) = eR$. Since $I \leq^e R_R$, $e = 1$ and so $x^n = 0$. This is a contradiction. □

References

[1] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules", New York, Springer-Verlag, 1992.

[2] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Principally quasi-Baer rings,* Comm. in Algebra, **29**(2)(2001), 639-660.

[3] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Quasi-Baer ring extensions and biregular rings,* Bull. of the Australian Math. Soc., **61**(1)(2000), 39-52.

[4] A. W. Chatters and C. R. Hajarnavis, "Rings with Chain Conditions", Research Notes in Math., Vol. 44, Pitman Advanced Publishing Program, Pitman, Boston-London-Melbourne, 1980.

[5] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, "Extending modules", Pitman Research Notes in Math. 313, Longman, 1994.

[6] B. D. Dung and L. V. Thuyet, *On strictly generalized PP rings,* Preprint.

[7] S. Endo, *Note on p.p. rings,* Nagoya Math. J., (17) (1960), 167-170.

[8] C. Faith, "Algebra II", Springer-Velag, New York, 1976.

[9] Y. Hirano, *On generalized PP-rings,* Math. J. Okayama Univ., (25) (1983), 7-11.

[10] C. Huh, H. K. Kim and Y. Lee , *PP rings and generalized PP rings,* J. of Pure and App. Algebra, (167)(2002), 37 - 52.

[11] D. V. Huynh and L. V. Thuyet, *On PP rings and generalized PP rings,* Preprint.

[12] K. R. Goodearl, "Von Neumann Regular Rings", Krieger, Malabar, 1991.

[13] T. K. Kwak, *Generalized Baer rings,* Inter. J. of Math. and Math. Sciences, Vol. 2006, Article ID 45837, 1-11.

[14] T. Y. Lam, "Lectures on Modules and Rings", Graduate Texts in Mathematics, Vol. 189, Springer-Verlag, New York, 1998.

[15] M. Ôhori, *On noncommutative generalized PP rings,* Math. J. Okayama, (25) (1984), 157-167.

[16] R. Wisbauer, "Foundations of module and ring theory", Gordon and Breach, 1991.