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# ON FUNCTORIAL PROPERTIES OF BCK AND BCI-STRUCTURES

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**Abstract**

In this paper, we study functors by retractions and co-retractions on BCI-algebras. Further, some functors by filters, prime ideals and self maps on BCK-algebras are discussed.

## 1. Introduction

The notion of BCK-algebra was originated by Imai and Iseki [6]. Later on, Iseki [7] introduced the notion of a BCI-algebra as a generalization of a BCK-algebra. A series of interesting notions concerning BCI-algebras were introduced. Several papers have been written on various aspects of these algebras and their relations to other algebraic structures have been studied (see [9,10,11,12]).

Iseki and Thaheem [14] proved that if $X$ is an associative BCI-algebra, then $End(X)$; the set of all BCI-homomorphisms on $X$, is also a BCI-algebra. They also proved that for any $\theta_1, \theta_2 \in End(X)$, $\theta_1 \star \theta_2 \in End(X)$ together with the mapping $\theta_1 \star \theta_2 : X \to X$ defined by $(\theta_1 \star \theta_2)(x) = \theta_1(x) \star \theta_2(x)$ for all $x \in X$.

In developing an algebraic theory of BCK-algebras, the notion of ideals has played an important role. The theory of ideals has been developed by Iseki [8] and extensively studied by several authors. Deeba [4] introduced the notion of filters as a dual to the concept of ideals of BCK-algebras and proved that if $P_1$, $P_2$ and $P_3$ are prime ideals of a bounded commutative BCK-algebra $X$, then

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$X{-}P_1$, $X{-}P_2$ and $X{-}P_3$ will be the corresponding filters of $X$. Thus there is a one to one correspondence between the ideals and filters of a bounded commutative BCK-algebra. Moreover, if $P$ is a prime ideal of $X$ and $x \in P$, then $Nx \in (X{-}P)$, the corresponding filter of the ideal $P$.

Kondo [15] defined a left map $l_x$ on a positive implicative BCK-algebra $X$ and proved that if $X$ is a positive implicative BCK-algebra, then the set of all left maps $L(X)$ is also a positive implicative BCK-algebra.

Motivated from [4, 14, 15], we study functorial properties of BCK and BCI-structures in view of the fact that a structure has a functorial property if a functor can be defined by it. In particular, we construct some functors using the functorial properties of BCK and BCI-structures such as retractions, co-retractions, filters, prime ideals and self maps.

## 2. Preliminaries

In this section, we recall the following aspects of the theory of BCK and BCI-algebras that are necessary for the development of this paper from [9, 13, 14, 16]. For background information on categories and functors we refer the reader to [1].

Let $X$ be a set with binary operation '$\star$' and a constant 0, then $X$ is called BCI-algebra if the following axioms are satisfied for all $x, y, z \in X$:

$(i)$ $(x * y) * (x * z) \leq z * y$,

$(ii)$ $x * (x * y) \leq y$,

$(iii)$ $x \leq x$,

$(iv)$ $x \leq 0 \implies x = 0$,

$(v)$ $x \leq y$ and $y \leq x \Longrightarrow x = y$,

$(vi)$ $x \leq y \Longleftrightarrow x \star y = 0$.

If we replace axiom$(iv)$ by $0 \leq x$, $X$ is called BCK-algebra.

A BCK-algebra $X$ is said to be commutative if $x \wedge y = y \wedge x$, where $x \wedge y = y \star (y \star x)$.

A BCK-algebra $X$ is said to be bounded if there is an element 1 in $X$ such that $x \leq 1$ for all $x \in X$. In a bounded BCK-algebra, we denote $1 \star x$ by $N_x$.

A BCK-algebra $X$ is said to be positive implicative if $(x \star z) \star (y \star z) = (x \star y) \star z$ for all $x, y, z \in X$.

Let $X$ be a positive implicative BCK-algebra. A self map $l_x : X \to X$ defined by $l_x(t) = x \star t$ for all $t \in X$, is called a left map of $X$. The composition of left maps is defined by $l_x o l_y = l_{x \star y}$ for all $x, y \in X$.

A non-empty subset $I$ of BCK-algebra $X$ is said to be an ideal of $X$ if

$(i)$ $0 \in I$,

$(ii)$ $x \in I$ and $y \star x \in I$ imply that $y \in I$.

An ideal $I$ of a commutative BCK-algebra $X$ is said to be prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

A non-empty set $F$ of BCK-algebra $X$ is said to be a filter of $X$ if

$(i)$ $x \in F$ and $y \geq x$ imply that $y \in F$

$(ii)$ $x \in F$ and $y \in F$ imply that $glb\{x, y\} \in F$.

Let $X$ and $Y$ be BCI-algebras (BCK-algebras). Then a mapping $f : X \to Y$ is called BCI-homomorphism (BCK-homomorphism) if

$$f(x \star y) = f(x) \star f(y) \quad \text{for all } x, y \in X.$$

The category of BCI-algebras can be constructed by taking the class of all BCI-algebras as the class of objects of the category and the class of all BCI-homomorphisms as the class of morphisms of the category. We shall denote the category of BCI-algebras by $B_{CI}$.

Similarly, the category of BCK-algebras is constructed and denote it by $B_{CK}$.

## 3. Functors by Retractions and Co-retractions

Consider the category $B_{(r)}$ in which the objects of the category are associative BCI-algebras and morphisms of the category are those BCI-homomorphisms which are retractions.

**Theorem 3.1.** Let $f : X \to Y$ be a morphism in the category $B_{(r)}$ and $g$ be any right inverse of $f$. Then the mapping $T_f : End(X) \to End(Y)$ defined by $T_f(\theta) = fo\theta og$ for all $\theta \in End(X)$, is a BCI-homomorphism.

**Proof.** If $\theta_1, \theta_2 : X \to X$, then we get $T_f(\theta_1 \star \theta_2) = fo(\theta_1 \star \theta_2)og$ and $T_f(\theta_1) \star T_f(\theta_2) = (fo\theta_1 og) \star (fo\theta_2 og)$. Clearly, $T_f(\theta_1 \star \theta_2) : Y \to Y$ and for any $y \in Y$, one gets

$$\begin{aligned} T_f(\theta_1 \star \theta_2)(y) &= (fo(\theta_1 \star \theta_2)og)(y) \\ &= (fo(\theta_1 \star \theta_2)o(g(y)) \\ &= f[(\theta_1 \star \theta_2)(g(y))] \qquad \text{(as } g(y) \in X) \\ &= (fo\theta_1 og)(y) \star (fo\theta_2 og)(y) \\ &= T_f(\theta_1)(y) \star T_f(\theta_2)(y) \end{aligned}$$

which yields $T_f(\theta_1 \star \theta_2) = T_f(\theta_1) \star T_f(\theta_2)$ and henceforth $T_f$ is a BCI-homomorphism.

**Proposition 3.1.** If $g : X \to Y$ and $h : Y \to Z$ are retractions, then $hog : X \to Z$ is also a retraction.

**Proof** Since $g : X \to Y$ and $h : Y \to Z$ are retractions, then there exist $g' : Y \to X$ and $h' : Z \to Y$ such that $gog' = I_Y$ and $hoh' = I_Z$.
Now, $(hog)o(g'oh') = ho(gog')oh' = ho(I_Y)oh' = (hoI_Y)oh' = hoh' = I_Z$ which implies $(hog)o(g'oh') = I_Z$ and so $hog : X \to Z$ is a retraction.

**Proposition 3.2.** If $g : X \to Y$ and $h : Y \to Z$ are morphisms in $B_{(r)}$, then $T_{hog} = T_h o T_g$.

**Proof** In view of Theorem 3.1 and Proposition 3.1, $T_{hog} : End(X) \to End(Z)$ is a BCI-homomorphism. For any $\theta \in Hom(X)$, we get

$$\begin{aligned} T_{hog}(\theta) &= (hog)o\theta o(g'oh') \\ &= h(go\theta)o(g'oh') \\ &= ho(go\theta og')oh' \\ &= ho(T_g(\theta))oh' \\ &= T_h(T_g(\theta)) \\ &= (T_h o T_g)(\theta) \end{aligned}$$

implying thereby $T_{hog} = T_h o T_g$.

**Corollary 3.1.** If $I_X : X \to X$ is an identity morphism in $B_{(r)}$, then $T_{I_X} : End(X) \to End(X)$ is an identity morphism in $B_{CI}$.

In view of Theorem 3.1, Propositions 3.1, 3.2 and Corollary 3.1, we can define a functor $T : B_{(r)} \to B_{CI}$ such that

($i$) $T(X) = End(X)$ for all $X \in B_{(r)}$,

($ii$) for any morphism $f : X \to Y$ in $B_{(r)}$, we have $T(f) = T_f : End(X) \to End(Y)$ in $B_{CI}$.

Now, consider the category $B_{(cr)}$ in which the objects of the category are associative BCI-algebras and morphisms of the category are those BCI-homomorphisms which are co-retractions.

**Theorem 3.2.** Let $f : X \to Y$ be a morphism in the category $B_{(cr)}$ and $g$ be any left inverse of $f$. Then the mapping $G_f : End(Y) \to End(X)$ defined by $G_f(\beta) = go\beta of$ for all $\beta \in End(Y)$, is a BCI-homomorphism.

**Proof** If $\beta_1, \beta_2 \in End(Y)$, then $\beta_1 \star \beta_2 \in End(Y)$. For any $x \in X$, we get

$$\begin{aligned} G_f(\beta_1 \star \beta_2)(x) &= (go(\beta_1 \star \beta_2)of)(x) \\ &= [go(\beta_1 \star \beta_2)]f(x) \\ &= g[\beta_1(f(x)) \star \beta_2(f(x))] \\ &= g(\beta_1(f(x))) \star g(\beta_2(f(x))) \\ &= (go\beta_1 of)(x) \star (go\beta_2 of)(x) \\ &= G_f(\beta_1)(x) \star G_f(\beta_2)(x) \\ &= (G_f(\beta_1) \star G_f(\beta_2))(x) \end{aligned}$$

which yields $G_f(\beta_1 \star \beta_2) = G_f(\beta_1) \star G_f(\beta_2)$. Therefore, $G_f$ is a BCI-homomorphism.

**Proposition 3.3.** If $g : X \to Y$ and $h : Y \to Z$ are co-retractions, then $hog : X \to Z$ is a co-retraction.

**Proof** The proof follows on the similar lines of Proposition 3.1.

**Proposition 3.4.** If $g : X \to Y$ and $h : Y \to Z$ are morphisms in $B_{(cr)}$, then $G_{hog} = G_g o G_h$.

**Proof** In view of Theorem 3.2 and Proposition 3.3, $G_{hog} : End(Z) \to End(X)$ is a BCI-homomorphism and for any $\mu \in End(Z)$, one gets

$$\begin{aligned} G_{hog}(\mu) &= (g'oh')o\mu o(hog) \qquad \text{(as } g, h \text{ are co–retractions)} \\ &= (g'oh')o(\mu oh)og \\ &= g'o(h'o\mu oh)og \\ &= g'o(G_h(\mu))og \\ &= G_g(G_h(\mu)) \\ &= (G_g oG_h)(\mu) \end{aligned}$$

which implies $G_{hog} = G_g oG_h$.

**Corollary 3.2.** If $I_X : X \to X$ is an identity morphism in $B_{(cr)}$, then $G_{I_X} : End(X) \to End(X)$ is an identity morphism in $B_{CI}$.

In view of Theorem 3.2, Propositions 3.3, 3.4 and Corollary 3.2, we can define a contravariant functor $G : B_{(cr)} \to B_{CI}$ such that

(*i*) $G(X) = End(X)$ for all $X \in B_{(cr)}$,

(*ii*) for any morphism $f : X \to Y$ in $B_{(cr)}$, we have $G(f) = G_f : End(Y) \to End(X)$ in $B_{CI}$.

## 4. Functors by Filters and Prime Ideals

Let $P(X)$ represents the category of prime ideals of $X$ in which objects of the category are the prime ideals and morphisms of the category are the BCK-homomorphisms between them. Further, let $F(X)$ denotes the category of filters of $X$ in which objects of the category are the filters and the morphisms of the category are the functions between them.

Let $P_1$, $P_2$ be the prime ideals. Then for any morphism $f : P_1 \to P_2$ in $P(X)$, we can define a function $S_f : (X - P_1) \to (X - P_2)$ in $F(X)$ by $S_f(N_x) = N_{f(x)}$ for all $x \in X$.

**Proposition 4.1.** If $f : P_1 \to P_2$ and $g : P_2 \to P_3$ are BCK-homomorphisms where $P_1$, $P_2$ and $P_3$ are prime ideals of the bounded commutative BCK-algebra $X$, then $S_{g\circ f} = S_g \circ S_f$.

**Proof** Let $X$ be a bounded commutative BCK-algebra. Let $f : P_1 \to P_2$ and $g : P_2 \to P_3$ be the morphisms in $P(X)$. Then for the composite morphism $gof : P_1 \to P_3$ in $P(X)$, the mapping $S_{gof} : (X–P_1) \to (X–P_3)$ is a morphism in $F(X)$. For any $N_x \in (X–P_1)$, we get

$$\begin{aligned} S_{gof}(N_x) &= N_{(gof)(x)} \\ &= N_{g(f(x))} \\ &= S_g(N_{f(x)}) \\ &= S_g(S_f(N_x)) \\ &= (S_g o S_f)(N_x) \end{aligned}$$

yielding thereby $S_{gof} = S_g o S_f$.

**Corollary 4.1.** If $I_P : P \to P$ is an identity homomorphism in $P(X)$, then $S_{I_P} : (X–P) \to (X–P)$ is an identity morphism in $F(X)$.

In view of the above discussion, we can define a covariant functor $S : P(X) \to F(X)$ such that

$(i)$ $S(P) = (X–P)$, for all $P \in P(X)$,

$(ii)$ for any morphism $f : P_1 \to P_2$ in $P(X)$, we have $S(f) = S_f : (X–P_1) \to (X - P_2)$ in $F(X)$.

## 5. Functors by Self Maps

Consider the category $B_{CK(i_+)}$ in which the objects of the category are positive implicative BCK-algebras and morphisms of the category are the BCI-homomorphisms between them.

**Proposition 5.1.** If $f : X \to Y$ is a BCK-homomorphism, then the map $L_f : L(X) \to L(Y)$ defined by $L_f(l_x) = l_{f(x)}$ for all $x \in X$, is a BCK-homomorphism.

**Proof** For any $l_{x_1}, l_{x_2} \in L(X)$, we have

$$\begin{aligned} L_f(l_{x_1} o l_{x_2}) &= L_f(l_{x_1 \star x_2}) \\ &= l_{f(x_1 \star x_2)} \\ &= l_{f(x_1) \star f(x_2)} \\ &= l_{f(x_1)} o l_{f(x_2)} \text{ [By the definition of composition of maps in L(X)]} \\ &= L_f(l_{x_1}) o L_f(l_{x_2}) \end{aligned}$$

which shows that $L_f : L(X) \to L(Y)$ is a BCK-homomorphism.

**Theorem 5.1.** If $f : X \to Y$ and $g : Y \to Z$ are BCK-homomorphisms in $B_{CK(i_+)}$, then $L_{gof} = L_g o L_f$.

**Proof** In view of Proposition 5.1, the maps $L_f : L(X) \to L(Y)$, $L_g : L(Y) \to L(Z)$ and their composition $L_{gof} : L(X) \to L(Z)$ are morphisms in the category $B_{CK(i_+)}$.
For any $l_x \in L(X)$, we have

$$\begin{aligned} L_{gof}(l_x) &= l_{gof(x)} \\ &= l_{g(f(x))} \\ &= L_g(l_{f(x)}) \\ &= L_g(L_f(l_x)) \\ &= (L_g o L_f)(l_x) \end{aligned}$$

yielding thereby $L_{gof} = L_g o L_f$.

**Corollary 5.1.** If $I_X : X \to X$ is an identity homomorphism in $B_{CK(i_+)}$, then $L_{I_X} : L(X) \to L(X)$ is also an identity homomorphism in $B_{CK(i_+)}$.

With the help of Proposition 5.1, Theorem 5.1 and Corollary 5.1, we can define a functor $F : B_{CK(i_+)} \to B_{CK(i_+)}$ such that

(*i*) $F(X) = L(X)$ for all $X \in B_{CK(i_+)}$,

(*ii*) for any morphism $f : X \to Y$ in $B_{CK(i_+)}$, we have $F(f) = L_f : L(X) \to L(Y)$ in $B_{CK(i_+)}$.

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