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SOME ASPECTS OF τ-FULL MODULES

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Abstract

Let τ be a hereditary torsion theory on Mod-R. For a right τ-full R-module M, we establish that $[\tau, \tau \vee \xi(M)]$ is a boolean lattice; we find necessary and sufficient conditions for the interval $[\tau, \tau \vee \xi(M)]$ be atomic, and we give conditions for the atoms be of some specific type in terms of the internal structure of M.

We also prove that there are lattice isomorphisms between the lattice $[\tau, \tau \vee \xi(M)]$ and the lattice of τ-pure fully invariant submodules of M, under the additional assumption that M is absolutely τ-pure.

With the aid of these results, we get a decomposition of a τ-full and absolutely τ-pure R-module M as a direct sum of τ-pure fully invariant submodules N and N' with different atomic characteristics on the intervals $[\tau, \tau \vee \xi(N)]$ and $[\tau, \tau \vee \xi(N')]$, respectively.

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1 Introduction

Let R be an associative ring with unit. Mod-R denotes the category of unitary right R-modules and R-tors denotes the frame of all hereditary torsion theories on Mod-R.

For a hereditary torsion theory $\tau \in R$-tors, William George Lau studied the τ-full modules, that is, τ-torsion-free modules which have the property that every essential submodule is τ-dense. The latest condition was named the τ-large condition by Lau, [12]. Earlier on, this notion was studied by Ann K. Boyle [4] in connection with her work on modules having Krull dimension and also, Robert Wisbauer worked with them in [16]. Later, some properties about these modules were pointed out in [8]. Zelmanowitz defined polyform modules in [18] which were proved to be full modules by Wisbauer in [17]. Other works concerned with these modules can be found in [14] and [15].

In this paper, for a τ-full module $M \in$ Mod-R, we investigate the behavior of the fully invariant submodules N such that M/N is τ-torsion-free. We establish a lattice isomorphism between the set of these submodules and a sublattice of R-tors determined by τ and M, considering that M be also relatively injective. Therefore, we can get some results about the structure of this modules. In order to do this, we have divided the paper in three sections: in Section 2 we give the concepts, characterizations and some results related to τ-full modules. In Section 3, we establish the lattice isomorphism between the lattice $[\tau, \tau \vee \xi(M)]$ and the lattice of τ-pure fully invariant submodules of M, assuming, in addition, that M is absolutely τ-pure. Under these conditions, it was proved, in [8, Proposition 15.6], that every τ-pure submodule of M is a direct summand of M; in this section we prove that if N is a τ-pure fully invariant submodule of M. there is another τ-pure fully invariant submodule of M which is complement of N to get M. Also, we get a decomposition of M in terms of some τ-pure fully invariant submodules N of M with different atomic structure on their intervals $[\tau, \tau \vee \xi(N)]$. In Section 4, we prove some equivalent statements so that interval $[\tau, \tau \vee \xi(M)]$ be atomic, for a τ-full module M, and give conditions on the internal structure of M in order that atoms be of some specific type. Among these conditions we get some decompositions of $\chi(M)$.

For $M, N \in$ Mod-R, the notation $N \leq M$ $(N < M)$ means that N is a (proper) submodule of M. If N is an essential submodule of M, we write $N \underset{ess}{\leq} M$. Also we use this symbols $\leq$ $(<)$ for the partial order in the lattice R-tors. For $\tau, \sigma \in R$-tors with $\tau \leq \sigma$, $[\tau, \sigma] = \{\gamma \in R\text{-tors} \mid \tau \leq \gamma \leq \sigma\}$. When we mean that X is a (proper) subset or a (proper) subclass of Y, we write $X \subseteq Y$ $(X \subset Y)$. For a family of right R-modules $\{M_a\}$, let $\chi(\{M_a\})$ be the torsion theory cogenerated by the family $\{M_a\}$, i.e. the maximal element of R-tors for which all the M_α are torsion free; and let $\xi(\{M_a\})$ be the torsion theory generated by the family $\{M_a\}$, i.e. the minimal element of R-tors for which all the M_α are torsion. In particular, we write $\chi(M)$ and $\xi(M)$ instead of $\chi(\{M\})$ and $\xi(\{M\})$, respectively. The greatest element of R-tors is denoted by χ and the least by ξ. For $\tau \in R$-tors, $\mathbb{T}_\tau$, $\mathbb{F}_\tau$ and t_τ denotes the torsion class, the torsion free class and the torsion functor associated to τ, respectively.

We give some concepts and results that we will refer to throughout this paper.

Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice. A non-zero element $a \in L$ is an *atom* if $x < a$ implies $x = 0$, for each $x \in L$. The lattice L is said to be *atomic* if for every $0 \neq y \in L$, there is an atom $a \in L$ such that $a \leq y$. L is said to be *locally atomic* if every non-zero element in L is a join of atoms. If L is a complete Boolean lattice, then L is atomic if and only if L is locally atomic if and only if the element 1 is a join of atoms of L. We also observe that if L is Boolean and if $a, b \in L$ are such that $a < b$, then $[a, b]$ is Boolean. For other concepts and terminology about lattice theory, the reader is referred to [5, 10].

Let $\tau \in R$-tors and $M \in$ Mod-R, a submodule N of M is said to be τ-*dense* in M if $M/N \in \mathbb{T}_\tau$. N is τ-*pure* in M if $M/N \in \mathbb{F}_\tau$. M is called τ-*cocritical* if $M \in \mathbb{F}_\tau$ and every $0 \neq N \leq M$ is τ-*dense* in M. M is *cocritical* if there is $\tau \in R$-tors such that M is τ-cocritical. We say that M is a τ-$\mathcal{A}$-module if $M \in \mathbb{F}_\tau$ and $\tau \vee \xi(M)$ is an atom in $[\tau, \chi]$. We write $E(M)$ for the injective hull of M, and for a $\tau \in R$-tors, we denote $E_\tau(M)$ the τ-injective hull of M which can be described as $E_\tau(M)/M = t_\tau(E(M)/M)$.

$\tau \in R$-tors is said to be *irreducible* if for $\tau', \tau'' \in R$-tors with $\tau' \wedge \tau'' = \tau$, we have that $\tau' = \tau$ or $\tau'' = \tau$. The element τ is *strongly irreducible* if $\wedge U \leq \tau$ implies that there exists $\sigma \in U$ such that $\sigma \leq \tau$, for each $\phi \neq U \subseteq R$-tors. We say that τ is *prime* if it is of the form $\chi(M)$ for some cocritical right R-module.

For all other concepts and terminology concerning torsion theories, the reader is referred to [8, 13].

2 τ-full modules

Definition 1. Let $\tau \in R$-tors. A nonzero right R-module M is said to be a τ-full module if $M \in \mathbb{F}_\tau$ and for every $0 \neq N \underset{ess}{\leq} M$, we have that $M/N \in \mathbb{T}_\tau$.[1]

Examples 2.

1. If M is τ-cocritical, then M is a τ-full module.
2. If M is a semisimple τ-torsion free module, then M is a τ-full module.
3. Let τ_g denote the Goldie torsion theory and $M \in$ Mod-R. Then M is τ_g-torsion free if and only if M is a τ_g-full module.
4. Let $\tau \in R$-tors be a hereditary torsion theory. τ is said to be spectral if the class of τ-injective and τ-torsion free right R-modules is a spectral

[1] The concept of τ-full module can also be defined for modules that are not necessarily τ-torsion free, as it is in [1].

category, i.e. a Grothendieck category where every short exact sequence splits. If τ is a spectral torsion theory and $M \in \mathbb{F}_{\tau}$, then M is a τ-full module. For further details see [2, Proposition 1.1], [3], and [13].

5. Let $M \in$ Mod-R. M is a ξ-full module if and only if M is a semisimple module.

6. Let τ_{sp} be the hereditary torsion theory whose torsion class consists of all semisimple and projective modules. For each $M \in$ Mod-R, $t_{\tau_{sp}}(M) = \sum\{ S \leq M \mid S$ is simple and projective$\}$. Then $M \in$ Mod-R is a τ_{sp}-full module if and only if M is semisimple and singular.

7. Let $\tau \in R$-tors. If R is τ-full, then $\tau = \tau_g$.

$\square$

In order to make this work self-contained we include the following results from [8, Chapter 15].

Proposition 3. *Let M be a τ-full module. Then the following conditions hold.*

1. *If $0 \neq N \leq M$, then N is also τ-full.*
2. *If N is a τ-pure submodule of M, then M/N is τ-full.*

The next proposition shows that the property of being τ-full of the module M_R, extends to any generalization σ of τ, when M is σ-torsion free.

Proposition 4. *Let $\tau, \sigma \in R$-tors such that $\tau \leq \sigma$. If $M \in$ Mod-R is τ-full and $M \in \mathbb{F}_{\sigma}$, then M is σ-full.*

Proof *Let $0 \neq N \underset{ess}{\leq} M$, then $M/N \in \mathbb{T}_{\tau}$. Therefore, $M/N \in \mathbb{T}_{\sigma}$ and M is σ-full.* $\square$

Corollary 5. *If $M \in$ Mod-R is τ-full for $\tau \in R$-tors, then M is a $\chi(M)$-full module.*

Remark 6. As a consequence of Proposition 4 it can be proved that $M \in$ Mod-R is τ-full if and only if the restriction of the torsion theory τ to the category $\sigma[M]$ is a spectral torsion theory.

Proposition 7. *Let $M \in$ Mod-R and $\tau, \sigma \in R$-tors. If M is τ-full and $M \in \mathbb{T}_{\sigma}$, then M is $(\tau \wedge \sigma)$-full.*

Proof *As $(\tau \wedge \sigma) \leq \tau$ and $M \in \mathbb{F}_{\tau}$ we see that $M \in \mathbb{F}_{\tau \wedge \sigma}$. If $N \underset{ess}{\leq} M$, then $M/N \in \mathbb{T}_{\tau} \cap \mathbb{T}_{\sigma}$. Hence M is $(\tau \wedge \sigma)$-full.* $\square$

Definition 8. A module M is called full if there exists $\tau \in R$-tors such that M is τ-full.

Remark 9. By Corollary 5 we see that a module M is full if and only if M is $\chi(M)$-full.

Now, for each R-module M we write $\xi_M = \xi(\{ M/N \mid N \underset{ess}{\leq} M\})$. Note that if M is a full module, then $M/N \in \mathbb{T}_{\chi(M)}$, for each $N \underset{ess}{\leq} M$; thus $\xi_M \leq \chi(M)$.

In the next result we assume that M is a full module. In Example 13 we shall see that this is a necessary condition.

Proposition 10. *Let M be a full R-module and $\tau \in R$-tors. Then M is τ-full if and only if $\tau \in [\xi_M, \chi(M)]$.*

Proof $\Rightarrow]$ *Let $\tau \in R$-tors such that M is τ-full, then $\tau \leq \chi(M)$, and if $N \underset{ess}{\leq} M$, we have that $M/N \in \mathbb{T}_\tau$; therefore $\xi_M \leq \tau$.*

$\Leftarrow]$ *Now, let $\pi \in [\xi_M, \chi(M)]$. Since $\xi_M \leq \pi$, $M/N \in \mathbb{T}_\pi$, for every $N \underset{ess}{\leq} M$. On the other hand, $\pi \leq \chi(M)$ tells us that $M \in \mathbb{F}_\pi$. Thus, M is π-full.* □

Corollary 11. *Let $\{\tau_\alpha\}_{\alpha\in I} \subseteq R$-tors and $M \in$ Mod-R. If M is τ_α-full for every $\alpha \in I$, then M is $\underset{\alpha\in I}{\wedge}\tau_\alpha$-full and $\underset{\alpha\in I}{\vee}\tau_\alpha$-full.*

Proof *If M is τ_α-full, then $\tau_\alpha \in [\xi_M, \chi(M)]$ for every $\alpha \in I$. So $\underset{\alpha\in I}{\wedge}\tau_\alpha$ and $\underset{\alpha\in I}{\vee}\tau_\alpha$ are in the interval $[\xi_M, \chi(M)]$. The result follows straightforwardly from the above proposition.* □

The next proposition is an immediate result from the definitions.

Proposition 12. *Let $\tau \in R$-tors and $M \in$ Mod-R. M is τ-cocritical if and only if M is τ-full and uniform.*

The following example shows that the injective hull of a full module is not always a full module.

Example 13. Let $R = \mathbb{Z}$, $p \in R$ be a prime number and $M = \mathbb{Z}_p$. M is simple and $\chi(\mathbb{Z}_p)$-torsion free module, so it is a $\chi(\mathbb{Z}_p)$-full module. However, $E(\mathbb{Z}_p) = \mathbb{Z}_{p^\infty}$ is not full since for every essential submodule $\mathbb{Z}_{p^k}$ we have that $\mathbb{Z}_{p^\infty}/\mathbb{Z}_{p^k} \simeq \mathbb{Z}_{p^\infty} \notin \mathbb{T}_{\chi(\mathbb{Z}_{p^\infty})}$. Notice that in this case $\xi_{\mathbb{Z}_{p^\infty}} \nleq \chi(\mathbb{Z}_{p^\infty})$, since $\xi_{\mathbb{Z}_{p^\infty}} = \xi(\mathbb{Z}_{p^\infty})$. □

Remark 14. In the following proposition, which was proved in [17], condition 2. is Zelmanowitz' definition of polyform module. So, this proposition says that a module $M \in$ Mod-R is full if and only if M is polyform.

Proposition 15. *Let $M \in$ Mod-R. The following conditions are equivalent.*

1. *M is full.*

2. *For every submodule N of M and every morphism $f : N \to M$ such that $\ker(f) \underset{ess}{\leq} N$, we have that $f = 0$.*

Proposition 16. *Let $\tau \in R$-tors and $M \in$ Mod-R a τ-full module. If $N \in$ Mod-R is such that $\chi(N) = \chi(M)$, then N contains a τ-full submodule.*

Proof *Since M is τ-full, $\tau \in [\xi_M, \chi(M)]$ by Proposition 10, and thus $\tau \leq \chi(N)$. As $M \in \mathbb{F}_{\chi(N)}$, then $Hom_R(M, E(N)) \neq 0$. Let $0 \neq f : M \to E(N)$, then there is a non-zero submodule $M' \leq M$ such that $0 \neq f(M') \leq N$. Therefore $f(M') \in \mathbb{F}_{\chi(N)} \subseteq \mathbb{F}_\tau$. By Proposition 3, we can conclude that $f(M')$ is τ-full.* □

Proposition 17. *Let $\tau \in R$-tors and M a τ-full R-module. Then the following conditions hold.*

1. *$E_\tau(M)$ is a τ-full R-module.*
2. *$E_\tau(M)$ is the greatest τ-full submodule of $E(M)$.*

Proof *1. It is a consequence of [8, Proposition 15.4].*

2. Let K be a τ-full submodule of $E(M)$, then $K \neq 0$ and thus $K \cap M \neq 0$. Moreover $K \cap M \underset{ess}{\leq} K$. Note that $K/K \cap M \in \mathbb{T}_\tau$ since K is a τ-full R-module. As $E(M)/E_\tau(M) \in \mathbb{F}_\tau$, then the morphism $f : K/K \cap M \to E(M)/E_\tau(M)$ defined by $f((x + K \cap M)) = x + E_\tau(M)$ must be zero. Hence $K \subseteq E_\tau(M)$.
□

Let $\tau \in R$-tors. A right R-module M is said to be *absolutely τ-pure* if it is τ-torsion free and τ-injective.

Remark 18. Let $M \in$ Mod-R and $\sigma = \chi(M) \wedge \chi(E(M)/M)$. As $\sigma \leq \chi(M)$, then $M \in \mathbb{F}_\sigma$; on the other hand $\sigma \leq \chi(E(M)/M)$ implies that $E(M)/M \in \mathbb{F}_\sigma$, i.e. $E_\sigma(M)/M = t_\sigma(E(M)/M) = 0$, which means that M is σ-injective. Therefore, M is absolutely σ-pure. So, if $\tau \in R$-tors, then M is absolutely τ-pure if and only if $\tau \in [\xi, \chi(M) \wedge \chi(E(M)/M)]$. (See [8, Chapter 10] for further details about absolutely τ-pure modules.)

From Proposition 10, we can conclude that for a full module M, if $\tau \in R$-tors is such that M is absolutely τ-pure and τ-full, then $\tau \in [\xi_M, \chi(M) \wedge \chi(E(M)/M)]$. However, it is not enough that M be a full module to have that $\xi_M \leq \chi(M) \wedge \chi(E(M)/M)$, as we can see in the following example. Thus the converse is not true in general.

Example 19. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}$, then $E(M) = \mathbb{Q}$, M is τ_g-full, $\chi(M) = \chi(\mathbb{Z}) = \tau_g$, $\chi(E(M)/M) = \chi(\mathbb{Q}/\mathbb{Z}) = \xi$ and $\xi_M = \xi_\mathbb{Z} = \tau_g$. Thus $\xi_M \nleq \chi(M) \wedge \chi(E(M)/M)$. □

3 Structure of $Sub_{P_\tau FI}(M)$ and $[\tau, \tau \vee \xi(M)]$

Let $\tau \in R$-tors and $M \in$ Mod-R. In this section we are going to study some properties of the set $\{N \leq M \mid N$ is τ-pure and fully invariant in $M\}$, henceforth we shall denote it as $Sub_{P_\tau FI}(M)$.

We begin with a characterization of the τ-pure submodules of a τ-full R-module.

Proposition 20. *Let $\tau \in R$-tors and M a τ-full R module. Then $N \leq M$ is τ-pure in M if and only if N is essentially closed in M.*

Proof $\Rightarrow]$ *Let N be a τ-pure submodule of M If $N \underset{ess}{\leq} N' < M$, then $N'/N \in \mathbb{T}_\tau$ since N' is τ-full; on the other hand $M/N \in \mathbb{F}_\tau$ implies that $N'/N \in \mathbb{F}_\tau$, hence, $N' = N$.*

$\Leftarrow]$ *Let $N \leq M$ essentially closed in M and let $N' \leq M$ be a pseudocomplement of N in M. Then N must be also a pseudocomplement of N' in M. Therefore we have an essential monomorphism $N' \simeq N \oplus N'/N \overset{ess}{\hookrightarrow} M/N$. So, we can deduce that $M/N \in \mathbb{F}_\tau$, since $N' \in \mathbb{F}_\tau$.* $\square$

Remark 21. We see, by Proposition 20, that the set $Sub_{P_\tau FI}$ does not depend on τ when M is a τ-full module, i.e. $Sub_{P_\tau FI}(M) = \{N \leq M \mid N$ is fully invariant and essentially closed in $M\}$.

Theorem 22. *Let $\tau \in R$-tors, $M \in$ Mod-R and $\varphi : [\tau, \tau \vee \xi(M)] \to Sub_{P_\tau FI}(M)$ defined by $\varphi(\sigma) = t_\sigma(M)$. Then the following conditions hold.*

1. *If M is a τ-full module, then φ is injective.*
2. *If M is a τ-full and absolutely τ-pure module, then φ is bijective.*

Proof *1. We first claim that for every $\sigma \in [\tau, \tau \vee \xi(M)]$, $\sigma = \tau \vee \xi(t_\sigma(M))$. Let $N = t_\sigma(M)$, then $\tau \leq \tau \vee \xi(N) \leq \sigma \leq \tau \vee \xi(M)$. Assume that $\tau \vee \xi(N) < \sigma$; then there exists $0 \neq K \in$ Mod-R such that $K \in \mathbb{T}_\sigma$ and $K \in \mathbb{F}_{\tau \vee \xi(N)}$. Therefore, $K \in \mathbb{F}_\tau$ and $K \in \mathbb{F}_{\xi(N)}$; so $Hom_R(N, E(K)) = 0$. On the other hand, $K \in \mathbb{T}_\sigma \subseteq \mathbb{T}_{\tau \vee \xi(M)}$ implies that $Hom_R(M, E(K)) \neq 0$. Let $0 \neq f \in Hom_R(M, E(K))$, then $N \leq \ker(f)$, and so there is a morphism $0 \neq \overline{f} : M/N \to E(K)$. It follows that there exists submodules $H/N < L/N \leq M/N$ and a monomorphism $L/H \hookrightarrow K \in \mathbb{T}_\sigma$, then $L/H \in \mathbb{T}_\sigma$. Since $L/N \leq M/N = M/t_\sigma(M) \in \mathbb{F}_\sigma$, it follows that $H/N \underset{ess}{\leq} L/N$ by [8, Proposition 5.7], thus $H \underset{ess}{\leq} L$. As L is τ-full, we have that $L/H \in \mathbb{T}_\tau$; but this is a contradiction because $L/H \hookrightarrow K \in \mathbb{F}_\tau$. Hence, $\sigma = \tau \vee \xi(t_\sigma(M))$.*

Now, let $\sigma, \sigma' \in [\tau, \tau \vee \xi(M)]$ such that $\varphi(\sigma) = \varphi(\sigma')$, then $t_\sigma(M) = t_{\sigma'}(M)$. Using the above equality, we have that $\sigma = \tau \vee \xi(t_\sigma(M)) = \tau \vee \xi(t_{\sigma'}(M)) = \sigma'$. Therefore, φ is injective.

2. By 1. we already know that φ is injective. Now, let $N \in Sub_{P_\tau FI}(M)$ and $\sigma = \tau \vee \xi(N)$. We claim that $t_\sigma(M) = N$.

As $t_\sigma(M)$ is τ-pure in M, there exists $L \leq M$ such that $M = t_\sigma(M) \oplus L$ by [8, Proposition 15.6]. Thus, $t_\sigma(M)$ is absolutely τ-pure and τ-full. Notice that $N \leq t_\sigma(M)$, even more, N is τ-pure in $t_\sigma(M)$. Then N is a direct summand of $t_\sigma(M)$; so there exists $K \leq t_\sigma(M)$ such that $t_\sigma(M) = N \oplus K$. Inasmuch as $K \simeq t_\sigma(M)/N \in \mathbb{T}_\sigma \cap \mathbb{F}_\tau$, we have that $K \notin \mathbb{F}_{\xi(N)}$; therefore $Hom_R(N, E(K)) \neq 0$. Let $0 \neq g : N \to E(K)$ and let $N_0 = g^{-1}(K)$, then there is a morphism $f : N/N_0 \to E(K)/K$ defined by $f(x + N_0) = g(x) + K$. We can see that f is a monomorphism. On the other hand, as K is a direct summand of $t_\sigma(M)$, K is τ-injective from where we get that $E(K)/K \in \mathbb{F}_\tau$; hence, $N/N_0 \in \mathbb{F}_\tau$. Since N is a τ-full and absolutely τ-pure module, there exists $N_1 \leq N$ such that $N = N_0 \oplus N_1$. In this way we have that $M = t_\sigma(M) \oplus L = N \oplus K \oplus L = N_0 \oplus N_1 \oplus K \oplus L$. Consequently, unless $K = 0$, it can be defined an endomorphism $0 \neq h : M \to M$ in such a way that $0 \neq h(N) \subseteq K$, which is a contradiction since N is fully invariant. Thus $N = t_\sigma(M)$, that is, φ is surjective. □

Now, considering Remarks 18 and 21 we have the following corollary.

Corollary 23. *Let $\tau \in R$-tors. If M is a τ-full and absolutely τ-pure R-module , then $[\sigma, \sigma \vee \xi(M)] \simeq Sub_{P_\tau FI}(M)\ \forall \sigma \in [\xi_M, \chi(M) \wedge \chi(E(M)/M)]$, where $\xi_M = \xi(\{ M/N |\ N \underset{ess}{\leq} M\})$.*

Corollary 24. *If M is a τ-full and absolutely τ-pure module, then φ is an isomorphism of complete lattices.*

Proof *It follows from the fact that φ preserves order and arbitrary meets.* □

The following examples show that the hypothesis of M be τ-full in Theorem 22,1 is not superfluous, neither the hypothesis of M be absolutely τ-pure in Theorem 22,2.

Example 25. Let $R = \mathbb{Z}$, $\tau = \xi$ and $M = \mathbb{Z}$, then $[\xi, \xi(\mathbb{Z})] = \mathbb{Z}$-tors. In this case M is not ξ-full, nor $\varphi : \mathbb{Z}\text{-tors} \to Sub_{P_\tau FI}(\mathbb{Z})$ such that $\varphi(\sigma) = t_\sigma(\mathbb{Z})$ is an injective function, since $t_\sigma(\mathbb{Z}) = 0\ \forall \sigma \in \mathbb{Z}$-tors with $\sigma < \chi$.

Example 26. Let F be a field, $A = F^{(\aleph_0)}$ and P the subalgebra of $F^{\aleph_0}$ generated by $\overline{1}$ and A, where $\overline{1}$ denotes the unitary element in the ring $F^{\aleph_0}$. Note that A is a maximal ideal of P, and that $A \in$ Mod-P is faithful and semisimple. We can see F as a unital subring of P if we consider $F_0 = \{(a) = (a, a, a, ...) |\ a \in F\}$. Now, let $Q = \mathcal{M}_{2\times 2}(P)$, the ring of all 2×2 matrices over P and R the subring $\begin{pmatrix} P & A \\ 0 & F_0 \end{pmatrix}$ of Q. The minimal right ideals of R are of the form $\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ where $S \leq A$ is a minimal ideal of P, and

$\begin{pmatrix} 0 & 0 \\ 0 & F_0 \end{pmatrix}$. So, the right socle of R is $soc_r(R) = \begin{pmatrix} 0 & A \\ 0 & F_0 \end{pmatrix}$; it is an essential right ideal of R since for every $0 \neq r \in R$ there is an element $s \in R$ such that $0 \neq rs \in soc_r(R)$. Moreover, if $x \in R$ is such that $x(soc_r(R)) = 0$, then $x = 0$; thus, R is a right non-singular R-module. Hence $R \in \mathbb{F}_{\tau_g}$, which means that R is τ_g-full. On the other hand, R is not absolutely τ_g-pure since $M = \begin{pmatrix} F^{\aleph_0} & A \\ 0 & F_0 \end{pmatrix} \in$ Mod-R and $R \underset{ess}{\leq} M$.

Now, set $H = \begin{pmatrix} P & A \\ 0 & 0 \end{pmatrix}$. H is a two-sided ideal of R, so it is a τ_g-torsion-free fully invariant submodule of R; furthermore, H is τ_g-pure in R since $R = H \oplus \begin{pmatrix} 0 & 0 \\ 0 & F_0 \end{pmatrix}$ as a right R-module. Let $x = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \in H$ with $e_1 = (1, 0, 0, ...)$, and $y = \begin{pmatrix} 0 & 0 \\ 0 & \overline{1} \end{pmatrix} \in R - H$. We can verify that $(0 : x) = \{r \in R|\ xr = 0\} = H = \{r \in R|\ yr \in H\} = (H : y)$; hence, $H \neq t_\sigma(R)\ \forall \sigma \in R$-tors by [11, Corollary of Proposition 2.1], specially, $H \neq t_\sigma(R)$ for every $\sigma \in [\tau_g, \tau_g \vee \xi(R)] = [\tau_g, \chi]$. □

Now for $\tau, \sigma \in R$-tors, we shall write $\tau \ll \sigma$ if $\tau \leq \sigma$ and for every $\alpha \in R$-tors such that $\sigma \wedge \alpha \leq \tau$, we have that $\alpha \leq \tau$.

Using this, we are going to prove that when we have a τ-full R-module, the interval $[\tau, \tau \vee \xi(M)]$ is a Boolean lattice.

Definition 27. The Cantor-Bendixson derivative on R-tors is the function d_{cb} from R-tors to itself given by $d_{cb}(\tau) = \wedge\{\sigma|\ \tau \ll \sigma\}$.

The following result has been already stated in [9, Proposition 1.10]; here we give a different proof.

Proposition 28. *If $\tau \in R$-tors and M is τ-full, then $\tau \vee \xi(M) \leq d_{cb}(\tau)$.*

Proof *Let M be a τ-full module. We are going to prove that $\tau \vee \xi(M) \leq \rho$, for every $\rho \in R$-tors such that $\tau \ll \rho$.*

Assume that there is a $\rho \in R$-tors such that $\tau \ll \rho$ and $\tau \vee \xi(M) \nleq \rho$. Since $\tau \leq \rho$, then $M \notin \mathbb{T}_\rho$. Let $\overline{M} = M/t_\rho(M)$, then $\overline{M} \neq 0$ and $\overline{M} \in \mathbb{F}_\rho \subseteq \mathbb{F}_\tau$; so $\overline{M}$ is τ-full and ρ-full, because M is τ-full. As $\overline{M} \in \mathbb{F}_\rho$, then $\tau \vee \xi(\overline{M}) \nleq \rho$.

Now, we claim that $\rho \wedge \xi(\overline{M}) \leq \tau$. Let $L \in \mathbb{T}_{\rho \wedge \xi(\overline{M})}$, then $L \in \mathbb{T}_\rho$ and $L \in \mathbb{T}_{\xi(\overline{M})}$; so $Hom_R(\overline{M}, E(L)) \neq 0$. Then there exists submodules $H \subset T \subseteq \overline{M}$ and a monomorphism $T/H \hookrightarrow L \in \mathbb{T}_\rho$, which means that $T/H \in \mathbb{T}_\rho$. As $T \in \mathbb{F}_\rho$, we have that $H \underset{ess}{\leq} T$. Since $\overline{M}$ is τ-full, T is τ-full, and then $T/H \in \mathbb{T}_\tau$. Hence $t_\tau(L) \neq 0$. So we have proved that any $(\rho \wedge \xi(\overline{M}))$-torsion module has non-zero τ-torsion. So, if $L \neq t_\tau(L)$, $0 \neq L' = L/t_\tau(L) \in \mathbb{T}_{\rho \wedge \xi(\overline{M})}$, then L' must have non-zero τ-torsion, which is impossible. Therefore $L \in \mathbb{T}_\tau$. It proves *that $\rho \wedge \xi\left(\overline{M}\right) \leq \tau$, then $\xi\left(\overline{M}\right) \leq \tau$ because $\tau \ll \rho$; but this is a contradiction since $\tau \leq \rho$ and $\tau \vee \xi\left(\overline{M}\right) \nleq \rho$.* □

Corollary 29. *Let $\tau \in R$-tors and M a τ-full R-module. Then $[\tau, \tau \vee \xi(M)]$ is a Boolean lattice.*

Proof *It follows from the fact that the interval $[\tau, d_{cb}(\tau)]$ is Boolean [9, Proposition 1.2] and from the above result.* □

Corollary 30. *Let $\tau \in R$-tors and $M \in$ Mod-R. If M is τ-full and absolutely τ-pure, then $Sub_{P_\tau FI}(M)$ is a Boolean lattice.*

Now, using the fact that $[\tau, \tau \vee \xi(M)]$ is a Boolean lattice and the bijective correspondence between $[\tau, \tau \vee \xi(M)]$ and $Sub_{P_\tau FI}(M)$ when M is a τ-full and an absolutely τ-pure module, we shall establish some equivalent conditions among the lattice $[\tau, \tau \vee \xi(M)]$, the module M and the hereditary torsion theory $\chi(M)$. Also, considering that $E_\tau(M)$ is τ-full and absolutely τ-pure module, when M is a τ-full, we shall give some properties of $Sub_{P_\tau FI}(E_\tau(M))$.

Proposition 31. *Let $\tau \in R$-tors, and let $M \in$ Mod-R an absolutely τ-pure and τ-full module. If $K_1, K_2, \ldots, K_n$ are τ-pure submodules of M, then $\sum_{i=1}^{n} K_i$ is absolutely τ-pure.*

Proof *It is enough to prove for $n = 2$. Let K_1 and K_2 be τ-pure submodules of M, then there exists H_1 and H_2 submodules of M such that $M = K_1 \oplus H_1$ and $M = K_2 \oplus H_2$, by [8, Proposition 15.6]. Therefore K_1 and K_2 are absolutely τ-pure and τ-full modules and $K_1 \cap K_2$ is a τ-pure submodule of K_1 and K_2. So, there exists $L_1 \leq K_1$ and $L_2 \leq K_2$ such that $K_1 = (K_1 \cap K_2) \oplus L_1$ and $K_2 = (K_1 \cap K_2) \oplus L_2$. Then $K_1 + K_2 = (K_1 \cap K_2) \oplus L_1 \oplus L_2$. As $K_1 \cap K_2$, L_1 and L_2 are τ-injective modules, $K_1 + K_2$ is an absolutely τ-pure module.* □

Corollary 32. *Let $\tau \in R$-tors, and let $M \in$ Mod-R an absolutely τ-pure and τ-full module. If $K_1, K_2, \ldots, K_n$ are τ-pure submodules of M, then $\sum_{i=1}^{n} K_i$ is a τ-pure submodule of M.*

Proof *Since $K_1 + K_2$ is absolutely τ-pure, by the above proposition, and $M \in \mathbb{F}_\tau$, then $K_1 + K_2$ is a τ-pure submodule of M, by [8, Proposition 10.1].* □

Proposition 33. *Let $\tau \in R$-tors, and let $M \in$ Mod-R be an absolutely τ-pure and τ-full module. If $N \in Sub_{P_\tau FI}(M)$, then there exists $N' \in Sub_{P_\tau FI}(M)$ such that $N \oplus N' = M$.*

Proof *Let $N \in Sub_{P_\tau FI}(M)$ and $\sigma = \tau \vee \xi(N) \in [\tau, \tau \vee \xi(M)]$. Since $[\tau, \tau \vee \xi(M)]$ is Boolean, there exists $\sigma^c \in [\tau, \tau \vee \xi(M)]$, the complement of σ in this lattice. By Theorem 22, there is a τ-pure fully invariant submodule N' of M such that $\sigma^c = \tau \vee \xi(N')$. Then $\tau \vee \xi(M) = \sigma \vee \sigma^c = (\tau \vee \xi(N)) \vee (\tau \vee \xi(N')) = \tau \vee \xi(N \oplus N')$.*

Now, we claim that $N \oplus N' = M$. As N is a τ-pure submodule of M, there exists $K \leq M$ such that $M = N \oplus K$, by [8, Proposition 15.6]. This implies that $N' = t_{\sigma^c}(M) = t_{\sigma^c}(N) \oplus t_{\sigma^c}(K) = t_{\sigma^c}(K) \leq K$. Similarly, as K is a τ-full and absolutely τ-pure R-module, and N' is τ-pure in K, then N' is a direct summand of K, that is, $K = N' \oplus K'$ where $K' \leq M$. Therefore $M = N \oplus N' \oplus K'$ and thus $N \oplus N' \in Sub_{P_\tau FI}(M)$. Since $\tau \vee \xi(N \oplus N') = \tau \vee \xi(M)$, it must happen that $K' = 0$; so $N \oplus N' = M$. □

Remark 34. As we can see in the Example 26, the only complement of H in R is $\begin{pmatrix} 0 & 0 \\ 0 & F_0 \end{pmatrix} \notin Sub_{P_{\tau_g} FI}(R)$, so, we cannot avoid the hypothesis that M be absolutely τ-pure in Proposition 33.

Remark 35. Let $\tau \in R$-tors and $M \in$ Mod-R such that M is a τ-full and absolutely τ-pure module. Then the following conditions hold.

1. If $K, N \in Sub_{P_\tau FI}(M)$, then $K \cap N \in Sub_{P_\tau FI}(N)$.
2. If $N \in Sub_{P_\tau FI}(M)$, then $Sub_{P_\tau FI}(N) \subseteq Sub_{P_\tau FI}(M)$.

   – If $K \in Sub_{P_\tau FI}(N)$, then there is $K' \in Sub_{P_\tau FI}(N)$ such that $K \oplus K' = N$, by Proposition 33, since N is also a τ-full and absolutely τ-pure module. By the same Proposition we know that there is $N' \in Sub_{P_\tau FI}(M)$ such that $N \oplus N' = M$; thus $K \oplus K' \oplus N' = M$. Therefore, K is τ-pure in M. On the other hand, for any morphism $f : M \to M$, $f(N) \subseteq N$, then if we take the restriction to N, we have that $f(K) \subseteq K$. Hence $Sub_{P_\tau FI}(N) \subseteq Sub_{P_\tau FI}(M)$.

Considering this, from Theorem 22 we get a decomposition of a τ-full and absolutely τ-pure module M as a direct sum of absolutely τ-pure fully invariant submodules.

Proposition 36. *Let $\tau \in R$-tors and let $M \in$ Mod-R be τ-full and absolutely τ-pure. If $N, K, K' \in Sub_{P_\tau FI}(M)$ are such that $N \oplus K = N \oplus K' = M$, then $K = K'$.*

Proof *Let $\sigma = \tau \vee \xi(N)$. By Theorem 22, $K = t_\rho(M)$ and $K' = t_{\rho'}(M)$ where $\rho = \tau \vee \xi(K)$ and $\rho' = \tau \vee \xi(K')$. As $M = N \oplus K = N \oplus K'$, we have that ρ and ρ' are complements of σ in $[\tau, \tau \vee \xi(M)]$. Since this interval is Boolean, $\rho = \rho'$, which means that $K = K'$.* □

Corollary 37. *Let $\tau \in R$-tors and let $M \in$ Mod-R be τ-full and absolutely τ-pure. If $N, K, K' \in Sub_{P_\tau FI}(M)$ are such that $N \oplus K = N \oplus K'$, then $K = K'$.*

Proof *Let $L = N \oplus K$, then $L \in Sub_{P_\tau FI}(M)$, by Corollary 32. Since $N \in Sub_{P_\tau FI}(M)$, then $N = N \cap L \in Sub_{P_\tau FI}(L)$. Analogously, it happens that $K, K' \in Sub_{P_\tau FI}(L)$. So, we can conclude that $K = K'$.* □

Now, we shall prove some results about the internal structure of a τ-full and absolutely τ-pure module.

Theorem 38. *Let N be a τ-full and absolutely τ-pure module such that $[\tau, \tau \vee \xi(N)]$ is atomic. Then there is a unique decomposition of N as $N = K \oplus K'$, where $K, K' \in Sub_{P_\tau FI}(N)$ and satisfy the following properties:*

a) K contains an independent family of uniform submodules $\{U_\alpha\}_{\alpha\in A}$ such that $\bigoplus_{\alpha\in A} U_\alpha \underset{ess}{\leq} K$,

b) K' does not contain any uniform submodule.

Proof *Let $\{\sigma_i\}_{i\in I}$ be the set of atoms in $[\tau, \tau \vee \xi(N)]$, then $\sigma_i = \tau \vee \xi(N_i)$ with $N_i \leq N$. Now, let $J = \{j \in I |$ exists U_j uniform such that $U_j \leq N_j\}$.*

If $J = \phi$, then N does not contain a uniform submodule; so the claim is satisfied. Let us suppose that $J \neq \phi$ and let $\sigma = \bigvee_{j\in J} \sigma_j$, then $N = K \oplus K'$ where $K = t_\sigma(N)$, $K' = t_{\sigma^c}(N)$ and $\sigma^c = \bigvee_{i\in I-J} \sigma_i$. Therefore, K' does not contain uniform submodules and we claim that for each $0 \neq H \leq K$ there is a uniform module $U \leq H$. As $H \in \mathbb{T}_\sigma = \mathbb{T}_{\bigvee_{j\in J}\sigma_j}$, there is $j_0 \in J$ such that $H \notin \mathbb{F}_{\sigma_{j_0}}$, where $\sigma_{j_0} = \tau \vee \xi(U_{j_0})$ with $U_{j_0} \leq N_{j_0}$ a uniform submodule. But $H \in \mathbb{F}_\tau$ implies that $H \notin \mathbb{F}_{\xi(U_{j_0})}$ which means that $Hom_R(U_{j_0}, E(H)) \neq 0$. Since $E(H) \in \mathbb{F}_\tau$ and U_{j_0} is τ-cocritical, we have that there is a submodule $0 \neq U'_{j_0} \leq U_{j_0}$ and a monomorphism $U'_{j_0} \hookrightarrow H$. Whence, each non-zero submodule of K contains a uniform submodule.

Now, let $\{U_\alpha\}_{\alpha\in A}$ a maximal independent family of uniform submodules of K, then $\bigoplus_{\alpha\in A} U_\alpha \underset{ess}{\leq} K$ because, as before, if there were a non-zero pseudocomplement of $\bigoplus_{\alpha\in A} U_\alpha$ it should contain a uniform submodule, which is impossible.

To see uniqueness, suppose that $N = L \oplus L'$ with $L, L' \in Sub_{P_\tau FI}(N)$ be such that they satisfy conditions a) and b), respectively. Then we have that $K = t_\sigma(N) = t_\sigma(L) \oplus t_\sigma(L') = t_\sigma(L)$, by definition of σ; thus $K \leq L$.

Let $\left\{U'_\beta\right\}_{\beta\in B}$ be an independent family of uniform submodules of L, such that $\bigoplus_{\beta\in B} U'_\beta \underset{ess}{\leq} L$; again, by definition of σ, we have that $U'_\beta \in \mathbb{T}_\sigma$ $\forall \beta \in B$. As L is τ-full, we conclude that $L \in \mathbb{T}_\sigma$; thus $L \leq K$. Therefore, $L = K$. Then, we get that $L' = K'$, by Proposition 36. This proves that the decomposition is unique. $\square$

Theorem 39. *Let $\tau \in R$-tors and let M be a τ-full and absolutely τ-pure R-module. Then there exist unique submodules $N, N' \in Sub_{P_\tau FI}(M)$ such that $M = N \oplus N'$ with $[\tau, \tau \vee \xi(N)]$ atomic and $[\tau, \tau \vee \xi(N')]$ atomless.*

Proof *Let* $\{\sigma_i\}_{i\in I}$ *be the set of atoms in* $[\tau, \tau\vee\xi(M)]$, *then* $\sigma_i = \tau\vee\xi(N_i)$ *where* $N_i = t_{\sigma_i}(M)$. *Let* $\sigma = \bigvee_{i\in I}\sigma_i = \bigvee_{i\in I}(\tau\vee\xi(N_i))$, *then* $\sigma\in[\tau,\tau\vee\xi(M)]$. *Thus* $\sigma = \tau\vee\xi(N)$ *with* $N = t_\sigma(M)$. *Then there is* $N'\in Sub_{P_\tau FI}(M)$ *such that* $N\oplus N' = M$, *by Proposition 33. Observe that* $\{\sigma_i\}_{i\in I}\subseteq[\tau,\tau\vee\xi(N)]$ *and that* $\bigvee_{i\in I}\sigma_i = \tau\vee\xi(N)$, *then* $[\tau,\tau\vee\xi(N)]$ *is atomic.*

Now, we claim that $[\tau,\tau\vee\xi(N')]$ *is atomless since any atom in this lattice would be an atom in* $[\tau,\tau\vee\xi(M)]$, *that is a* σ_i *for some* $i\in I$.

It can be proved uniqueness with a similar argument as the one used in Theorem 38. □

As a consequence of theorems 38 and 39 we have the following result.

Corollary 40. *Let* $\tau\in R$*-tors and let* M *be a* τ*-full and absolutely* τ*-pure* R*-module. Then there exists* $N, N', N''\in Sub_{P_\tau FI}(M)$ *unique submodules of* M *such that* $M = N\oplus N'\oplus N''$ *where* $[\tau,\tau\vee\xi(N'')]$ *is atomless,* N' *contains no uniform submodules and* N *is an essential extension of a direct sum of uniform submodules.*

4 Structure of $[\tau,\tau\vee\xi(M)]$ and decompositions of the torsion theory $\chi(M)$

As we mentioned in the Introduction, a right R-module M is said to be a τ-$\mathcal{A}$-module, with $\tau\in R$-tors, if it is τ-torsion free and $\tau\vee\xi(M)$ is an atom in $[\tau,\chi]$. The next proposition involves this concept.

Proposition 41. *Let* M *be a* τ*-full* R*-module. Then the following conditions are equivalent.*

1. $\tau\vee\xi(M)$ *is an atom in* $gen(\tau)$.
2. $E_\tau(M)$ *is a* τ*-*$\mathcal{A}$*-module.*
3. $\chi(M)$ *is an irreducible element of* R*-tors.*
4. *The only* τ*-pure fully invariant submodules of* $E_\tau(M)$ *are* 0 *and* $E_\tau(M)$.

Proof $1.\Leftrightarrow 2.$ *It follows from [6, Propositions 2.4, 2.9].*

$1.\Rightarrow 3.$ *It follows from [6, Corollary 2.17].*

$3.\Rightarrow 4.$ *Suppose that* $0\le N\le E_\tau(M)$ *is a* τ*-pure fully invariant submodule of* $E_\tau(M)$. *Then there is* $\sigma\in[\tau,\tau\vee\xi(E_\tau(M))]$ *such that* $t_\sigma(E_\tau(M)) = N$, *by Theorem 22. Now, by Corollary 29,* σ *has a complement in* $[\tau,\tau\vee\xi(E_\tau(M))]$

which we denote by σ^c. If $N' = t_{\sigma^c}(E_\tau(M))$, then $E_\tau(M) = N \oplus N'$, by Proposition 33. It means that $\chi(M) = \chi(E_\tau(M)) = \chi(N \oplus N') = \chi(N) \wedge \chi(N')$. Since $\chi(M)$ is irreducible, $\chi(M) = \chi(N)$ or $\chi(M) = \chi(N')$.

If $\chi(M) = \chi(N)$, then $N' \in \mathbb{F}_{\chi(N)}$; so there exists a submodule $0 \neq N'' \leq N'$ and a non-zero morphism $f : N'' \to N$ such that $0 \neq f(N'') \in \mathbb{T}_\sigma \cap \mathbb{T}_{\sigma^c} = \mathbb{T}_{\sigma \wedge \sigma^c} = \mathbb{T}_\tau$. Then $f(N'') \in \mathbb{T}_\tau \cap \mathbb{F}_\tau = 0$, which is a contradiction unless $N' = 0$, and thus $N = E_\tau(M)$. Similarly, if $\chi(M) = \chi(N')$, we prove that $N = 0$.

4. $\Rightarrow$ 1. *Let $\rho \in [\tau, \tau \vee \xi(M)] = [\tau, \tau \vee \xi(E_\tau(M))]$. If $N = t_\rho(E_\tau(M))$, then N is a τ-pure fully invariant submodule of $E_\tau(M)$. Therefore, $N = 0$ or $N = E_\tau(M)$. Hence $\rho = \tau$ or $\rho = \tau \vee \xi(M)$, respectively, by Theorem 22.* $\square$

Theorem 42. *Let M be a τ-full R-module. Then the following conditions are equivalent.*

1. *$[\tau, \tau \vee \xi(M)]$ is an atomic lattice.*
2. *There is an independent family $\{M_i\}_{i \in I}$ of submodules of M such that M_i is a τ-$\mathcal{A}$-module for every $i \in I$, $\tau \vee \xi(M_i) \neq \tau \vee \xi(M_j)$ if $i \neq j$, and $\bigoplus_{i \in I} M_i \underset{ess}{\leq} M$.*
3. *$\chi(M)$ decomposes as the meet of an irredundant family of torsion theories $\{\chi(E_i)\}_{i \in I}$, where $E_i \leq M$ and E_i is a τ-$\mathcal{A}$-module $\forall i \in I$.*
4. *$\chi(M)$ uniquely decomposes as the meet of an irredundant family of irreducible torsion theories.*
5. *If $0 \neq N \leq M$, then $\chi(N)$ uniquely decomposes as the meet of an irredundant family of irreducible torsion theories.*

Proof 1. $\Rightarrow$ 2. *Let $\{\sigma_i\}_{i \in I}$ be the set of atoms in $[\tau, \tau \vee \xi(M)]$. If $M_i = t_{\sigma_i}(M)$, then $\sigma_i = \tau \vee \xi(M_i)$ and M_i is a τ-$\mathcal{A}$-module, since σ_i is an atom. We claim that $\{M_i\}_{i \in I}$ is an independent family in M. Let $j \in I$ and $N = M_j \cap \left(\sum_{i \neq j} M_i\right)$, then $N \in \mathbb{T}_{\sigma_j \wedge \bigvee_{i \neq j} \sigma_i} = \mathbb{T}_{\bigvee_{i \neq j} (\sigma_j \wedge \sigma_i)} = \mathbb{T}_\tau$, because $\sigma_j \wedge \sigma_i = \tau$ $\forall i \neq j$. As $N \leq M \in \mathbb{F}_\tau$, we have that $N \in \mathbb{T}_\tau \cap \mathbb{F}_\tau = \{0\}$, so $N = 0$. Therefore, $\{M_i\}_{i \in I}$ is an independent family of submodules of M which are τ-$\mathcal{A}$-modules. Also, by construction, $\tau \vee \xi(M_i) \neq \tau \vee \xi(M_j)$ if $i \neq j$. Now, we just need to prove that $\bigoplus_{i \in I} M_i \underset{ess}{\leq} M$.*

Suppose that $\bigoplus_{i \in I} M_i$ is not an essential submodule of M. Let $0 \neq K \leq M$ a pseudocomplement of $\bigoplus_{i \in I} M_i$, then $\tau \vee \xi(K) \in [\tau, \tau \vee \xi(M)]$. Since $[\tau, \tau \vee \xi(M)]$ is a locally atomic lattice, then $\tau \vee \xi(K) = \bigvee_{j \in J} \sigma_j$ for some $J \subseteq I$. Thus,

$K \in \mathbb{T}_{\bigvee\limits_{j\in J}\sigma_j}$, which means that there is $j_0 \in J$ such that $K \notin \mathbb{F}_{\sigma_{j_0}}$. But $t_{\sigma_{j_0}}((\bigoplus\limits_{i\in I} M_i)\oplus K) = t_{\sigma_{j_0}}(\bigoplus\limits_{i\in I} M_i)\oplus t_{\sigma_{j_0}}(K) \le t_{\sigma_{j_0}}(M) = M_{j_0}$, then $t_{\sigma_{j_0}}(K) = 0$. Hence $K = 0$, which is a contradiction. Therefore $\bigoplus\limits_{i\in I} M_i \underset{ess}{\le} M$.

2. $\Rightarrow$ 3. *Let $\{M_i\}_{i\in I}$ be an independent family of submodules of M such that M_i is a τ-$\mathcal{A}$-module for every $i \in I$, $\tau \vee \xi\,(M_i) \neq \tau \vee \xi\,(M_j)$ if $i \neq j$, and $\bigoplus\limits_{i\in I} M_i \underset{ess}{\le} M$. The last condition implies that $\bigwedge\limits_{i\in I} \chi\,(M_i) = \chi(\bigoplus\limits_{i\in I} M_i) = \chi\,(M)$. As $\tau \vee \xi\,(M_i) \neq \tau \vee \xi\,(M_j)$ if $i \neq j$, then $\chi\,(M_i) \neq \chi\,(M_j)$ if $i \neq j$, by [6, Corollary 2.16]. Now, suppose that there exists $j \in I$ such that $\bigwedge\limits_{i\neq j} \chi\,(M_i) = \chi\,(M)$. Then $M_j \in \mathbb{F}_{\chi(M)} = \mathbb{F}_{\bigwedge\limits_{i\neq j}\chi(M_i)}$; so there is $k \in I$ such that $k \neq j$ and $M_j \notin \mathbb{T}_{\chi(M_k)}$. Therefore, $Hom_R\,(M_j, E\,(M_k)) \neq 0$, which means that there are submodules $M_j'' < M_j' \le M_j$ and a monomorphism $M_j'/M_j'' \hookrightarrow M_k \in \mathbb{F}_\tau$. Hence, by [6, Proposition 2.4], $\tau \vee \xi\,(M_j) = \tau \vee \xi\left(M_j'\right) = \tau \vee \xi\left(M_j'/M_j''\right) = \tau \vee \xi\,(M_k)$, that is a contradiction.*

3. $\Rightarrow$ 1. *Let $\chi\,(M) = \bigwedge\limits_{i\in I} \chi\,(E_i)$ be an irredundant meet, with E_i a τ-$\mathcal{A}$-module and $E_i \le M$ for every $i \in I$. We are going to prove that $\tau \vee \xi\,(M)$ is a join of atoms.*

Let $\sigma_i = \tau\vee\xi\,(E_i)$ and $M_i = t_{\sigma_i}\,(M)$, then $\sigma_i = \tau\vee\xi\,(M_i)$, by Theorem 22,1; furthermore, if $N = M_j \cap \sum_{i\neq j} M_i$ for some $j \in I$, then $N \in \mathbb{T}_{\sigma_j\wedge(\bigvee\limits_{i\neq j}\sigma_i)} = \mathbb{T}_{\bigvee\limits_{i\neq j}(\sigma_i\wedge\sigma_j)} = \mathbb{T}_\tau$, i.e. $N \in \mathbb{T}_\tau \cap \mathbb{F}_\tau = \{0\}$; thus the sum $\sum_{i\in I} M_i$ is direct.

Now we claim that $\tau\vee\xi(\bigoplus\limits_{i\in I} M_i) = \tau\vee\xi\,(M)$. Let $\sigma = \tau\vee\xi(\bigoplus\limits_{i\in I} M_i)$ and suppose that $\sigma < \tau \vee \xi\,(M)$. As $\sigma \in [\tau, \tau \vee \xi\,(M)]$, there exists $\sigma^c \in [\tau, \tau \vee \xi\,(M)]$, and $\sigma^c = \tau \vee \xi\,(K)$ where $K = t_{\sigma^c}\,(M)$, by Theorem 22. However, $K \le M$ implies that $K \in \mathbb{F}_{\chi(M)} = \mathbb{F}_{\bigwedge\limits_{i\in I}\chi(E_i)} = \mathbb{F}_{\bigwedge\limits_{i\in I}\chi(M_i)}$, by [6, Corollary 2.16]. Then there exists $j \in I$ such that $K \notin \mathbb{T}_{\chi(M_j)}$; so there are submodules $K'' < K' \le K$ and a monomorphism $K'/K'' \hookrightarrow M_j \in \mathbb{T}_\sigma$. But as $K \in \mathbb{T}_{\sigma^c}$, $K'/K'' \in \mathbb{T}_{\sigma^c} \cap \mathbb{T}_\sigma = \mathbb{T}_\tau$. Therefore, $K'/K'' \in \mathbb{T}_\tau \cap \mathbb{F}_\tau = \{0\}$, since $M_j \in \mathbb{F}_\tau$; thus $K' = K''$ which is a contradiction; whence $K = 0$ and $\sigma = \tau \vee \xi\,(M)$. Hence, $\tau \vee \xi\,(M) = \tau \vee \xi(\bigoplus\limits_{i\in I} M_i) = \bigvee\limits_{i\in I}\,(\tau \vee \xi\,(M_i))$ is a join of atoms, which is equivalent to $[\tau, \tau \vee \xi\,(M)]$ be atomic.

3. $\Rightarrow$ 4. *The decomposition of $\chi\,(M)$ as a meet of an irredundant family of irreducible torsion theories is an immediate consequence of 3, since if $\chi\,(M) = \bigwedge\limits_{i\in I} \chi\,(E_i)$, where $E_i \le M$ and E_i is a τ-$\mathcal{A}$-module $\forall\, i \in I$, which means that $\chi\,(E_i)$ is an irreducible element of R-tors [6, Corollary 2.17].*

Now, suppose that there is $\{\alpha_j\}_{j\in J} \subseteq R$*-tors an irredundant family of irreducible torsion theories such that* $\chi(M) = \bigwedge_{j\in J} \alpha_j$*. For any* $i \in I$, $E_i \in \mathbb{F}_{\chi(M)} = \mathbb{F}_{\bigwedge_{j\in J}\alpha_j}$*, which means that there is a* $j_i \in J$ *such that* $E_i \notin \mathbb{T}_{\alpha_{j_i}}$*. Let* $\alpha_{j_i} = \chi(L_{j_i})$ *with* L_{j_i} *an injective* R*-module. Then* $Hom_R(E_i, L_{j_i}) \neq 0$*; thus there are submodules* $E_i'' < E_i' \leq E_i$ *and a monomorphism* $E_i'/E_i'' \hookrightarrow L_{j_i}$*. Whence there is a* τ*-full submodule* N_i *of* L_{j_i}*. Let us take* $N = \sum\{N_\gamma \leq L_{j_i} \mid N_\gamma \text{ is } \tau\text{-full}\}$ *and* $K \leq L_{j_i}$ *such that* $N \oplus K \underset{ess}{\leq} L_{j_i}$*. Then* $\chi(N) \wedge \chi(K) = \chi(L_{j_i}) = \alpha_{j_i}$*. Since* α_{j_i} *is irreducible,* $\alpha_{j_i} = \chi(N)$ *or* $\alpha_{j_i} = \chi(K)$*. If* $\alpha_{j_i} = \chi(K)$*, using a similar argument as above, we can prove that there is a* τ*-full submodule of* K*. But this is not possible, by definition of* N*. Then* $\chi(N) = \chi(L_{j_i}) = \alpha_{j_i}$*; thus* $E_\tau(N)$ *is a* τ*-*$\mathcal{A}$*-module, by Proposition 41. Since every submodule of a* τ*-*$\mathcal{A}$*-module cogenerates the same, we have that* $\chi(N_i) = \chi(E_\tau(N)) = \chi(N) = \chi(L_{j_i})$*; but* $\chi(N_i) = \chi(E_i)$ *implies that* $\chi(E_i) = \alpha_{j_i}$*. Therefore, since both meets are irredundant we have that for each* $\chi(E_i)$ *there is an* α_{j_i} *such that* $\chi(E_i) = \alpha_{j_i}$*.*

4. $\Rightarrow$ 3. *Let* $\chi(M) = \bigwedge_{i\in I} \chi(E_i)$ *where* $\chi(E_i)$ *is irreducible for every* $i \in I$*. We can assume that each* E_i *is injective.*

We claim that $M \notin \mathbb{T}_{\chi(E_i)}$*,* $\forall i \in I$*. Suppose that there is* $j \in I$ *such that* $M \in \mathbb{T}_{\chi(E_j)}$*. Since* $M \in \mathbb{F}_{\chi(M)} = \mathbb{F}_{\bigwedge_{i\in I}\chi(E_i)} = \mathbb{F}_{\chi(E_j)\wedge(\bigwedge_{i\neq j}\chi(E_i))}$*, then* $M \in \mathbb{F}_{\bigwedge_{i\neq j}\chi(E_i)}$*; therefore,* $\bigwedge_{i\neq j}\chi(E_i) \leq \chi(M)$*. But, as* $\bigwedge_{i\in I}\chi(E_i)$ *is an irredundant meet,* $\chi(M) = \bigwedge_{i\in I}\chi(E_i) < \bigwedge_{i\neq j}\chi(E_i)$ *which is a contradiction. Hence,* $M \notin \mathbb{T}_{\chi(E_i)}$*,* $\forall i \in I$*. It means that* $Hom_R(M, E_i) \neq 0$*; then there are submodules* $K_i < N_i \leq M$ *and a monomorphism* $N_i/K_i \hookrightarrow E_i$*. Since* M *is* τ*-full and* $N_i/K_i \in \mathbb{F}_{\chi(E_i)} \subseteq \mathbb{F}_\tau$*,* N_i/K_i *is* τ*-full; then each* E_i *contains a* τ*-full submodule.*

Let $M_i = \sum\{L \leq E_i \mid L \text{ is } \tau\text{-full}\}$*, then* M_i *is the greatest* τ*-full submodule of* E_i *[14, Proposition 1.7]. We claim that* $\chi(M_i) = \chi(E_i)$ $\forall i \in I$*. If* $M_i \underset{ess}{\leq} E_i$*, the assertion is satisfied. If* M_i *is not essential in* E_i*, then there is* $0 \neq K_i \leq E_i$ *a pseudocomplement of* M_i *in* E_i*; then* $M_i \oplus K_i \underset{ess}{\leq} E_i$*. Therefore,* $\chi(M_i) \wedge \chi(K_i) = \chi(M_i \oplus K_i) = \chi(E_i)$*. Since* $\chi(E_i)$ *is irreducible, we have that* $\chi(E_i) = \chi(M_i)$ *or* $\chi(E_i) = \chi(K_i)$*.*

If $\chi(E_i) = \chi(K_i)$*, then* $M \notin \mathbb{T}_{\chi(K_i)}$*, from what we proved above. Then, with a similar argument than the one used for* E_i*,* K_i *contains a* τ*-full module. But this is impossible, by the definition of* M_i*. Therefore,* $\chi(E_i) = \chi(M_i)$*.*

Now, since M_i *is* τ*-full and* $\chi(E_i)$ *is irreducible we have that* $E_\tau(M_i)$ *is a* τ*-*$\mathcal{A}$*-module, by Proposition 41. Hence* $\chi(M) = \bigwedge_{i\in I}\chi(E_i) = \bigwedge_{i\in I}\chi(M_i)$ *is an irredundant meet of torsion theories cogenerated by* τ*-*$\mathcal{A}$*-modules. Aside,* $M_i \in \mathbb{F}_{\chi(M)}$ *implies that* $Hom_R(M_i, E(M)) \neq 0$*, which means that there are* *submodules $K_i < K_i' \leq M_i$ and a monomorphism $K_i'/K_i \hookrightarrow M$. If $N_i = K_i'/K_i$, then $N_i \leq M$ is a τ-$\mathcal{A}$-module with $\chi(N_i) = \chi(M_i)$. Therefore, $\chi(M) = \bigwedge_{i\in I} \chi(N_i)$ is an irredundant meet of torsion theories cogenerated by τ-$\mathcal{A}$-modules that are submodules of M.*

1. $\Rightarrow$ 5. *Let $N \leq M$, then N is τ-full. Since $[\tau, \tau \vee \xi(N)] \subseteq [\tau, \tau \vee \xi(M)]$, then $[\tau, \tau \vee \xi(N)]$ is also an atomic lattice. Hence, $\chi(N)$ uniquely decomposes as the meet of an irredundant family of irreducible torsion, by* 1. $\Rightarrow$ 4.

5. $\Rightarrow$ 4. *It is immediate considering $N = M$.* □

Now, we fit the last theorem in case the decomposition of $\chi(M)$ can be done with strongly irreducible torsion theories. In order to do this, we give some concepts.

Definition 43. 1. A non-zero right R-module M is decisive if M is τ-torsion or τ-torsion free for every $\tau \in R$-tors.

2. Let $\tau \in R$-tors and $M \in$ Mod-R. M is a τ-$\mathcal{D}$-module if M is a τ-$\mathcal{A}$-module and there exists a decisive module D such that $\chi(M) = \chi(D)$. See [7] for details about these modules.

The following technical result will be used to prove the next theorem.

Lemma 44. *If N is a right τ-$\mathcal{D}$-module, then N contains a decisive submodule.*

Proof *As N is a τ-$\mathcal{D}$-module, there is a decisive module D such that $\chi(N) = \chi(D)$. Then $D \notin \mathbb{T}_{\chi(N)}$, which means that $Hom_R(D, E(N)) \neq 0$; hence, there are submodules $D'' < D' \leq D$ and a monomorphism $D'/D'' \hookrightarrow N$. We claim that D'/D'' is decisive. Let $\alpha \in R$-tors; since D is decisive, $D \in \mathbb{T}_\alpha$ or $D \in \mathbb{F}_\alpha$. In the first case, $D' \in \mathbb{T}_\alpha$ and thus $D'/D'' \in \mathbb{T}_\alpha$. In the second one, $\alpha \leq \chi(D) = \chi(N) = \chi(D'/D'')$ which implies that $D'/D'' \in \mathbb{F}_\alpha$.* □

Theorem 45. *Let M be a τ-full R-module. Then the following conditions are equivalent.*

1. *$[\tau, \tau \vee \xi(M)]$ is an atomic lattice, and every atom in this lattice can be written as $\tau \vee \xi(D)$ with D a decisive module.*

2. *There is an independent family $\{M_i\}_{i\in I}$ of submodules of M such that M_i is τ-$\mathcal{D}$-module for every $i \in I$, $\tau \vee \xi(M_i) \neq \tau \vee \xi(M_j)$ if $i \neq j$, and $\bigoplus_{i\in I} M_i \underset{ess}{\leq} M$.*

3. *$\chi(M)$ decomposes as the meet of an irredundant family of torsion theories $\{\chi(D_i)\}_{i\in I}$, where $D_i \leq M$ and D_i is a τ-$\mathcal{D}$-module $\forall\, i \in I$.*

4. *$\chi(M)$ uniquely decomposes as the meet of an irredundant family of strongly irreducible torsion theories.*

5. *If $0 \neq N \leq M$, then $\chi(N)$ uniquely decomposes as the meet of an irredundant family of strongly irreducible torsion theories.*

Proof 1. $\Rightarrow$ 2. *Let $\{\sigma_i\}_{i\in I}$ be the set of atoms in $[\tau, \tau \vee \xi(M)]$. If $M_i = t_{\sigma_i}(M)$, then $\sigma_i = \tau \vee \xi(M_i)$. By 1, $\sigma_i = \tau \vee \xi(D_i)$ with D_i a decisive module. As D_i is decisive, $D_i \in \mathbb{F}_\tau$; thus D_i is a τ-$\mathcal{D}$-module. Therefore, $\chi(M_i) = \chi(D_i)$, by [6, Corollary 2.16]. So, M_i is a τ-$\mathcal{D}$-module. Now, statement 2 follows with the same arguments of* 1. $\Rightarrow$ 2. *of Theorem 42.*

2. $\Rightarrow$ 3. *Let $\{M_i\}_{i\in I}$ be an independent family of submodules of M such that M_i is τ-$\mathcal{D}$-module for every $i \in I$, $\tau \vee \xi(M_i) \neq \tau \vee \xi(M_j)$ if $i \neq j$, and $\bigoplus_{i\in I} M_i \underset{ess}{\leq} M$. Then $\chi(M_i) = \chi(D_i)$ with D_i decisive $\forall\, i \in I$, and $\chi(M) = \bigwedge_{i\in I} \chi(M_i) = \bigwedge_{i\in I} \chi(D_i)$, where $\chi(D_i) \neq \chi(D_j)$ if $i \neq j$ by [6, Corollary 2.16]. Now, we can use the same argument as* 2. $\Rightarrow$ 3. *of Theorem 42 to deduce the irredundancy.*

3. $\Rightarrow$ 4. *Let $\chi(M) = \bigwedge_{i\in I} \chi(D_i)$ be an irredundant meet with $D_i \leq M$ and D_i a τ-$\mathcal{D}$-module $\forall\, i \in I$. As $\chi(D_i)$ is strongly irreducible $\forall\, i \in I$, by [8, Proposition 32.7], this is an irredundant meet of strongly irreducible torsion theories. The uniqueness of the decomposition can be proved with a similar argument as the one used in* 3. $\Rightarrow$ 4. *of the previous theorem.*

4. $\Rightarrow$ 1. *By* 4*, we know that $\chi(M) = \bigwedge_{i\in I} \chi(D_i)$ with D_i a decisive module $\forall\, i \in I$. We can use the same argument as in the proof of* 4. $\Rightarrow$ 3. *of Theorem 42 to prove that $M \notin \mathbb{T}_{\chi(D_i)}$, $\forall\, i \in I$. This means that $Hom(M, E(D_i)) \neq 0$ $\forall\, i \in I$ which implies that there is a submodule N_i of D_i which is τ-full and $\chi(N_i) = \chi(D_i)$ is irreducible. Then $E_\tau(N_i)$ is a τ-$\mathcal{A}$-module, by Proposition 41, in fact, $E_\tau(N_i)$ is a τ-$\mathcal{D}$-module. Analogously, we can argue as at the end of the proof of* 4. $\Rightarrow$ 3. *of Theorem 42 to prove that there is a τ-$\mathcal{D}$-module, $D'_i \leq M$, $\forall\, i \in I$ such that $\chi(D'_i) = \chi(N_i)$. Therefore, we have that $\chi(M) = \bigwedge_{i\in I} \chi(D'_i)$ with $D'_i \leq M$ and D'_i a τ-$\mathcal{D}$-module $\forall\, i \in I$; hence, the lattice $[\tau, \tau \vee \xi(M)]$ is atomic, by Theorem 42.*

Now, let $\sigma \in [\tau, \tau \vee \xi(M)]$ be an atom. If $N = t_\sigma(M)$, then $\sigma = \tau \vee \xi(N)$ and $N \in \mathbb{F}_{\chi(M)} = \mathbb{F}_{\bigwedge_{i\in I} \chi(D'_i)}$. Thus, considering that $N \notin \mathbb{T}_{\chi(D'_j)}$ for some $j \in I$, one can prove that $\chi(N) = \chi\left(D'_j\right)$. Then, N is a τ-$\mathcal{D}$-module. Otherwise, by Lemma 44 there exists a decisive submodule D of N and we conclude that $\sigma = \tau \vee \xi(N) = \tau \vee \xi(D)$.

1. $\Rightarrow$ 5. *Let $0 \neq N \leq M$. Then N is τ-full, and $[\tau, \tau \vee \xi(N)] \subseteq [\tau, \tau \vee \xi(M)]$ means that $[\tau, \tau \vee \xi(N)]$ is atomic, by 1. We also have, by 1, that each atom of $[\tau, \tau \vee \xi(N)]$ can be written as $\tau \vee \xi(D)$ with D a decisive module. Thence,*

$\chi(N)$ uniquely decomposes as the meet of an irredundant family of strongly irreducible torsion theories, by $1. \Rightarrow 4.$

$5. \Rightarrow 4.$ *It immediately holds.* □

Now we present the case where the atoms in $[\tau, \tau \vee \xi(M)]$ can be written as $\tau \vee \xi(C)$ with C a τ-cocritical module.

Theorem 46. *Let M be a τ-full R-module. Then the following conditions are equivalent.*

1. *$[\tau, \tau \vee \xi(M)]$ is an atomic lattice, and every atom in this lattice can be written as $\tau \vee \xi(C)$ with C a τ-cocritical module.*
2. *Every non-zero submodule of M contains a uniform submodule.*
3. *$\chi(M)$ decomposes as the meet of an irredundant family of torsion theories $\{\chi(C_i)\}_{i \in I}$, where $C_i \leq M$ and C_i is τ-cocritical $\forall\, i \in I$.*
4. *$\chi(M)$ uniquely decomposes as the meet of an irredundant family of prime torsion theories.*
5. *If $0 \neq N \leq M$, then $\chi(N)$ uniquely decomposes as the meet of an irredundant family of prime torsion theories.*

Proof $1. \Rightarrow 2.$ *Let $0 \neq N \leq M$. Since $[\tau, \tau \vee \xi(N)] \subseteq [\tau, \tau \vee \xi(M)]$, then $[\tau, \tau \vee \xi(N)]$ satisfies the same conditions of 1. for $[\tau, \tau \vee \xi(M)]$.*

Let $\sigma \in [\tau, \tau \vee \xi(N)]$ be an atom, then $\sigma = \tau \vee \xi(t_\sigma(N)) = \tau \vee \xi(C)$ with C a τ-cocritical module, by 1. and [6, Proposition 2.4, 2.]. Therefore, $t_\sigma(N) \in \mathbb{T}_{\tau \vee \xi(C)}$, which means that $t_\sigma(N) \notin \mathbb{F}_{\xi(C)}$. Thus, there is a morphism $0 \neq f : C \to E(t_\sigma(N))$. So, as C is τ-cocritical, there exists a submodule C' of C and a monomorphism $C' \hookrightarrow t_\sigma(N)$. Hence, N has a τ-cocritical submodule and consequently it has a uniform submodule.

$2. \Rightarrow 3.$ *As M has a uniform submodule, there must exist a maximal independent family $\{U_\lambda\}_{\lambda \in \Lambda}$ of uniform submodules of M. We claim that $\bigoplus_{\lambda \in \Lambda} U_\lambda$ is essential in M, since if it was no essential there should be a pseudocomplement $K \neq 0$ of $\bigoplus_{\lambda \in \Lambda} U_\lambda$ in M, which should contain a uniform submodule. This is not possible. So, by [6, Corollary 2.6] the family $\{U_\lambda\}_{\lambda \in \Lambda}$ satisfies condition 2 of Theorem 42 which implies that $\chi(M)$ can be expressed as an irredundant meet $\chi(M) = \bigwedge_{i \in I} \chi(E_i)$ with E_i a τ-$\mathcal{A}$-module and $E_i \leq M$ for every $i \in I$. By 2., each E_i contains a uniform submodule C_i. Hence, $C_i \leq M$ is τ-cocritical $\forall\, i \in I$ and $\chi(M) = \bigwedge_{i \in I} \chi(E_i) = \bigwedge_{i \in I} \chi(C_i)$ is an irredundant meet.*

Now, we can use similar arguments as in Theorem 45 to obtain the proofs of $3. \Rightarrow 4. \Rightarrow 1. \Rightarrow 5. \Rightarrow 4.$ □

Corollary 47. *Let M be a τ-full R-module. If $[\tau, \tau \vee \xi(M)]$ is an atomic lattice, and every atom in this lattice can be written as $\tau \vee \xi(C)$ with C a τ-cocritical module, then* $\sum \{U \leq M \mid U \text{ is uniform}\} \underset{ess}{\leq} M$.

References

[1] Albu, T., "*F-semicocritical modules, F-primitive ideals and prime ideals*", Rev. Roumaine Math. Pueres Appl. 31, No. 6, 449-459, (1986).

[2] Arroyo, M. J. and Ríos J., "*Some aspects of spectral torsion theories*", Comm. Algebra 22(12), 4991-5003, (1994).

[3] Arroyo, M. J., Ríos J. and Wisbauer, R., "*Spectral torsion theories in module categories*", Comm. Algebra 25(7), 2249-2270, (1997).

[4] Boyle, Ann K., "*The large condition for rings with Krull dimension*", Proc. Amer. Math. Soc. 72, 27-32, (1978).

[5] Călugăreanu, G., *Lattice Concepts of Module Theory*, Kluwer Academic Publishers, USA, (2000).

[6] Castro, J., Raggi, F., Ríos J. and Van den Berg, J., "*On the atomic dimension in module categories*", Comm. Algebra 33, 4679-4692, (2005).

[7] Castro, J., Raggi, F. and Ríos J., "*Decisive dimension and other related torsion theoretic dimensions*", to appear in Journal of Pure and Applied Algebra.

[8] Golan, J., *Torsion Theories*, Longman Scientific & Technical, Harlow, (1986).

[9] Golan, J. and Simmons, H., *Derivatives, nuclei and dimensions on the frame of torsion theories*, Longman Scientific & Technical, Harlow, (1988).

[10] Grätzer, G., *General Lattice Theory*, Second edition, Birkhäuser Verlag, Berlin, (1998).

[11] Lambek, J., *Torsion Theories, Additive Semantics, and Rings of Quotients*, Lecture Notes in Mathematics #177, Springer-Verlag, Berlin, (1971).

[12] Lau, William G., *Torsion Theoretic Generalizations of Semisimple Modules*, PhD Thesis, University of Wisconsin-Milwaukee, (1980).

[13] Stenström, B. *Rings of Quotients*, Die Grundlehren der Math. Wiss. in Eizeld, Vol. 217, Springer-Verlag, Berlin, (1975).

[14] Teply, M. L., *Semicocritical modules*, Secretariado de publicaciones e intercambio científico, Universidad de Murcia, España, (1988).

[15] Vachuska, P., *Applications of the τ-full socle*, PhD Thesis, University of Wisconsin-Milwaukee, (1992).

[16] Wisbauer, R., "*Localization of Modules and the Central Closure of Rings*", Comm. in Algebra 9(14), 1455-1493, (1981).

[17] Wisbauer, R., *Modules and Algebras: Bimodule Structure and Group Actions on Algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics 81, (1996).

[18] Zelmanowitz, J. M., "*Representation of Rings with faithful polyform modules*", Comm. in Algebra 14(6), 1141-1169, (1986).