East-West J. of Mathematics: Vol. 9, No 2 (2007) pp. 113-123

ON FILTER COREGULAR SEQUENCES AND CO-FILTER MODULES

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Abstract

In this paper, we introduce the notion of filter coregular sequence to study a class of Artinian modules called co-filter modules.

1 Introduction

Throughout this paper, let (R, m) be Noetherian local ring and A an Artinian R-module. The notion of coregular sequence introduced by Ooshi [12] is an useful tool to study the structure of Artinian modules. Note that, for any Artinian R-module A, any coregular sequence of A in m can be extended to a maximal one and all the coregular sequences of A in m have the same length. This common length is called the *width* of A and denoted by $\operatorname{Width} A$. In general we have $\text{N-dim}\, A \geq \operatorname{Width} A$, where $\text{N-dim}\, A$ is the Noetherian dimension of A introduced by R.N. Roberts [13]. An Artinian R-module A is called *co-Cohen-Macaulay* if $\text{N-dim}\, A = \operatorname{Width} A$. The class of co-Cohen-Macaulay modules plays a center role in the category of Artinian modules which is some sense the same as that of the class of Cohen-Macaulay modules in the category of Noetherian modules. Co-Cohen-Macaulayness can be characterized by the equality between the length and the multiplicities with respect to a system of parameters (cf. [4]), by the coregularness of systems of paramaters (cf. [16]), and by the vanishing of local homology modules $H_r^m(A)$ (cf. [3]).

The purpose of this paper is to introduce the notion of filter coregular sequence for Artinian modules as a generalization of the notion of coregular

Key words: Filter coregular sequence, co-filter modules, Noetherian dimension.
(2000) AMS Mathematics Subject Classification: 13D45.

sequence, in order to study a class of Artinian modules called "co-filter modules" satisfying the condition that any system of parameters is a filter coregular sequence. In some sense, the notions of filter coregular sequences and co-filter modules for the category of Artinian modules are respectively dual to that of filter regular sequences and filter modules introduced by Cuong-Schenzel-Trung [6] for the category of Noetherian modules.

This paper is devided into 3 sections. In section 2, we introduce the notion of filter coregular sequence and give characterizations of filter coregular sequences (Theorem 2.3). Some characterizations of co-filter modules via systems of parameters (Propositon 3.2), reduced systems of parameters (Theorem 3.7) and generalized co-Cohen-Macaulay modules (Proposition 3.8) are shown.

2 Filter coregular sequence

We always assume that (R, m) is Noetherian local ring and A is an Artinian R-module. Before introducing the notion of filter coregular sequence for Artinian modules, we recall the notion of coregular sequence was defined by Ooishi [12]. A sequence $x_1, \ldots, x_r$ of elements in m is said to be *coregular sequence* of A (or A-coregular sequence) if

$$x_i(0 :_A (x_1, \ldots, x_{i-1})R) = 0 :_A (x_1, \ldots, x_{i-1})R,$$

for all $i = 1, \ldots, r$. An element $x \in m$ is called *A-coregular element* if $xA = A$. It should be noted that $0 :_A (x_1, \ldots, x_r)R) \neq 0$. Therefore the notion of coregular sequence is defined in some sense dual to the known notion of regular sequence.

Definition 2.1. A sequence of elements $x_1, \ldots, x_r$ in m is called a *filter coregular (or f-coregular)* sequence with respect to A if for all $i = 1, \ldots r$

$$x_i(0 :_A (x_1, \ldots, x_{i-1})R) \supseteq \bigcap_{n \geq 0} m^n(0 :_A (x_1, \ldots, x_{i-1})R).$$

An element $x \in m$ is called *an f-coregular element* if $xA \supseteq \bigcap_{n \geq 0} m^n A$.

Remark 2.2. It is clear that every coregular sequence is a filter coregular sequence. The converse is not true in general. For example, let k be a field, $S = k[x_1, \ldots, x_d]$ - the polynomial ring and $R = k[[x_1, \ldots, x_d]]$ - the ring of formal power series of d variables over k. Set $m = (x_1, \ldots, x_d)R$ - the unique maximal ideal of R. Let $B = k[x_1^{-1}, \ldots, x_d^{-1}]$ be the $S-$module of inverse polynomials. Then B is an Artinian $S-$module, (cf.[9]), and hence B has a natural structure as an Artinian $R-$module. Let $n > 1$ be an integer, set $C = R/(x_1^n, x_2, \ldots, x_d)R$ and $A = B \oplus C$. Note that B is co-Cohen-Macaulay and $\ell_R(C) < \infty$. Hence $m^n C = 0$ for $n \gg 0$. Let an element $0 \neq y \in m$. We can check that y is an f-coregular element, but it is not an A-coregular element.

The theory of secondary representation was introduced by Macdonald in [10] is in some sense dual to the more known theory of primary decomposition of Noetherian modules. Note that every Artinian $R-$module A has a secondary representation $A = B_1 + \ldots + B_n$ of p_i-secondary submodules B_i. The set $\{p_1, \ldots, p_n\}$ is independent of the minimal secondary representation of A and denoted by $\mathrm{Att}_R A$. Note that $A \neq 0$ if and only if $\mathrm{Att}_R A \neq \emptyset$. In this case, the minimal elements in $\mathrm{Att}_R A$ are exactly the minimal prime ideals containing $\mathrm{Ann}_R A$. Moreover, if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of Artinian R- modules, then $\mathrm{Att}\, A'' \subseteq \mathrm{Att}\, A \subseteq \mathrm{Att}\, A' \cup \mathrm{Att}\, A''$. Denote by $\widehat{R}$ the m-adic completion of R. Then A has a natural struture as an $\widehat{R}$-module and with this struture, each subset of A is an R-submodule of A if and only if it is an $\widehat{R}$-submodule of A. Therefore A is an Artinian $\widehat{R}$-module. Moreover, cf. [14],

$$\mathrm{Att}_R A = \{\widehat{p} \cap R |\ \widehat{p} \in \mathrm{Att}_{\widehat{R}} A\}.$$

The following theorem is the main result of this section which gives us some characterizations of f-coregular sequences.

Theorem 2.3. *Let $x_1, \ldots, x_r \in m$. The following conditions are equivalent:*

(i) $x_1, \ldots, x_r$ is f-coregular with respect to A considering as an R-module.

(ii) $x_i \notin p$ for all $p \in \mathrm{Att}_R(0 :_A (x_1, \ldots, x_{i-1})R) \setminus \{m\}$, for all $i = 1, \ldots, r$.

(iii) $\ell_R\big(0 :_A (x_1, \ldots, x_{i-1})R)/x_i(0 :_A (x_1, \ldots, x_{i-1})R\big) < \infty$.

(iv) $x_1, \ldots, x_r$ is an f-coregular sequence with respect to $\bigcap\limits_{n \geq 0} m^n A$.

(v) $x_1, \ldots, x_r$ is f-coregular with respect to A considering as an $\widehat{R}$-module.

Proof (i)⇔(ii) is obvious.

(i)⇔(iii). It is sufficient to prove for the case $r = 1$. Assume that x is an f-coregular element with respect to A. As (i)⇔(ii), we have $x \notin p$ for all $p \in \mathrm{Att}_R A \setminus \{m\}$. If $m \notin \mathrm{Att}_R A$, then x is a A-coregular element, i.e. $xA = A$ and hence $\ell_R(A/xA) = 0 < \infty$. If $m \in \mathrm{Att}_R A$, then $xA \supseteq m^n A$ for $n \gg 0$. Hence

$$\ell_R(A/xA) \leqslant \ell_R(A/m^n A) < \infty.$$

Convesely, assume that $\ell_R(A/xA) < \infty$. Then $m^t A \subseteq xA$ for some $t \in \mathbb{N}$. Hence x is an f-coregular element with respect to A.

(i)⇔(iv). Let $A = B_0 + B_1 + \ldots + B_k$ be a minimal secondary representation of A, where $B_0 = 0$ or $m-$secondary. Note that $m^t B_0 = 0$ for some $t \in \mathbb{N}$. Therefore

$$\bigcap_{n \geq 0} m^n A = B_1 + \ldots + B_k = m^t A.$$

It follows that

$$0 :_{B_1 + \ldots + B_k} (x_1^t, \ldots, x_{i-1}^t)R + B_0 = 0 :_A (x_1^t, \ldots, x_{i-1}^t)R.$$

It implies the two following equalities

$$x_i^t(0:_A(x_1^t,\dots,x_{i-1}^t)R)=x_i^t(0:_{B_1+\dots+B_k}(x_1^t,\dots,x_{i-1}^t)R),$$

$$\bigcap_{n\geq 0} m^n(0:_A(x_1^t,\dots,x_{i-1}^t)R)=\bigcap_{n\geq 0} m^n(0:_{B_1+\dots+B_k}(x_1^t,\dots,x_{i-1}^t)R).$$

Hence $x_1^t,\dots,x_r^t$ is an f-coregular sequence with respect to A if and only if $x_1^t,\dots,x_r^t$ is an f-coregular sequence with respect to $\bigcap_{n\geq 0} m^nA$, if and only if $x_1,\dots,x_r$ is an f-coregular sequence with respect to $\bigcap_{n\geq 0} m^nA$, if and only if $x_1,\dots,x_r$ is an f-coregular sequence with respect to A.

(i)$\Leftrightarrow$(v). It is sufficient to prove for the case $r=1$. By using the equivalence of (i) and (ii) with note that

$$\operatorname{Att}_R A=\{\widehat{p}\cap R|\ \widehat{p}\in\ \operatorname{Att}_{\widehat{R}} A\}$$

we get the result. □

R. N. Roberts [13] introduced the notion of Krull dimension for Artinian modules. Later Kirby [9] changed the terminology of Roberts and referred to Noetherian dimension to avoid any confusion with Krull dimension for finitely generated modules. The *Noetherian dimension* $\text{N-dim}_R A$ of A is defined as follows: if $A=0$ we put $\text{N-dim}\,A=-1$. Then by induction, for an integer $d\geq 0$, we put $\text{N-dim}_R A=d$ if $\text{N-dim}_R A<d$ is false and for every ascending sequence $A_0\subseteq A_1\subseteq\dots$ of submodules of A, there exists n_0 such that $\text{N-dim}_R(A_{n+1}/A_n)<d$ for all $n>n_0$.

Note that if $0\longrightarrow A'\longrightarrow A\longrightarrow A''\longrightarrow 0$ is an exact sequence of Artinian $R-$modules then

$$\text{N-dim}_R A=\max\{\text{N-dim}_R A',\ \text{N-dim}_R A''\}.$$

Moreover, $\text{N-dim}_R A=\text{N-dim}_{\widehat{R}} A$. Therefore, without any confusion, we can write $\text{N-dim}\,A$ replaced by $\text{N-dim}_R A$ or $\text{N-dim}_{\widehat{R}} A$. There are many nice properties of Noetherian dimension for Artinian modules which are in some sense dual to that of Krull dimension for finitely generated modules. For example, $\ell(0:_A m^n)$ is a polynomial for $n\gg 0$ and

$$\text{N-dim}_R A=\deg\ell(0:_A m^n)=\min\{t:\exists x_1,...,x_t\in m,\ell(0:_A(x_1,...,x_t)R)<\infty\},$$

(cf.[9], [13]). Let $\text{N-dim}\,A=d$. By the above fact, there exists a system $(x_1,\dots,x_d)$ of d elements in m such that $\ell(0:_A(x_1,\dots,x_d)R)<\infty$. Such a system is called a *system of parameters* of A. A system $(x_1,\dots,x_t)$, where $t\leqslant d$, is called a *part of a system of parameters* of A if there exist elements $x_{t+1},\dots,x_d$ such that $(x_1,\dots,x_d)$ is a system of parameters of A.

The following facts will be often used in the sequel.

Lemma 2.4. *([5]) The following statements are true.*

(i) N-dim $A = 0$ *if and only if* $A \neq 0$ *and* $\ell(A) < \infty$. *In this case* $\operatorname{Att}_R A = \{m\}$.

(ii) $\text{N-dim}\, A \leqslant \dim\big(R/\operatorname{Ann}_R A\big) = \max\{\dim R/p : \ p \in \operatorname{Att}_R A\}$ *and there exists an Artinian module A such that* $\text{N-dim}\, A < \dim\big(R/\operatorname{Ann}_R A\big)$.

(iii) $\text{N-dim}\, A = \dim\big(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A\big) = \max\{\dim \widehat{R}/\widehat{p} : \ \widehat{p} \in \operatorname{Att}_{\widehat{R}} A\}$.

It follows by Lemma 2.4, (i) and Theorem 2.3, (i)$\Leftrightarrow$(iii) that $x_1, \ldots, x_r$ is an f-coregular sequence with respect to A if and only if

$$\text{N-dim}\big(0 :_A (x_1, \ldots, x_{i-1})R)/x_i(0 :_A (x_1, \ldots, x_{i-1})R\big) \leqslant 0$$

for all $i = 1, \ldots, r$.

Here are some properties of f-coregular sequences.

Proposition 2.5. *let* $\text{N-dim}\, A = d$. *The following statements are true.*

(i) If $x_1, \ldots, x_r$ *is an f-coregular sequence with respect to A then there exists an element* $y \in m^n$ *such that* $x_1, \ldots, x_r, y$ *is an f-coregular sequence with respect to A. In particular, for any integer* $n > 0$, *there exists an f-coregular sequence with respect to A of length n.*

(ii) If $x_1, \ldots, x_r$ *is an f-coregular sequence with respect to A then*

$$\text{N-dim}_R(0 :_A (x_1, \ldots, x_r)R) = \sup\{\text{N-dim}\, A - r, 0\}.$$

Therefore, any f-coregular sequence of length at most d is a part of system of parameters of A.

Proof

(i) It is easily derived from the Prime Avoidance Theorem.

(ii) The case $r = 0$ is trivial. Let $r > 0$. If $\text{N-dim}\, A = 0$, then there is nothing to prove. Assume that $\text{N-dim}\, A = d > 0$. Since x_1 is f-coregular, we have by Theorem 2.3, (i) $\Leftrightarrow$ (ii) that $x_1 \notin p$ for all $p \in \operatorname{Att}_R A \setminus \{m\}$. Set $A' = 0 :_A x_1$. Then by induction and by [4, Theorem 2.6], we have

$$\begin{aligned}\text{N-dim}(0 :_A (x_1, \ldots, x_r)R) &= \text{N-dim}(0 :_{A'} (x_2, \ldots, x_r)R)\\ &= \sup\{\text{N-dim}\, A' - (r-1), 0\}\\ &= \sup\{\text{N-dim}\, A - r, 0\}.\end{aligned}$$

□

3 Co-filter modules

Definition 3.1. A is called *a co-filter module* if every system of parameters of A is an f-coregular sequence.

The following result is a characterization of co-filter module via system of parameters.

Proposition 3.2. *The following conditions are equivalent:*

(i) A is a co-filter module.

(ii) For any part of system of parameters $x_1, \ldots, x_r$ of A and any minimal secondary representation $0 :_A (x_1, \ldots, x_r)R = B_1 + \ldots + B_k$ of $0 :_A (x_1, \ldots, x_r)R$ with B_i p_i-secondary, $\text{N-dim}\, B_i = d - r$ *for all i satisfying $p_i \neq m$.*

Proof (i)$\Rightarrow$(ii) Let $x_1, \ldots, x_r$ be a part of system of parameters of A and set $B = 0 :_A (x_1, \ldots, x_r)R$. Assume that there exists $p_i \in \operatorname{Att}_R B \setminus \{m\}$ such that $\text{N-dim}\, B_i < d - r$. So we can choose an element $y \in p_i$ such that $x_1, \ldots, x_r, y$ is a part of system of parameters of A. By [4, Lemma 2.10, (i)], $x_1, \ldots, x_r, y^t$ is also a part of system of parameters of A for all $t \geq 1$, and therefore it is an f-coregular sequence of A by the hypothesis. Thus, for $n \gg 0$, $y^t B \supseteq m^n B$. So we get from the surjection $B/m^n B \longrightarrow B/y^t B$ that

$$\operatorname{Att}_R B/y^t B \subseteq \operatorname{Att}_R B/m^n B \subseteq \{m\}.$$

On the other hand, since $y \in p_i$, it follows that $y^t B_i = 0$ for $t \gg 0$. Therefore $p_i \notin \operatorname{Att}_R(y^t B)$. From the exact sequence

$$0 \longrightarrow y^t B \longrightarrow B \longrightarrow B/y^t B \longrightarrow 0$$

we have the inclusion $\operatorname{Att}_R B \subseteq \operatorname{Att}_R(y^t B) \cup \{m\}$. Therefore $p_i \notin \operatorname{Att}_R B$. This gives a contradiction.

(ii)$\Rightarrow$(i) follows easily from Theorem 2.3, (i)$\Leftrightarrow$(ii) and [16, Lemma 2.14]. □

Let M be a Noetherian R-module. It should be noted that if the m-adic completion $\widehat{M}$ of M is an f-module then M is an f-module. The converse is true when R is a quotient of a Cohen-Macaulay ring (cf. [15, Appendix, Lemma 8]). For Artinian modules, from Theorem 2.3, (i)$\Leftrightarrow$(v), we have immediately the following result.

Proposition 3.3. *If A is a co-filter $\widehat{R}$-module then A is a co-filter R-module.*

Note that in general the converse of the above result is not true. Here is an example.

Example 3.4. *There exists an Artinian module A over local ring (R, m) such that A is a co-filter R-module but A is not a co-filter $\widehat{R}$-module.*

Proof Let (R, m) be the Noetherian local domain of dimension 2 constructed by D. Ferrand and M. Raynaud [7] for which the m-adic completion $\widehat{R}$ of R has an associated prime ideal $\widehat{q}$ of dimension 1. Set $B = H^1_m(R)$, $C = H^2_m(R)$ and $A = B \oplus C$. We get by [5, Example 4.1] that B is a co-Cohen-Macaulay module of Noetherian dimension 1, $\mathrm{Att}_R\, B = \{0\}$. Moreover, we have by [5, Theorem 3.5] that $\text{N-dim}\, C = 2$ and by [1, Theorem 7.3.2] that $\mathrm{Att}_R\, C = \mathrm{Assh}\, R = \{0\}$. Then we have

(i) $A = B \oplus C$ is an Artinian R-module of Noetherian dimension 2, the Krull dimension $\dim_R A = \dim R/\operatorname{Ann} A = \dim R = 2$ and $\mathrm{Att}_R\, A = \{0\}$.

(ii) Let (x, y) be a system of parameters of A. Then x is an f-coregular element with respect to A since $\mathrm{Att}_R\, A = \{0\}$. Moreover, since $\text{N-dim}(0 :_A x) = 1$ and y is a parameter element of $(0 :_A x)$, we have by [16, Lemma 2.14] that $y \notin p$, for all $p \in \mathrm{Att}_R(0 :_A x)$ such that the secondary component with respect to p has Noetherian dimension 1. Therefore, y is an f-coregular element with respect to $(0 :_A x)$ and hence (x, y) is an f-coregular sequence with respect to A, it means that A is a co-filter R-module.

However, A is not a co-filter $\widehat{R}$-module. In fact, according to the hypothesis and [1, 11.3.3], the associated prime ideal $\widehat{q}$ of dimension 1 belongs to $\mathrm{Att}_{\widehat{R}}\, B$. Therefore,

$$\mathrm{Att}_{\widehat{R}}\, A = \mathrm{Att}_{\widehat{R}}(B \oplus C) = \mathrm{Att}_{\widehat{R}}\, B \cup \mathrm{Att}_{\widehat{R}}\, C \supseteq \{\widehat{q}\} \cup \mathrm{Att}_{\widehat{R}}\, C.$$

Note that $\mathrm{Att}_{\widehat{R}}\, C \neq \emptyset$ and $\dim \widehat{R}/\widehat{p} = 2$ for all $\widehat{p} \in \mathrm{Att}_{\widehat{R}}\, C$, while $\dim \widehat{R}/\widehat{q} = 1$ with $\widehat{q} \in \mathrm{Att}_{\widehat{R}}\, B$. So, A is not a co-filter $\widehat{R}$-module by Propositon 3.2. □

Definition 3.5. A system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of A is called *reducing* if $x_i \notin p$, for all $p \in \mathrm{Att}_R(0 :_A (x_1, \ldots, x_{i-1})R)$ such that the secondary component with respect to p has Noetherian dimension more or equal $d - i$, for all $i = 1, \ldots, d-1$.

Next we will show the technical result which will be useful to prove a relation between f-coregular sequences and reduced systems of parameters.

Lemma 3.6. *(i) Let $\underline{x} = (x_1, \ldots, x_d) \in m$ be a system of parameters of A. Then $\underline{x}$ is reducing w.r.t $R-$module A if and only if it is reducing w.r.t $\widehat{R}$-module A.*

(ii) Let $x \in m$ be an f-coregular element of A, $\widehat{q} \in \mathrm{Att}_{\widehat{R}}\, A \setminus \{m\}$ and $\widehat{p}$ a minimal prime of $\widehat{R}$ containing (q, x). Then $\widehat{p} \in \mathrm{Att}_{\widehat{R}}(0 :_A x)$.

(iii) Let $p \in \mathrm{Att}_R\, A$ and $x \in p$. Then $p \in \mathrm{Att}_R(0 :_A x^n)$ for $n \gg 0$.

Proof (i) For all $i = 1, \ldots, d-1$, we set $B = 0 :_A (x_1, \ldots, x_{i-1})R$ and let $B = B_1 + \ldots + B_k$ be a minimal secondary representation of R-module B, with B_j p_j-secondary, $j = 1, \ldots, k$. Then by the similarly aguments in [2, Lemma

3.2], we have $B = \sum_{j \leqslant k, u \leqslant n_j} C_{u,j}$ is a minimal secondary representation of $\widehat{R}$-module B with $C_{u,j}$ $\widehat{p}_{u,j}$-secondary.

Suppose that $\underline{x}$ is reducing of $R-$module A but it is not reducing of $\widehat{R}-$module A. Then there exists an integer $i \in \{1, \ldots, d-1\}$ and $\widehat{p}_{u,j} \in \mathrm{Att}_{\widehat{R}} B$, such that $\text{N-dim}\, C_{u,j} \geqslant d - i$ and $x_i \in \widehat{p}_{u,j}$. Then $x_i \in q = \widehat{p}_{u,j} \cap R \in \mathrm{Att}_R B$. Set $C = \sum_{\widehat{p}_{u,j} \cap R = q} C_{u,j}$. Then $\text{N-dim}\, C \geqslant d - i$. It gives a contradiction. Conversely, suppose that $\underline{x}$ is reducing of $\widehat{R}-$module A but it is not reducing of R-module A. Then there exists an integer $i \in \{1, \ldots, d-1\}$ and $p_j \in \mathrm{Att}_R B$ such that $\text{N-dim}\, B_j \geqslant d-i$ and $x_i \in p_j, j = 1, \ldots, k$. Therefore there is at least one of secondary component $C_{u,j}$ of $\widehat{R}$-module B with respect to all the attached primes $\widehat{p}_{u,j} \in \mathrm{Att}_{\widehat{R}} B$ sastisfying $\widehat{p}_{u,j} \cap R = p_j$. Since $\text{N-dim}\, C_{u,j} \geqslant d-i$, it gives a contradiction.

(ii) Note by Theorem 2.3, (i)⇔(v) that x is also an f-coregular element of $\widehat{R}$-module A. Let $A = A_0 + A_1 + \ldots + A_k$ be a minimal secondary representation of $\widehat{R}$-module A, where $A_0 = 0$ or $m\widehat{R}$-secondary, and A_i $\widehat{p}_i$-secondary, for all $i = 1, \ldots, k$. Let $A' = A_1 + \ldots + A_k$. Then x is a coregular element of A' and $\widehat{q}$ is an element in $\mathrm{Att}_{\widehat{R}} A'$. Therefore $\widehat{p} \in \mathrm{Att}_{\widehat{R}}(0 :_{A'} x)$ by [12, Lemma 3.18]. Let $0 :_{A'} x = B_0 + B_1 + \ldots + B_t$ be a minimal secondary representation of $\widehat{R}$-module $0 :_{A'} x$, where $B_0 = 0$ or $m\widehat{R}$-secondary, B_j $\widehat{q}_j$-secondary, for all $j = 1, \ldots, t$. Let $C = A_0 + B_0 + B_1 + \ldots + B_t$. It is easily seen that $\widehat{p} \in \mathrm{Att}_{\widehat{R}} C$. Note that for $n \gg 0$, we have $0 :_A x \subseteq 0 :_A x^n = C$. Therefore we get the exact sequence $0 \longrightarrow 0 :_A x \longrightarrow C \longrightarrow C/(0 :_A x) \longrightarrow 0$. Since $C/(0 :_A x)$ is zero or $m\widehat{R}$-secondary and $\widehat{p} \in \mathrm{Att}_{\widehat{R}} C$, we get $\widehat{p} \in \mathrm{Att}_{\widehat{R}}(0 :_A x)$.

(iii) Let $A = A_0 + A_1 + \ldots + A_k$ be a minimal secondary representation of R-module A, where $A_0 = 0$ or m-secondary, and A_i p_i-secondary, for all $i = 1, \ldots, k$. Without loss of generality, we may assume that A_1 is p-secondary. From the exact sequence

$$0 \longrightarrow 0 :_A x^n \longrightarrow A \longrightarrow A/0 :_A x^n \longrightarrow 0,$$

we get $p \in \mathrm{Att}_R(0 :_A x^n) \cup \mathrm{Att}_R(A/0 :_A x^n)$. Since $x \in p$, we have $0 :_A x^n \supseteq A_1$ for $n \gg 0$, it follows that

$$\begin{aligned}\mathrm{Att}_R(A/0 :_A x^n) &= \mathrm{Att}_R\left((A_2 + \ldots + A_k)/\left((0 :_A x^n) \cap (A_2 + \ldots + A_k)\right)\right)\\ &\subseteq \mathrm{Att}_R(A_2 + \ldots + A_k).\end{aligned}$$

Therefore $p \notin \mathrm{Att}_R(A/0 :_A x^n)$. It follows that $p \in \mathrm{Att}_R(0 :_A x^n)$. □

Remark. The conclusion (ii) in Lemma 3.6 is not true in general if we work on attached primes of $R-$module A. It means that there exists an Artinian R-module A, an element $x \in m$ which is an f-coregular element with respect to A, and $q \in \mathrm{Att}_R A \backslash m$ such that p is a minimal prime of (q, x) but $p \notin \mathrm{Att}_R(0 :_A x)$. Indeed, let (R, m) be the Noetherian local domain of dimension 2 constructed by D. Ferrand and M. Raynaud [7] for which the m-adic completion $\widehat{R}$ of R has an associated prime ideal $\widehat{q}$ of dimension 1, let $A = H^1_m(R)$ and let $0 \neq x \in m$. Then $\mathrm{Att}_R A = \{0\}$ and $\ell_R(0 :_A x) < \infty$ by [5, Example 4.3]. Take $q = 0$ and $0 \neq p$ be a minimal prime ideal of R containing x. Clearly we have $p \supset (q, x)$ but $p \notin \mathrm{Att}_R(0 :_A x) = \{m\}$.

Recall that a system $\underline{x} = (x_1, \ldots, x_t)$ of elements in m is called a *multiplicative system* of A if $\ell_R(0 :_A \underline{x}R) < \infty$. The *multiplicity* $e(\underline{x}; A)$ of A with respect to the multiplicative system $\underline{x}$ is defined by the obvious way in [4]. It has been shown in [4] many properties of the multiplicity for Artinian modules which are similar to that of multiplicity for Noetherian modules over local rings. For example, $0 \leqslant e(\underline{x}; A) \leqslant \ell(0 :_A \underline{x}R)$ and $e(x_1, \ldots, x_t; A) > 0$ if and only if $t = \text{N-dim}\, A$ (see [4, Corollary 4.5]). Especially, Lemma 5.4 in [4] gives us a result dual to that shown by Auslander-Buchsbaum: let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters of A, then

$$\ell(0 :_A \underline{x}R) - e(\underline{x}; A) = \sum_{i=1}^{d} e(x_{i+1}, \ldots, x_d; C_i/x_iC_i),$$

where $C_i = 0 :_A (x_1, \ldots, x_{i-1})R$, for $i = 1, \ldots, d$.

The following theorem is a relation between co-filter module and reduced system of parameters of A.

Theorem 3.7. *For an Artinian R-module A with* $\text{N-dim}\, A = d \geq 1$, *the following conditions are equivalent:*

(i) A is a co-filter module.

(ii) Each system of parameters of A is reducing.

(iii) For each system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of A, we have

$$\ell_R(0 :_A \underline{x}R) - e(\underline{x}; A) = \ell_R(0 :_A (x_1, \ldots, x_{d-1})R / x_d(0 :_A (x_1, \ldots, x_{d-1})R).$$

Proof (i)⇒(ii) It follows easily by Proposition 3.2, (i)⇒(ii) and Theorem 2.3, (i)⇔(ii).

(ii)⇔(iii) For a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of A, for $i = 1, \ldots, d$, we set $B_i = 0 :_A (x_1, \ldots, x_{i-1})R$. By Lemma 3.6, (i), we can assume that $R = \widehat{R}$. It follows by [4, Corollary 4.5] that $e(x_{i+1}, \ldots, x_d; B_i/x_iB_i) = 0$ if and only if $\text{N-dim}(B_i/x_iB_i) < d - i$, for all $i = 1, \ldots, d-1$. Note that $\underline{x}$ is reducing if and only if $\text{N-dim}_R\, B_i/x_iB_i < d - i$, for all $i = 1, \ldots, d-1$. Now the result follows by [4, Lemma 5.4].

(ii)⇒(i) It is clear for the case $d \leqslant 2$. Assume that $d \geqslant 3$ and the claim true for every Artinian module with Noetherian dimesion less than d. By Theorem 2.3, (i)⇔(v) and Lemma 3.6, (i) we can assume that $R = \widehat{R}$. Then the condition (ii) remains valid for $0 :_A x_1$. Therefore we only have to prove that $x_1 \notin p$ for all $p \in \operatorname{Att} A \setminus \{m\}$. Assume that $x_1 \in p$ for some $p \in \operatorname{Att}_R A \setminus \{m\}$. Then the secondary component B of A with respect to p has Noetherian dimension strictly less than $d-1$ by assumption (ii). By Lemma 3.6, (iii), there exists n_0 such that $p \in \operatorname{Att}(0 :_A x_1^n)$, for all $n \geqslant n_0$. Let $A' = 0 :_A x_1^n$. Since each system of parameters of A' is an f-coregular sequence, the secondary component C of A' with respect to p has Noetherian dimension $d-1$ by Proposition 3.2. Since $R = \widehat{R}$, it follows by Lemma 2.4, (iii) that

$$d - 1 > \text{N-dim}\, B = \dim B = \dim R/p = \dim C = \text{N-dim}\, C = d - 1.$$

This gives a contradiction. □

The class of generalized co-Cohen-Macaulay modules was introduced in [2]. Recall that an Artinian R-module A is called *generalized co-Cohen-Macaulay* if $I(A) < \infty$, where we set $I(\underline{x}; A) = \ell_R(0 :_A \underline{x}R) - e(\underline{x}; A)$ and $I(A) = \sup_{\underline{x}} I(\underline{x}; A)$, for every system of parameters $\underline{x}$ of A in m. The structure of these modules is known by the properties of co-standard system of parameters, multiplicity, local homology modules (cf. [2]). Especially, let q be an m-primary ideal of R. A sequence $(x_1, \ldots, x_r)$ of elements in m is called a *q-weak co-sequence of A* if

$$x_i(0 :_A (x_1, \ldots, x_{i-1})R) \supseteq q(0 :_A (x_1, \ldots, x_{i-1})R) \text{ for all } i = 1, \ldots, r,$$

where we mean $x_1 A \supseteq qA$ when $i = 1$. Clearly, a q-weak co-sequence is alway f-coregular sequence.

Proposition 3.8. *If A is a generalized co-Cohen-Macaulay module then A is a co-filter module.*

Proof It follows easily by the characterization of generalized co-Cohen-Macaulay module via q-weak co-sequence in [2, Theorem 4.4] and above comment. □

Note that the converse of Proposition 3.8 is not true in general. Below we give a counterexample for this.

Example 3.9. *There exists an Artinian module A over local ring (R, m) such that A is a co-filter R-module but A is not a generalized co-Cohen-Macaulay module.*

Proof We consider the co-filter R-module $A = H^1_m(R) \oplus H^2_m(R)$ as in Example 3.4. Suppose that A is a generalized co-Cohen-Macaulay R-module, then we have by [2, Corollary 4.9, (iii)] that A is generalized co-Cohen-Macaulay $\widehat{R}$-module. Denote by $D(A) = \operatorname{Hom}_R(A, E)$ the Matlis dual of A, where E is the injective envelope of the residue field R/m. Then $D(A)$ is a generalized Cohen-Macaulay $\widehat{R}$-module, and hence it is a filter $\widehat{R}$-module by [15, Proposition 16, (i)⇔(ii)]. Since $\mathrm{Ass}_{\widehat{R}}(D(A)) = \mathrm{Att}_{\widehat{R}}\, A$, we have A is also a co-filter $\widehat{R}$-module, a contradiction (cf. Example 3.4). □

Acknowledgment. The author would like to thank Prof. Le Thanh Nhan for her useful suggestions.

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