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UNIQUE RANGE SETS FOR p-ADIC MEROMORPHIC FUNCTIONS IN SEVERAL VARIABLES

Tran Dinh Duc

Institute of Mathematics,
P.O. Box 631 Bo Ho, 10000 Hanoi, Vietnam

Abstract

In this paper we give some unique range sets for p-adic meromorphic functions sharing four values in several variables.

1. Introduction.

In 1926, Nevanlinna proved that two nonconstant meromorphic functions of one complex variable which attain same five distinct values at the same points, must be identical.

It is observed that p-adic entire functions of one variable behave in many ways more like polynomials than like entire functions of one complex variable. In 1971, Adams and Straus [1] proved the following theorem.

Theorem A. *Let f, g be two nonconstant p-adic entire functions such that for two distinct (finite) values a, b we have $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. Then $f \equiv g$.*

For p-adic meromorphic functions, Adams and Straus [1] obtained the following result similar to Nevanlinna's.

Theorem B. *Let f, g be two nonconstant p-adic meromorphic functions such that for four distinct values a_1, a_2, a_3, a_4 we have $f(x) = a_i \Leftrightarrow g(x) = a_i, i = 1, 2, 3, 4$. Then $f \equiv g$.*

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Ru [9] and Hu and Yang [4] extended Theorem B to p-adic holomorphic curves.

The main tool in the cited above papers is the Nevanlinna theory in one variable for the non-Archimedean case. The aim of this paper is to extend Theorem B to the case of p-adic meromorphic functions in several variables .

In this paper by using the p-adic Nevanlinna theory in high dimension, developed in [2], [3], [5], [7],[8] , we givesome range sets for p-adic meromorphic functions in several variables .

2. Height of $p-$adic holomorphic functions of several variables

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p. We use the notations $b_{(m)} = (b_1, ..., b_m)$, $b_i(b) = (b_1, ..., b_{i-1}, b, b_{i+1}, ..., b_m)$, $b_{(m,i_s)} = b_i(b_{i_s})$, $\widehat{(b_i)} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_m)$, $D_r = \{z \in \mathbb{C}_p : |z| \leqslant r, r > 0\}$, $D_{<r>} = \{z \in \mathbb{C}_p : |z| = r, r > 0\}$, $D_{r_{(m)}} = D_{r_1} \times \cdots \times D_{r_m}$, where $r_{(m)} = (r_1, \ldots, r_m)$ for $r_i \in \mathbb{R}^*_+$, $D_{<r_{(m)}>} = D_{<r_1>} \times \cdots \times D_{<r_m>}$, $|\gamma| = \gamma_1 + \cdots + \gamma_m$, $z^\gamma = z_1^{\gamma_1}...z_m^{\gamma_m}$, $r^\gamma = r_1^{\gamma_1}...r_m^{\gamma_m}$, $\gamma = (\gamma_1, ..., \gamma_m)$, where $\gamma_i \in \mathbb{N}$, $| \, . \, | = | \, . \, |_p$, $\log = \log_p$. Notice that the set of $(r_1, ..., r_m) \in \mathbb{R}_+^{*m}$ such that there exist $x_1, ..., x_m \in \mathbb{C}_p$ with $|x_i| = r_i, i = 1, ..., m$, is dense in $\mathbb{R}_+^{*m}$. Therefore, without loss of generality one may assume that $D_{<r_{(m)}>} \neq \emptyset$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ and

$$f = \sum_{|\gamma| \geq 0} a_\gamma z^\gamma, \quad |z_i| \leqslant r_i \text{ for } i = 1, \ldots, m.$$

Then we have

$$\lim_{|\gamma| \to \infty} |a_\gamma| r^\gamma = 0.$$

Hence, there exists an $(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m$ such that $|a_\gamma| r^\gamma$ is maximum.

Define

$$|f|_{r_{(m)}} = \max_{0 \leqslant |\gamma| < \infty} |a_\gamma| r^\gamma.$$

Lemma 2.1 ([7]) *For each $i = 1, \ldots, m$, let $r_{i_1}, \ldots, r_{i_q}$ be positive real numbers such that $r_{i_1} \geq \cdots \geq r_{i_q}$. Let $f_s(z_{(m)}), s = 1, 2, \ldots, q$, be q non-zero holomorphic functions on $D_{r_{(m,i_s)}}$. Then there exist $u_{(m,i_s)} \in D_{r_{(m,i_s)}}$ such that*

$$|f_s(u_{(m,i_s)})| = |f_s|_{r_{(m,i_s)}}, s = 1, 2, \ldots, q.$$

Definition 2.2. *The height* of the function $f(z_{(m)})$ is defined by

$$H_f(r_{(m)}) = \log |f|_{r_{(m)}}.$$

If $f(z_{(m)}) \equiv 0$, then set $H_f(r_{(m)}) = -\infty$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ and

$$f = \sum_{|\gamma|\geq 0} a_\gamma z^\gamma, \quad |z_i| \leqslant r_i \text{ for } i = 1, \ldots, m.$$

Write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}\widehat{(z_i)} z_i^k, \quad i = 1, 2, \ldots, m.$$

Set

$$\begin{aligned} I_f(r_{(m)}) &= \Big\{(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m : |a_\gamma| r^\gamma = |f|_{r_{(m)}}\Big\}, \\ n_{1i,f}(r_{(m)}) &= \max\Big\{\gamma_i : \exists\ (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(r_{(m)})\Big\}, \\ n_{2i,f}(r_{(m)}) &= \min\Big\{\gamma_i : \exists\ (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(r_{(m)})\Big\}, \\ n_{i,f}(0,0) &= \min\Big\{k : f_{i,k}\widehat{(z_i)} \not\equiv 0\Big\}, \\ \nu_f(r_{(m)}) &= \sum_{i=1}^{m} \big(n_{1i,f}(r_{(m)}) - n_{2i,f}(r_{(m)})\big). \end{aligned}$$

Call $r_{(m)}$ a *critical point* if $\nu_f(r_{(m)}) \neq 0$.

For a fixed $i \quad (i = 1, \ldots, m)$ we set for simplicity

$$n_{i,f}(0,0) = \ell, k_1 = n_{1i,f}(r_{(m)}),\ k_2 = n_{2i,f}(r_{(m)}).$$

Then there exist multi-indices $\gamma = (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(r_{(m)})$ and $\mu = (\mu_1, \ldots, \mu_i, \ldots, \mu_m) \in I_f(r_{(m)})$ such that $\gamma_i = k_1$, $\mu_i = k_2$.

We consider the following holomorphic functions on $D_{r_{(m)}}$

$$f_\ell(z_{(m)}) = f_{i,\ell}\widehat{(z_i)} z_i^\ell, f_{k_1}(z_{(m)}) = f_{i,k_1}\widehat{(z_i)} z_i^{k_1}, f_{k_2}(z_{(m)}) = f_{i,k_2}\widehat{(z_i)} z_i^{k_2}.$$

The functions are not identically zero.

Set

$$\begin{aligned} U_{if,r_{(m)}} = \{u = u_{(m)} \in D_{r_{(m)}} :& |f_\ell(u)| = |f_\ell|_{r_{(m)}}, |f(u)| = |f|_{r_{(m)}}, \\ & |f_{k_1}(u)| = |f_{k_1}|_{r_{(m)}}, |f_{k_2}(u)| = |f_{k_2}|_{r_{(m)}}\}, \end{aligned}$$

where $i = 1, \ldots, m$. By Lemma 2.1, $U_{if,r_{(m)}}$ is a non-empty set. For each $u \in U_{if,r_{(m)}}$, set

$$f_{i,u}(z) = f(u_1, \ldots, u_{i-1}, z, u_{i+1}, \ldots, u_m),\ z \in D_{r_i}.$$

Theorem 2.3. *Let $f(z_{(m)})$ be a holomorphic function on $D_{r_{(m)}}$. Assume that $f(z_{(m)})$ is not identically zero. Then for each $i = 1, \ldots, m$, and for all $u \in U_{if,r_{(m)}}$, we have*

1) $H_f(r_{(m)}) = H_{f_{i,u}}(r_i)$,

2) $n_{1i,f}(r_{(m)})$ is equal to the number of zeros of $f_{i,u}$ in D_{r_i},

3) $n_{1i,f}(r_{(m)}) - n_{2i,f}(r_{(m)})$ is equal to the number of zeros of $f_{i,u}$ on $D_{<r_i>}$.

For the proof, see [7, Theorem 3.1].

From Theorem 2.3 we see that $f(z_{(m)})$ has zeros on $D_{<r_{(m)}>}$ if and only if $r_{(m)}$ is a critical point.

For a an element of $\mathbb{C}_p$ and f a holomorphic function on $D_{r_{(m)}}$, which is not identically equal to a, define

$$n_{i,f}(a, r_{(m)}) = n_{1i,f-a}(r_{(m)}), \quad i = 1, \ldots, m.$$

Fix real numbers $\rho_1, \ldots, \rho_m$ with $0 < \rho_i \leqslant r_i,\ i = 1, \ldots, m$.

For each $x \in \mathbb{R}$, set $A_i(x) = (\rho_1, \ldots, \rho_{i-1}, x, r_{i+1}, \ldots, r_m),\ i = 1, \ldots, m$, $B_i(x) = (\rho_1, \ldots, \rho_{i-1}, x, \rho_{i+1}, \ldots, \rho_m), i = 1, \ldots, m$.

Define *the counting function* $N_f(a, r_{(m)})$ by

$$N_f(a, r_{(m)}) = \frac{1}{\ln p} \sum_{i=1}^{m} \int_{\rho_i}^{r_i} \frac{n_{i,f}(a, A_i(x))}{x} dx.$$

If a=0, then set $N_f(r_{(m)}) = N_f(0, r_{(m)})$.

Then

$$N_f(a, B_i(r_i)) = \frac{1}{\ln p} \int_{\rho_i}^{r_i} \frac{n_{i,f}(a, B_i(x))}{x} dx.$$

For each $i = 1, 2, ..., m$, set

$$\begin{aligned} k_{1,i} &= n_{1i,f}(A_i(r_i)), k_{2,i} = n_{2i,f}(A_i(r_i)), \\ U^i_{if,A_i(r_i)} &= \{u^i = u^i_{(m)} \in D_{A_i(r_i)} : |f_\ell(u^i)| = |f_\ell|_{A_i(r_i)}, \\ &\quad |f(u^i)| = |f|_{A_i(r_i)}, |f_{k_{1,i}}(u^i)| = |f_{k_{1,i}}|_{A_i(r_i)}, \\ &\quad |f_{k_{2,i}}(u^i)| = |f_{k_{2,i}}|_{A_i(r_i)}\}, \\ \Gamma_i &= \{A_i(x) : A_i(x) \text{ is a critical point}, 0 < x \leqslant r_i\}. \end{aligned}$$

By Lemma 2.1 and Theorem 2.3, Γ_i is a finite set. Suppose that Γ_i, $i = 1, \dots, m$, contains n elements $A_i(x^j)$, $j = 1, \dots, n$. From this and Lemma 2.1 it follows that

$$\mathcal{U}^i_{if,A_i(r_i)} = \{u^i = u^i_{(m)} \in U^i_{if,A_i(r_i)} : \exists u^i_i(u^j) \in U^i_{if,A_i(x^j)},\ j = 1, \dots, n\} \neq \emptyset,$$

$i = 1, \dots, m$.

Lemma 2.4.

1) Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then for each $i = 1, 2, ..., m$, and for all $u^i \in \mathcal{U}^i_{if,A_i(r_i)}$, we have

$$n_{f_{i,u^i}}(x) = n_{i,f} \circ A_i(x), \rho_i \leqslant x \leqslant r_i,$$

2) Let $f_s(z_{(m)}), s = 1, 2, \dots, q$, be q non-zero holomorphic functions on $D_{r_{(m)}}$. Then for each i = 1,2,..., m, there exists $u^i \in \mathcal{U}^i_{if_s,A_i(r_i)}$ for all $s = 1, \dots, q$.

The result can be proved easily by using Lemma 2.1 and Theorem 2.3.

Theorem 2.5. *Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$H_f(r_{(m)}) - H_f(\rho_{(m)}) = N_f(r_{(m)}).$$

The proof of Theorem 2.5 follows immediately from [7, Theorem 3.2].

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$, $a = (a_1, \dots, a_m) \in D_{r_{(m)}}$, and

$$f = \sum_{|\gamma|=0}^{\infty} a_\gamma (z_1 - a_1)^{\gamma_1} \dots (z_m - a_m)^{\gamma_m}, \quad z_{(m)} \in D_{r_{(m)}}.$$

Set

$$v_f(a) = \min\ \{|\gamma| : a_\gamma \neq 0\}.$$

For each $i = 1, 2, \dots, m$, write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{z_i - a_i})(z_i - a_i)^k.$$

Set

$$g_{i,k}(z_1, ..., z_{i-1}, z_{i+1}, ..., z_m) = f_{i,k}(\widehat{z_i - a_i}),$$
$$b_{i,k} = g_{i,k}(a_1, ..., a_{i-1}, a_{i+1}, ..., a_m).$$

Then

$$f_{i,a}(z) = \sum_{k=0}^{\infty} b_{i,k}(z_i - a_i)^k.$$

Set

$$v_{i,f}(a)=\begin{cases}\min\ \{k: b_{i,k}\neq 0\} & \text{if } f_{i,a}(z)\not\equiv 0\\ +\infty & \text{if } f_{i,a}(z)\equiv 0,\end{cases}$$

$$ord_{i,f}(a)=\begin{cases}\min\ \{k: g_{i,k}\widehat{(z_i)}\not\equiv 0\}\\ +\infty \quad \text{if } g_{i,k}\widehat{(z_i)}\equiv 0 \ for\ all\ k.\end{cases}$$

If $f(a)=0$, then a (resp.,a_i) is a zero of $f(z_{(m)})$ (resp., $f_{i,a}(z)$). Then the numbers $v_f(a), v_{i,f}(a), ord_{i,f}(a)$ are called *multiplicity, $i-th$ partial multiplicity, $i-th$ partial order,* respectively, of a.
Set

$$v=(u^1,\dots,u^m), u^i\in \mathfrak{U}^i_{if,A_i(r_i)},$$
$$N_{f_v}(r_{(m)})=N_{f_{1,u^1}}(r_1)+\dots+N_{f_{m,u^m}}(r_m),$$
$$V=\{v: N_{f_v}(r_{(m)})=N_f(r_{(m)})\},$$

where $\overline{n}_{f_{i,u^i}}(r_i)$ be the number of distinct zeros of f_{i,u^i}. By Lemma 2.4 and [4], V is a non-empty set,

$$N_{f_v}(r_{(m)})=\sum\nolimits_{\substack{1\\ \rho_1<|a|\leqslant r_1}}(v(a)+\log r_1)+n_{f_1,u^1}(0,\rho_1)(\log r_1-\log\rho_1)$$
$$+\dots+\sum\nolimits_{\substack{m\\ \rho_m<|a|\leqslant r_m}}(v(a)+\log r_m)+n_{f_m,u^m}(0,\rho_m)(\log r_m-\log\rho_m), \qquad (2.1)$$

where

$$\sum\nolimits_{\substack{i\\ \rho_i<|a|\leqslant r_i}}(v(a)+\log r_i)$$

is taken on all of zeros a of f_{i,u^i} (counting multiplicity) with $\rho_i<|a|\leqslant r_i$, $i=1,2,...,m$. Notice that, the sums in (2.1) are finite sums.
Denote by $\overline{N}_{f_v}(r_{(m)})$ the sum (2.1), where every zero a of the functions f_{i,u^i}, $i=1,\dots,m$, is counted ignoring multiplicity. Set

$$\overline{N}_f(r_{(m)})=\max_{v\in\ V}\overline{N}_{f_v}(r_{(m)}).$$

From Lemma 2.4 it follows that one can find $u^i\in\ \mathfrak{U}^i_{if,A_i(r_i)}$ and $v=(u^1,\dots,u^m)$ such that $\overline{N}_f(r_{(m)})=\overline{N}_{f_v}(r_{(m)})$.

If γ is a multi-index and f is a meromorphic function of m variables, then we denote by ∂^γ_f the partial derivative

$$\frac{\partial^{|\gamma|}f}{\partial z_1^{\gamma_1}\dots\partial z_m^{\gamma_m}}.$$

Theorem 2.6. *Let f be a non-zero entire function on $\mathbb{C}_p^m$ and γ a multi-index with $|\gamma| > 0$. Then*

$$H_{\partial^\gamma f}(B_e(r_e)) - H_f(B_e(r_e)) \leqslant -|\gamma| \log r_e + O(1).$$

The proof of Theorem 2.6 follows immediately from [3, Lemma 4.1].

3.Unique range sets for p-adic meromorphic functions in several variables

Let $f = \frac{f_1}{f_2}$ be a meromorphic function on $D_{r_{(m)}}$ (resp., $\mathbb{C}_p^m$), where f_1, f_2 be two holomorphic functions on $D_{r_{(m)}}$ (resp., $\mathbb{C}_p^m$), have no common zeros, and $a \in \mathbb{C}_p$.

We set

$$H_f(r_{(m)}) = \max_{1 \leqslant i \leqslant 2} H_{f_i}(r_{(m)}),$$

and

$$N_f(a, r_{(m)}) = N_{f_1 - af_2}(r_{(m)}).$$

For a point $d \in \mathbb{C}_p$ we define the function $v_f^d : \mathbb{C}_p^m \to (\mathbb{N} \cup \{+\infty\})^m$ by $v_f^d(a_{(m)}) = v^0_{f_1 - df_2}(a_{(m)})$ and write $v_f^d(a_{(m)}) = (v^d_{1,f}(a_{(m)}), \ldots, v^d_{m,f}(a_{(m)}))$, and $v_f^\infty(a_{(m)}) = v^0_{f_2}(a_{(m)})$ and write $v_f^\infty(a_{(m)}) = (v^\infty_{1,f}(a_{(m)}), \ldots, v^\infty_{m,f}(a_{(m)}))$.

For a subset S of $\mathbb{C}_p$ we set

$$E_i(f, S) = \bigcup_{d \in S} \Big\{(q_i, a_{(m)}) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{C}_p^m | f(a_{(m)}) - d = 0,\ v^d_{i,f}(a_{(m)}) = q_i\Big\},$$

$$E_i(f, S \cup \{\infty\}) = E_i(f, S) \bigcup \{(q_i, a_{(m)}) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{C}_p^m | v^\infty_{i,f}(a_{(m)}) = q_i\Big\},$$

$i = 1, 2..., m$.

$$\overline{E}_f(a) = \big\{z \in \mathbb{C}_p : f_1 - af_2 = 0 \ \text{ ignoring multiplicities}\big\},$$
$$\overline{E}_f(\infty) = \big\{z \in \mathbb{C}_p : f_2 = 0 \ \text{ ignoring multiplicities}\big\}.$$

Lemma 3.1. *Let $f = \frac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p^m$. Then there exists a multi-index $\gamma_1 = (0, \ldots, 0, \gamma_{1e}, 0, \ldots, 0)$ such that $\gamma_{1e} = 1$*

and $\partial_f^{\gamma_1} = \frac{\partial_{f_1}^{\gamma_1}.f_2 - \partial_{f_2}^{\gamma_1}.f_1}{f_2^2}$ *and the Wronskians*

$$W(f) = W(f_1, f_2) = det\begin{pmatrix} f_1 & f_2 \\ \partial_{f_1}^{\gamma_1} & \partial_{f_2}^{\gamma_1} \end{pmatrix}$$

are not identically zero.

For the proof, see [3, Lemma 4.2].

Theorem 3.2. *Let* f *be a non-constant meromorphic function on* $\mathbb{C}_p^m$ *and* $a_i \in \mathbb{C}_p, i = 1, \ldots, q$. *Then*

$$(q-1)H_f(B_e(r_e)) \leqslant \sum_{j=1}^{q} \overline{N}_f(a_j, B_e(r_e)) + \overline{N}_f(\infty, B_e(r_e)) - \log r_e + O(1).$$

Proof Set $G = \{G_{\beta_1} \ldots G_{\beta_{q-1}}\}$, where $(\beta_1, \ldots, \beta_{q-1})$ is taken on all different choices of $q-1$ numbers in the set $\{1, \ldots, q+1\}$, and $G_i = f_1 - a_i f_2,\ i = 1, \ldots, q$, and $G_{q+1} = f_2$. Set $H_G(B_e(r_e)) = \max_{(\beta_1 \ldots \beta_{q-1})} H_{G_{\beta_1} \ldots \beta_{q-1}}(B_e(r_e))$. We need the following

Lemma 3.3. *We have* $H_G(B_e(r_e)) \geq (q-1)H_f(B_e(r_e)) + O(1)$, *where* $O(1)$ *does not depend on* r_e.

Proof We have

$$\begin{aligned} H_G(B_e(r_e)) &= \max_{(\beta_1, \ldots, \beta_{q-1})} H_{G_{\beta_1} \ldots G_{\beta_{q-1}}}(B_e(r_e)) \\ &= \max_{(\beta_1, \ldots, \beta_{q-1})} \sum_{1 \leqslant i \leqslant q-1} H_{G_{\beta_i}}(B_e(r_e)). \end{aligned}$$

Assume that for a fixed r_e, the following inequalities hold

$$H_{G_{\beta_1}}(B_e(r_e)) \geq H_{G_{\beta_2}}(B_e(r_e)) \geq \ldots \geq H_{G_{\beta_{q+1}}}(B_e(r_e)).$$

Then

$$H_G(B_e(r_e)) = H_{G_{\beta_1}}(B_e(r_e)) + H_{G_{\beta_2}}(B_e(r_e)) + \cdots + H_{G_{\beta_{q-1}}}(B_e(r_e)). \quad (3.1)$$

Since $a_1, \ldots, a_q$ are distinct numbers in $\mathbb{C}_p$, then

$$f_i = b_{i_0} G_{\beta_q} + b_{i_1} G_{\beta_{q+1}},\ i = 1, 2,$$

where b_{i_0}, b_{i_1} are constants, which do not depend on r_e. It follows that

$$H_{f_i}(B_e(r_e)) \leqslant \max_{0 \leqslant j \leqslant 1} H_{G_{\beta_{q+j}}}(B_e(r_e)) + O(1).$$

Therefore, we obtain

$$H_{f_i}(B_e(r_e)) \leqslant H_{G_{\beta_j}}(B_e(r_e)) + O(1),$$

for $j = 1, \dots, q-1$ and $i = 1, 2$. Hence,

$$H_f(B_e(r_e)) = \max_{1 \leqslant i \leqslant 2} H_{f_i}(B_e(r_e)) \leqslant H_{G_{\beta_j}}(B_e(r_e)) + O(1), \tag{3.2}$$

for $j = 1, \dots, q-1$. Summarizing $(q-1)$ inequalities (3.2) and by (3.1), we have

$$H_G(B_e(r_e)) \geq (q-1)H_f(B_e(r_e)) + O(1).$$

Now we prove Theorem 3.2. Denote by $W(g_1, g_2)$ the Wronskian of two entire functions g_1, g_2 with respect to the γ_1 as in 3.1.
Since f is non-constant , we have $W(f_1, f_2) \not\equiv 0$. Let (α_1, α_2) be distinct two numbers in $\{1, \dots, q+1\}$, and $(\beta_1, \dots, \beta_{q-1})$ be the rest. Note that the functions f_i can be represented as linear combinations of $G_{\alpha_1}, G_{\alpha_2}$. Then we have

$$W(G_{\alpha_1}, G_{\alpha_2}) = c_{(\alpha_1, \alpha_2)} W(f_1, f_2),$$

where $c_{(\alpha_1, \alpha_2)} = c$ is a constant, depending only on (α_1, α_2). We denote

$$\begin{aligned} A = A(\alpha_1, \alpha_2) &= \frac{W(G_{\alpha_1}, G_{\alpha_2})}{G_{\alpha_1} G_{\alpha_2}} \\ &= \det \begin{pmatrix} 1 & 1 \\ \frac{\partial^{\gamma_1}_{G_{\alpha_1}}}{G_{\alpha_1}} & \frac{\partial^{\gamma_1}_{G_{\alpha_2}}}{G_{\alpha_2}} \end{pmatrix}. \end{aligned}$$

Hence

$$\frac{G_1 \dots G_{q+1}}{W(f_1, f_2)} = \frac{c G_{\beta_1} \dots G_{\beta_{q-1}}}{A}. \tag{3.3}$$

Set $L_i = \frac{\partial^{\gamma_1}_{G_{\alpha_i}}}{G_{\alpha_i}}$, $i = 1, 2$. Then

$$\log |A|_{B_e(r_e)} \leqslant \max_{1 \leqslant i \leqslant 2} \log |L_i|_{B_e(r_e)}.$$

By Theorem 2.6

$$\log |L_i|_{B_e(r_e)} \leqslant -|\gamma_1| \log r_e + O(1).$$

Because $|\gamma_1| = 1$

$$\log |L_i|_{B_e(r_e)} \leqslant -\log r_e + O(1). \tag{3.4}$$

By (3.3), we obtain

$$\sum_{i=1}^{q+1} H_{G_i}(B_e(r_e)) - H_W(B_e(r_e))$$
$$= H_{G_{\beta_1}\ldots G_{\beta_{q-1}}}(B_e(r_e)) - \log|A|_{B_e(r_e)} + O(1).$$

From this and (3.4), we have

$$\begin{aligned} H_G(B_e(r_e)) &= \max_{(\beta_1,\ldots,\beta_{q-1})} H_{G_{\beta_1}\ldots G_{\beta_{q-1}}}(B_e(r_e)) \\ &\leqslant \sum_{i=1}^{q+1} H_{G_i}(B_e(r_e)) - H_W(B_e(r_e)) - \log r_e + O(1). \end{aligned}$$

By Lemma 3.3

$$(q-1)H_f(B_e(r_e)) \leqslant \sum_{i=1}^{q+1} H_{G_i}(B_e(r_e)) - H_W(B_e(r_e)) - \log r_e + O(1).$$

Thus

$$(q-1)H_f(B_e(r_e)) + H_W(B_e(r_e)) \leqslant \sum_{j=1}^{q+1} H_{G_j}(B_e(r_e)) - \log r_e + O(1). \quad (3.5)$$

By Theorem 2.5

$$H_W(B_e(r_e)) = N_W(B_e(r_e)) + O(1),$$
$$H_{G_j}(B_e(r_e)) = N_{G_j}(B_e(r_e)) + O(1).$$

Therefore and (3.5) we obtain

$$(q-1)H_f(B_e(r_e)) + N_W(B_e(r_e)) \leqslant \sum_{j=1}^{q+1} N_{G_j}(B_e(r_e)) - \log r_e + O(1). \quad (3.6)$$

For a fixed $B_e(r_e)$, we consider non-zero entire functions $W, G_1, \ldots, G_q$ on $D_{B_e(r_e)}$. From Lemma 2.4 it follows that one can find $u^e \in \mathcal{U}^e_{G_j,B_e(r_e)}$ and $u^e \in \mathcal{U}^e_{W,B_e(r_e)}, j = 1, \ldots, q$, such that

$N_W(B_e(r_e)) = N_{W_{e,u^e}}(r_e), N_{G_j}(B_e(r_e)) = N_{(G_j)_{e,u^e}}(r_e).$

Now let $u^e_e(x)$ be a zero of G_j, having the $e-th$ partial multiplicity equal to $k, (k \neq +\infty), k \geq 2$. Since $\gamma_1 = (0, \ldots, 0, \gamma_{1e}, 0, \ldots, 0)$ with $\gamma_{1e} = 1$, we have $v_{i,\partial^{\gamma_1}_{G_j}}(u^e_e(x)) = k-1$ if $i = e$.

On the other hand

$$W(G_{\alpha_1}, G_{\alpha_2}) = c_{(\alpha_1,\alpha_2)}W,$$

where (α_1, α_2) are distinct two numbers in $\{1, \ldots, q+1\}$. Therefore $u_e^e(x)$ is a zero of W having $e-th$ partial multiplicity at least $k-1$.
Now we consider the function $F = \prod_{i=1}^{q} G_i$.
Because F is not a constant, F has zeros. Let $u_e^e(x)$ be a zero of F. By the hypothesis, $a_1, \ldots, a_q$ are distinct numbers, from this it follows that there exists only one function G_j such that $G_j(u_e^e(x)) = 0$. Therefore

$$\sum_{j=1}^{q} N_{(G_j)_{e,u^e}}(r_e) - N_{W_{e,u^e}}(r_e) \leqslant \sum_{j=1}^{q} \overline{N}_{(G_j)_{e,u^e}}(r_e).$$

Thus

$$\sum_{j=1}^{q} N_{G_j}(B_e(r_e)) - N_W(B_e(r_e)) \leqslant \sum_{j=1}^{q} \overline{N}_{(G_j)_{e,u^e}}(r_e) = \sum_{j=1}^{q} \overline{N}_{G_j}(B_e(r_e)).$$

From this and (3.6) we obtain Theorem 3.2.

Theorem 3.4. *Let f, g be two nonconstant meromorphic functions on $\mathbb{C}_p^m$ such that $\overline{E}(f, a_i) = \overline{E}(g, a_i), a_i \in \mathbb{C}_p \cup \{\infty\}, i = 1, 2, ..., q$. If $q \geq 4$, then $f \equiv g$.*

Proof Assume, on the contrary, that $f \not\equiv g$. Set

$$\varphi = \frac{1}{f} - \frac{1}{g}.$$

Then, $\varphi \not\equiv 0$ and $H_\varphi\Big(B_e(r_e)\Big) \leqslant H_f\Big(B_e(r_e)\Big) + H_g\Big(B_e(r_e)\Big)$.

Therefore, applying Theorem 3.2 to the function f and values $a_1, \ldots a_q$ we have

$$\Big(q-2\Big)H_f\Big(B_e(r_e)\Big) \leqslant \sum_{j=1}^{q} \overline{N}_f\Big(a_j, B_e(r_e)\Big) - \log r_e + O(1)$$
$$\leqslant \overline{N}_\varphi\Big(B_e(r_e)\Big) - \log r_e + O(1).$$

Similarly

$$\Big(q-2\Big)H_g\Big(B_e(r_e)\Big) \leqslant \overline{N}_\varphi\Big(B_e(r_e)\Big) - \log r_e + O(1).$$

Summing up these inequalities and using Theorem 2.5, we obtain

$$\Big(q-2\Big)\Big(H_f\Big(B_e(r_e)\Big) + Hg\Big(B_e(r_e)\Big) \leqslant 2\Big(H_f\Big(B_e(r_e)\Big)$$
$$+ H_g\Big(B_e(r_e)\Big) - 2\log r_e + O(1).$$

Therefore

$$\Big(q-4\Big)\Big(H_f\Big(B_e(r_e)\Big)+H_g\Big(B_e(r_e)\Big)+2\log r_e \leqslant O(1).$$

It implies $q-4<0$, a contradiction. Theorem 3.4 is proved.

Theorem 3.5. *Let f, g be two nonconstant meromorphic functions on $\mathbb{C}_p^m$ such that $E_i(f,a_j)=E_i(g,a_j), i=1,2,...,m, a_j\in\mathbb{C}_p\cup\{\infty\}, j=1,2,3$. Then $f\equiv g$.*
Proof We need the following

Lemma 3.6. *Let f, g be two non-zero entire funtions on $\mathbb{C}_p^m$ such that $v_f^0=v_g^0$ on $\mathbb{C}_p^m$. Then $f=cg$ where c is a non-zero constant in $\mathbb{C}_p$.*

Proof Take $r_1,\dots,r_m>0$ such that f,g have no zeros in $D_{<r_{(m)}>}$. If f is a non-zero constant then so is g. Therefore $f=cg$. Assume that f is non-constant. Since $v_f^0=v_g^0$, g is non-constant. Let $a=(a_1,\dots,a_m)$, $b=(b_1,\dots,b_m)$ be two any elements of $D_{<r_{(m)}>}$. Set

$$C_i(b_i)=(b_1,\dots,b_i,a_{i+1},\dots,a_m),\ i=1,\dots,m.$$

By $v_f^0=v_g^0$, $v_{i,f}(z_{(m)})=v_{i,g}(z_{(m)}),\ i=1,\dots,m$. Thus

$$f_{i,C_i(b_i)}=c_i g_{i,C_i(b_i)},\text{ with } c_i=\frac{f(a)}{g(a)}=\frac{f(C_i(b_i))}{g(C_i(b_i))},\text{and } c_i=c_{i+1},$$

$i=1,2,...,m-1$. From this we have

$$\frac{f(a)}{g(a)}=\frac{f(b)}{g(b)}\quad\text{for all}\quad a,b\in D_{r_{<m>}}.$$

Set

$$c=\frac{f(a)}{g(a)},\ a\in D_{<r_{(m)}>},\ h=f-cg.$$

Asume that h is not identically zero. Consider h,f,g in $D_{<r_{(m)}>}$. By Lemma 2.2, there exists $u\in D_{<r_{(m)}>}$ such that $h_{i,u},\ f_{i,u},\ g_{i,u}$ are not identically zero, $i=1,2,\dots,m$. We have $f_{i,u}=c'g_{i,u}, c'=\frac{f(u)}{g(u)}$. Therefore $c=c'$ and $h_{i,u}=f_{i,u}-cg_{i,u}$ identically zero. From this we a obtain contradiction. So, $f=cg$.

Now we prove Theorem 3.5. Write $f=\frac{f_1}{f_2}$, $g=\frac{g_1}{g_2}$, where f_1,f_2 are two holomorphic functions on $\mathbb{C}_p^m$), having no common zeros, and g_1,g_2 are too. Applying Lemma 3.6 to $f_1+a_jf_2$ and $g_1+a_jg_2$, $j=1,2,3$ we have $f_1+a_jf_2=c_j(g_1+a_jg_2), c_j\neq 0, j=1,2,3$. From this we obtain $c_1=c_2=c_3$ and $f\equiv g$.

Theorem 3.7. *Let f,g be two nonconstant meromorphic functions on $\mathbb{C}_p^m$ such that $\overline{E}(f,a_i)\subset\overline{E}(g,a_i), a_i\in\mathbb{C}_p\cup\{\infty\}, i=1,2,...,q$, and $E_i(f,b_j)=E_i(g,b_j), b_j\in\mathbb{C}_p\cup\{\infty\}, j=1,2$, and $a_i\neq b_j$ for all i,j. If $q\geq 4$, then $f\equiv g$.*

Proof Similarly as in the proof of Theorem 3.5, we have $f = cg, c \neq 0$. Assume, on the contrary, that $f \not\equiv g$. Set

$$\varphi = \frac{1}{f} - \frac{1}{g}.$$

Then, $\varphi \not\equiv 0$ and $H_\varphi\Big(B_e(r_e)\Big) \leqslant H_f\Big(B_e(r_e)\Big) + H_g\Big(B_e(r_e)\Big)$.

Applying Theorem 3.2 to the function f and values $a_1, \dots a_q$ we have

$$\Big(q-2\Big)H_f\Big(B_e(r_e)\Big) \leqslant \sum_{j=1}^{q} \overline{N}_f\Big(a_j, B_e(r_e)\Big) - \log r_e + O(1)$$
$$\leqslant \overline{N}_\varphi\Big(B_e(r_e)\Big) - \log r_e + O(1).$$

Using Theorem 2.5, we obtain

$$\Big(q-2\Big)\Big(H_f\Big(B_e(r_e)\Big)\Big) \leqslant \Big(H_f\Big(B_e(r_e)\Big) + H_g\Big(B_e(r_e)\Big) - 2\log r_e + O(1).$$

By $H_f(B_e(r_e)) = H_g(B_e(r_e))$

$$\Big(q-4\Big)H_f(B_e(r_e) + 2\log r_e \leqslant O(1).$$

It implies $q - 4 < 0$, a contradiction. Theorem 3.7 is proved.

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