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ON THE COMPOSITION OF THE DISTRIBUTIONS $x_+^\lambda \ln^m x_+$ AND x_+^μ

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Abstract

Let F be a distribution and let f be a locally summable function. The neutrix composition $F(f)$, of F and f, is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The neutrix composition of the distributions $x_+^\lambda \ln^m x_+$ and x_+^μ is evaluated for $-1 < \lambda < 0,\ \mu > 0,\ \lambda\mu \neq -1, -2, \ldots$ and $m = 0, 1, 2, \ldots$.

1. Introduction

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$ and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

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We define the locally summable function $x_+^\lambda \ln^m x_+$ for $\lambda > -1$ and $m = 0, 1, 2, \dots$ by

$$x_+^\lambda \ln^m x_+ = \begin{cases} x^\lambda \ln^m x, & x > 0, \\ 0, & x < 0. \end{cases}$$

The distribution $x_+^\lambda \ln^m x_+$ is then defined inductively for $\lambda < -1$, $\lambda \neq -2, -3, \dots$ and $m = 0, 1, 2, \dots$ by the equation

$$(x_+^\lambda \ln^{m+1} x_+)' = \lambda x_+^{\lambda-1} \ln^{m+1} x_+ + (m+1) x_+^{\lambda-1} \ln^m x_+.$$

The distribution $x_-^\lambda \ln^m x_-$ is then defined for $\lambda \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$ by

$$x_-^\lambda \ln^m x_- = (-x)_+^\lambda \ln^m (-x)_+,$$

and the distribution $|x|^\lambda \ln^m |x|$ is defined for $\lambda \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$ by

$$|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ + x_-^\lambda \ln^m x_-.$$

It follows that if r is a positive integer and $-r - 1 < \lambda < -r$, then

$$\langle x_+^\lambda \ln^m x_+, \varphi(x) \rangle = \int_0^\infty x^\lambda \ln^m x \Big[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \Big] dx$$

for arbitrary φ in $\mathcal{D}$.

In particular, if φ has its support contained in the interval $[-1, 1]$, then

$$\begin{aligned} \langle x_+^\lambda \ln^m x_+, \varphi(x) \rangle &= \int_0^1 x^\lambda \ln^m x \Big[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \Big] dx \\ &\quad + \sum_{k=0}^{r-1} \frac{(-1)^m m! \varphi^{(k)}(0)}{k! (\lambda + k + 1)^{m+1}} \end{aligned} \tag{1}$$

for $-r - 1 < \lambda < -r$, and

$$\begin{aligned} \langle |x|^\lambda \ln^m |x|, \varphi(x) \rangle &= \int_{-1}^1 |x|^\lambda \ln^m |x| \Big[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(2k)!} x^{2k} \Big] dx \\ &\quad + \sum_{k=0}^{r-1} \frac{2(-1)^m m! \varphi^{(2k)}(0)}{(2k)! (\lambda + 2k + 1)^{m+1}} \end{aligned} \tag{2}$$

for $-2r - 1 < \lambda < -2r + 1$ and $\lambda \neq -2r$.

We now let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n: \qquad \lambda > 0, \ r = 1, 2, \dots,$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [2], and was originally called the composition of distributions.

Definition 1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b), with $-\infty < a < b < \infty$, if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$ where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$.

The following two theorems were proved in [2] and [3] respectively:

Theorem 1 *The neutrix compositions* $(x_-^\mu)_-^\lambda$ *and* $(x_+^\mu)_-^\lambda$ *exist and*

$$(x_-^\mu)_-^\lambda = (x_+^\mu)_-^\lambda = 0$$

for $\mu > 0$ *and* $\lambda\mu \neq -1, -2, \ldots$ *and*

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu}(x_+^\mu)_-^\lambda = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu - 1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ and $\lambda\mu = -1, -2, \ldots$.

Theorem 2 *The neutrix composition $(x_+^r)_-^{-s}$ exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s} c(\rho)}{r(rs-1)!} \delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \ldots$, where $c(\rho) = \int_0^1 \ln t \rho(t)\, dt$.

In the previous theorem, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \ldots$, and not as in Gel'fand and Shilov [6].

The next two theorems were proved in [4] and [5] respectively.

Theorem 3 *The neutrix composition $(x_+^r)^{-1}$ exists and*

$$(x_+^r)^{-1} = x_+^{-r} + (-1)^r \frac{2c(\rho) - r\phi(r-1)}{r!} \delta^{(r-1)}(x)$$

for $r = 1, 2, \ldots$ where

$$c(\rho) = \int_0^1 \ln t \rho(t)\, dt, \quad \phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

Theorem 4 *The neutrix composition $(x_+^\mu)_+^\lambda$ exists and*

$$(x_+^\mu)_+^\lambda = x_+^{\lambda\mu}$$

for $\lambda < 0$, $\mu > 0$ and λ, $\lambda\mu \neq -1, -2, \ldots$.

To prove the next theorem, we need the following lemma which can easily be proved by induction.

Lemma

$$\int_1^n v^\alpha \ln^r v\, dv = \frac{(-1)^r r!(n^{\alpha+1} - 1)}{(\alpha+1)^{r+1}} + \sum_{i=0}^{r-1} \frac{(-1)^i r! n^{\alpha+1} \ln^{r-i} n}{(r-i)!(\alpha+1)^{i+1}}$$

for $r = 1, 2, \ldots$ and $-1 < \alpha < 0$.

We now prove

Theorem 5 *If $F_{m,\lambda}(x)$ denotes the distribution $x_+^\lambda \ln^m x_+$, then the neutrix composition $F_{m,\lambda}(x_+^\mu)$ exists and*

$$F_{m,\lambda}(x_+^\mu) = \mu^m x_+^{\lambda\mu} \ln^m x_+ \tag{3}$$

for $-1 < \lambda < 0$, $\mu > 0$, $\lambda\mu \neq -1, -2, \ldots$ *and* $m = 0, 1, 2, \ldots$.

Proof For $m = 0$, this is just theorem 4. We then assume that $m \geq 1$. We put

$$[F_{m,\lambda}(x)]_n = (x_+^\lambda \ln^m x_+) * \delta_n(x)$$

and so

$$[F_{m,\lambda}(x)]_n = \begin{cases} \int_{-1/n}^{1/n} (x-t)^\lambda \ln^m(x-t)\delta_n(t)\,dt, & 1/n < x, \\ \int_{-1/n}^{x} (x-t)^\lambda \ln^m(x-t)\delta_n(t)\,dt, & -1/n \leq x \leq 1/n, \\ 0, & x < -1/n. \end{cases}$$

Then

$$[F_{m,\lambda}(x_+^\mu)]_n = \begin{cases} \int_{-1/n}^{1/n} (x^\mu-t)^\lambda \ln^m(x^\mu-t)\delta_n(t)\,dt, & 1/n < x^\mu, \\ \int_{-1/n}^{x^\mu} (x^\mu-t)^\lambda \ln^m(x^\mu-t)\delta_n(t)\,dt, & 0 \leq x^\mu \leq 1/n, \\ \int_{-1/n}^{0} (-t)^\lambda \ln^m(-t)\delta_n(t)\,dt, & x < 0. \end{cases} \quad (4)$$

It follows that

$$\begin{aligned} \int_{-1}^{1} x^k [F_{m,\lambda}(x_+^\mu)]_n\,dx &= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^\mu} (x^\mu - t)^\lambda \ln^m(x^\mu - t)\delta_n(t)\,dt\,dx \\ &\quad + \int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} (x^\mu - t)^\lambda \ln^m(x^\mu - t)\delta_n(t)\,dt\,dx \\ &\quad + \int_{-1}^{0} x^k \int_{-1/n}^{0} (-t)^\lambda \ln^m(-t)\delta_n(t)\,dt\,dx \\ = \frac{n^{-(\lambda\mu+k+1)/\mu}}{\mu} &\int_0^1 v^{(k+1)/\mu-1} \int_{-1}^{v} (v-u)^\lambda [\ln(v-u) - \ln n]^m \rho(u)\,du\,dv \\ + \frac{n^{-(\lambda\mu+k+1)/\mu}}{\mu} &\int_{-1}^{1} \rho(u) \int_1^n v^{(k+1)/\mu-1} (v-u)^\lambda [\ln(v-u) - \ln n]^m\,dv\,du \\ + n^{-\lambda} &\int_{-1}^{0} x^k \int_{-1}^{0} (-u)^\lambda \ln^m(-u)\rho(u)\,du\,dx \\ = I_1 + I_2 + I_3 & \end{aligned} \quad (5)$$

where the substitutions $u = nt$ and $v = nx^\mu$ have been made.

It follows immediately that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = \operatorname*{N-lim}_{n\to\infty} I_3 = 0 \quad (6)$$

for $k = 0, 1, 2, \ldots$.

Further,

$$\begin{aligned}
&\int_1^n v^{(k+1)/\mu-1}(v-u)^\lambda[\ln(v-u)-\ln n]^m\,dv \\
&= \sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}\ln^{m-s}n\int_1^n v^{(k+1)/\mu-1}(v-u)^\lambda\ln^s(v-u)\,dv \\
&= \sum_{s=0}^{m-1}\sum_{i=1}^{s}\binom{m}{s}\binom{s}{i}(-1)^{m-s}\ln^{m-s}n\int_1^n v^{(k+1)/\mu+\lambda-1}(1-u/v)^\lambda \\
&\qquad\qquad\qquad\qquad \times\ln^i(1-u/v)\ln^{s-i}v\,dv \\
&\qquad + \sum_{s=0}^{m-1}\binom{m}{s}(-1)^{m-s}\ln^{m-s}n\int_1^n v^{(k+1)/\mu+\lambda-1}(1-u/v)^\lambda\ln^s v\,dv \\
&\qquad + \sum_{i=1}^{m}\binom{m}{i}\int_1^n v^{(k+1)/\mu+\lambda-1}(1-u/v)^\lambda\ln^i(1-u/v)\ln^{m-i}v\,dv \\
&\qquad + \int_1^n v^{(k+1)/\mu+\lambda-1}(1-u/v)^\lambda\ln^m v\,dv \\
&= \sum_{s=0}^{m-1}\sum_{i=1}^{s}(-1)^{m-s+i}\binom{m}{s}\binom{s}{i}\ln^{m-s}n\int_1^n v^{(k+1)/\mu+\lambda-1} \\
&\qquad\qquad \times\Big[\frac{u^i}{v^i}+(\frac{i}{2}-\lambda)\frac{u^{i+1}}{v^{i+1}}+O(v^{-i-2})\Big]\ln^{s-i}v\,dv \\
&\qquad + \sum_{s=0}^{m-1}\binom{m}{s}(-1)^{m-s}\ln^{m-s}n\int_1^n v^{(k+1)/\mu+\lambda-1} \\
&\qquad\qquad \times\Big[1-\frac{\lambda u}{v}+O(v^{-2})\Big]\ln^s v\,dv \\
&\quad + \sum_{i=1}^{m}(-1)^i\binom{m}{i}\int_1^n v^{(k+1)/\mu+\lambda-1}\Big[\frac{u^i}{v^i}+(\frac{i}{2}-\lambda)\frac{u^{i+1}}{v^{i+1}}+O(v^{-i-2})\Big]\ln^{m-i}v\,dv \\
&\quad + \int_1^n v^{(k+1)/\mu+\lambda-1}(1-u/v)^\lambda\ln^m v\,dv. \qquad (7)
\end{aligned}$$

Using the lemma, it follows that

$$\begin{aligned}
&n^{-(k+1)/\mu-\lambda}\ln^{m-s}n\int_1^n v^{(k+1)/\mu+\lambda-1}\Big[\frac{u^i}{v^i}+(\frac{i}{2}-\lambda)\frac{u^{i+1}}{v^{i+1}}+\ldots\Big]\ln^{s-i}v\,dv \\
&\qquad = O(n^{-i}\ln^{m-i}n)+cn^{(k+1)/\mu-\lambda}\ln^{m-s}n \quad (8)
\end{aligned}$$

for some constant c, for $i = 1, \ldots, s$ and $s = 0, 1, \ldots, m-1$,

$$n^{-(k+1)/\mu-\lambda} \ln^{m-s} n \int_1^n v^{(k+1)/\mu+\lambda-1} \Big[1 - \frac{\lambda u}{v} + \ldots\Big] \ln^s v \, dv$$
$$= O(n^{-1} \ln^m n) + P(\ln n) + dn^{(k+1)/\mu-\lambda} \ln^{m-s} n \quad (9)$$

for some constant d, for $s = 0, \ldots, m-1$ where $P(\ln n)$ denotes a polynomial in $\ln n$ with positive powers,

$$n^{-(k+1)/\mu-\lambda} \int_1^n v^{(k+1)/\mu+\lambda-1} \Big[\frac{u^i}{v^i} + (\frac{i}{2} - \lambda)\frac{u^{i+1}}{v^{i+1}} + \ldots\Big] \ln^{m-i} v \, dv$$
$$= O(n^{-i} \ln^{m-i} n) + en^{(k+1)/\mu-\lambda} \quad (10)$$

for some constant e, with $i = 1, \ldots, m$, and

$$n^{-(k+1)/\mu-\lambda} \int_1^n v^{(k+1)/\mu+\lambda-1} (1 - u/v)^\lambda \ln^m v dv = \frac{(-1)^m m! (1 - n^{-(k+1)/\mu+\lambda})}{[(k+1)/\mu + \lambda]^{m+1}}$$
$$+ P(\ln n) + O(n^{-2} \ln^m n) \quad (11)$$

where $P(\ln n)$, once again, denotes a polynomial in $\ln n$ with positive powers.

It now follows from equations (5) and (8) to (11) that

$$\mathrm{N}\!-\!\lim_{n\to\infty} I_2 = \frac{(-1)^m m! \mu^m}{(\lambda\mu + k + 1)^{m+1}} \quad (12)$$

for $k = 0, 1, 2, \ldots$.

It now follows from equations (5), (6) and (12) that

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_{-1}^1 x^k [F_{m,\lambda}(x_+^\mu)]_n \, dx = \frac{(-1)^m m! \mu^m}{(\lambda\mu + k + 1)^{m+1}} \quad (13)$$

for $k = 0, 1, 2, \ldots$.

We now consider the case $k = r$, where r is chosen so that $0 < \lambda\mu + r + 1 < 1$, and let ψ be an arbitrary continuous function. When $0 \le x^\mu \le 1/n$, we have

$$\int_0^{n^{-1/\mu}} x^r \psi(x) [F_{m,\lambda}(x_+^\mu)]_n \, dx$$
$$= \frac{n^{-(\lambda\mu+r+1)/\mu}}{\mu} \int_0^1 \psi((v/n)^{\frac{1}{\mu}}) v^{(r+1)/\mu-1} \int_{-1}^v (v-u)^\lambda [\ln(v-u) - \ln n]^m \rho(u) \, du \, dv$$

and it follows that

$$\lim_{n\to\infty} \int_0^{n^{-1/\mu}} x^r \psi(x) [F_{m,\lambda}(x_+^\mu)]_n \, dx = 0. \quad (14)$$

When $x < 0$, we have

$$\int_{-1}^{0} x^r \psi(x)[F_{m,\lambda}(x_+^\mu)]_n\,dx = n^{-\lambda}\int_{-1}^{0} x^r\psi(x)\int_{-1}^{0}(-u)^\lambda \ln^m(-u)\rho(u)\,du\,dx$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^{0} x^r\psi(x)[F_{m,\lambda}(x_+^\mu)]_n\,dx = 0. \tag{15}$$

When $x^\mu > 1/n$, we have

$$\begin{aligned}
[F_{m,\lambda}(x_+^\mu)]_n &= \int_{-1/n}^{1/n}(x^\mu - t)^\lambda \ln^m(x^\mu - t)\delta_n(t)\,dt \\
&= \int_{-1}^{1}(x^\mu - u/n)^\lambda \ln^m(x^\mu - u/n)\rho(u)\,du \\
&= x^{\lambda\mu}\int_{-1}^{1}\Big[\ln^m x^\mu - \frac{\lambda u \ln^m x^\mu}{nx^\mu} - \frac{mu\ln^{m-1}x^\mu}{nx^\mu} + O(n^{-2})\Big]\rho(u)\,du \\
&= \mu^m x^{\lambda\mu}\ln^m x + O(n^{-2}).
\end{aligned} \tag{16}$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1,1]$. By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^r}{r!}\varphi^{(r)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\langle [F_{m,\lambda}(x_+^\mu)]_n, \varphi(x)\rangle &= \int_{-1}^{1}[F_{m,\lambda}(x_+^\mu)]_n\varphi(x)\,dx \\
&= \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}x^k[F_{m,\lambda}(x_+^\mu)]_n\,dx + \int_{n^{-1/\mu}}^{1}\frac{x^r}{r!}[F_{m,\lambda}(x_+^\mu)]_n\varphi^{(r)}(\xi x)\,dx \\
&\quad + \int_{0}^{n^{-1/\mu}}\frac{x^r}{r!}[F_{m,\lambda}(x_+^\mu)]_n\varphi^{(r)}(\xi x)\,dx + \int_{-1}^{0}\frac{x^r}{r!}[F_{m,\lambda}(x_+^\mu)]_n\varphi^{(r)}(\xi x)\,dx.
\end{aligned}$$

Using equations (13) to (16), it follows that

$$
\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty}\langle [F_{m,\lambda}(x_+^{\mu})]_n, \varphi(x)\rangle &= \sum_{k=0}^{r-1} \frac{(-1)^m m!\mu^m \varphi^{(k)}(0)}{k!(\lambda\mu+k+1)^{m+1}} \\
&\qquad + \mu^m \int_0^1 \frac{x^{\lambda\mu+r}\ln^m x}{r!}\varphi^{(r)}(\xi x)\,dx \\
&= \mu^m \int_0^1 x^{\lambda\mu}\ln^m x\Big[\varphi(x) - \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\Big]\,dx \\
&\qquad + \mu^m \sum_{k=0}^{r-1} \frac{(-1)^m m!\varphi^{(k)}(0)}{k!(\lambda\mu+k+1)^{m+1}} \\
&= \mu^m \langle x_+^{\lambda\mu}\ln^m x_+, \varphi(x)\rangle,
\end{aligned}
$$

on using equation (1). This proves equation (3) on the interval $[-1,1]$. However, equation (3) clearly holds on any interval not containing the origin, and the proof is complete.

Theorem 6. *The neutrix composition $F_{m,\lambda}(|x|^{\mu})$ exists and*

$$
F_{m,\lambda}(|x|^{\mu}) = \mu^m |x|^{\lambda\mu}\ln^m|x| \tag{17}
$$

for $-1<\lambda<0$, $\mu>0$, $\lambda\mu \neq -1,-2,\ldots$ and $m=0,1,2,\ldots$.

Proof It follows from equation (4) that

$$
[F_{m,\lambda}(|x|^{\mu})]_n = \begin{cases} \int_{-1/n}^{1/n}(|x|^{\mu}-t)^{\lambda}\ln^m(|x|^{\mu}-t)\delta_n(t)\,dt, & 1/n<|x|^{\mu}, \\ \int_{-1/n}^{|x|^{\mu}}(|x|^{\mu}-t)^{\lambda}\ln^m(|x|^{\mu}-t)\delta_n(t)\,dt, & 0\le |x|^{\mu}\le 1/n. \end{cases} \tag{18}
$$

Since $[F_{m,\lambda}(|x|^{\mu})]_n$ is an even function, it follows that

$$
\int_{-1}^{1} x^k [F_{m,\lambda}(|x|^{\mu})]_n\,dx = 0 \tag{19}
$$

for $k=1,3,\ldots$.

In general, we have

$$
\begin{aligned}
\int_0^1 x^k [F_{m,\lambda}(|x|^{\mu})]_n\,dx &= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^{\mu}} (x^{\mu}-t)^{\lambda}\ln^m(x^{\mu}-t)\delta_n(t)\,dt\,dx \\
&\qquad + \int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} (x^{\mu}-t)^{\lambda}\ln^m(x^{\mu}-t)\delta_n(t)\,dt\,dx \\
&= I_1 + I_2
\end{aligned}
$$

and it follows as above that

$$\operatorname*{N-lim}_{n\to\infty} \int_{-1}^{1} x^k [F_{m,\lambda}(|x|^\mu)]_n \, dx = \frac{2(-1)^m m! \mu^m}{(\lambda\mu + k + 1)^{m+1}} \tag{20}$$

for $k = 0, 2, 4, \ldots$ since the integrand is even.

We now consider the case $k = 2r$, where r is chosen so that $0 < \lambda\mu + 2r + 1 < 2$, and let ψ be an arbitrary continuous function. Then it follows as above that

$$\begin{aligned} \lim_{n\to\infty} \int_0^{n^{-1/\mu}} x^{2r}\psi(x)[F_{m,\lambda}(|x|^\mu)]_n \, dx &= \lim_{n\to\infty} \int_{-n^{-1/\mu}}^{0} x^{2r}\psi(x)[F_{m,\lambda}(|x|^\mu)]_n \, dx \\ &= 0 \end{aligned} \tag{21}$$

and

$$[F_{m,\lambda}(|x|^\mu)]_n = \mu^m |x|^{\lambda\mu} \ln^m |x| + O(n^{-2}) \tag{22}$$

if $|x|^\mu \geq 1/n$.

Again let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. Then

$$\varphi(x) = \sum_{k=0}^{2r-1} \frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^{2r}}{(2r)!}\varphi^{(2r)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned} \langle [F_{m,\lambda}(|x|^\mu)]_n, \varphi(x)\rangle &= \int_{-1}^{1} [F_{m,\lambda}(|x|^\mu)]_n \varphi(x) \, dx \\ &= \sum_{k=0}^{r-1} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \int_{-1}^{1} x^{2k+1}[F_{m,\lambda}(|x|^\mu)]_n \, dx \\ &\quad + \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(2k)!} \int_{-1}^{1} x^{2k}[F_{m,\lambda}(|x|^\mu)]_n \, dx \\ &\quad + \int_{n^{-1/\mu}}^{1} \frac{x^{2r}}{(2r)!}[F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) \, dx \\ &\quad + \int_{-1}^{-n^{-1/\mu}} \frac{x^{2r}}{(2r)!}[F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) \, dx \\ &\quad + \int_{0}^{n^{-1/\mu}} \frac{x^{2r}}{(2r)!}[F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) \, dx \\ &\quad + \int_{-n^{-1/\mu}}^{0} \frac{x^{2r}}{(2r)!}[F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) \, dx. \end{aligned}$$

Using equations (20) to (22), it follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty}\langle [F_{m.\lambda}(|x|^{\mu})]_n, \varphi(x)\rangle \;=\; & \sum_{k=0}^{r-1} \frac{2(-1)^m m!\mu^m \varphi^{(2k)}(0)}{(2k)!(\lambda\mu+2k+1)^{m+1}} \\
& + \mu^m \int_{-1}^{1} \frac{|x|^{\lambda\mu+2r}}{(2r)!} \ln^m |x| \varphi^{(2r)}(\xi x)\, dx \\
= \mu^m & \int_{-1}^{1} |x|^{\lambda\mu} \ln^m |x| \Big[\varphi(x) - \sum_{k=0}^{r-1} \frac{x^{2k}}{(2k)!}\varphi^{(2k)}(0)\Big] dx \\
& + \mu^m \sum_{k=0}^{r-1} \frac{2(-1)^m m! \varphi^{(2k)}(0)}{(2k)!(\lambda\mu+2k+1)^{m+1}} \\
= \mu^m & \langle |x|^{\lambda\mu} \ln^m |x|, \varphi(x)\rangle,
\end{aligned}$$

on using equation (2). This proves equation (17) on the interval $[-1, 1]$. However, equation (17) clearly holds on any interval not containing the origin, and the proof is complete.

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