

ON THE COMPOSITION OF THE DISTRIBUTIONS $x_+^\lambda \ln^m x_+$ AND x_+^μ

B. Fisher*, S. Orankitjaroen†, T. Kraiweeradechachai†,
G. Sritanratana† and K. Nonlaopon‡

**Department of Mathematics
University of Leicester, Leicester, LE1 7RH, England
e-mail: fbr@mcs.le.ac.uk*

†*Department of Mathematics
Mahidol University, Bangkok, Thailand
e-mail: yong.33@hotmail.com, scsok@mahidol.ac.th and scgst@mahidol.ac.th*

‡*Department of Mathematics,
Khonkaen University, Khonkaen, Thailand
e-mail: kamsingn@yahoo.com*

Abstract

Let F be a distribution and let f be a locally summable function. The neutrix composition $F(f)$, of F and f , is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The neutrix composition of the distributions $x_+^\lambda \ln^m x_+$ and x_+^μ is evaluated for $-1 < \lambda < 0$, $\mu > 0$, $\lambda\mu \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$

1. Introduction

In the following, we let \mathcal{D} be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$ and let \mathcal{D}' be the space of distributions defined on \mathcal{D} .

Key words and phrases: Distribution, delta-function, neutrix composition of distributions, neutrix, neutrix limit.

2000 AMS Mathematics Subject Classification: 46F10.

We define the locally summable function $x_+^\lambda \ln^m x_+$ for $\lambda > -1$ and $m = 0, 1, 2, \dots$ by

$$x_+^\lambda \ln^m x_+ = \begin{cases} x^\lambda \ln^m x, & x > 0, \\ 0, & x < 0. \end{cases}$$

The distribution $x_+^\lambda \ln^m x_+$ is then defined inductively for $\lambda < -1$, $\lambda \neq -2, -3, \dots$ and $m = 0, 1, 2, \dots$ by the equation

$$(x_+^\lambda \ln^{m+1} x_+)' = \lambda x_+^{\lambda-1} \ln^{m+1} x_+ + (m+1)x_+^{\lambda-1} \ln^m x_+.$$

The distribution $x_-^\lambda \ln^m x_-$ is then defined for $\lambda \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$ by

$$x_-^\lambda \ln^m x_- = (-x)_+^\lambda \ln^m (-x)_+,$$

and the distribution $|x|^\lambda \ln^m |x|$ is defined for $\lambda \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$ by

$$|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ + x_-^\lambda \ln^m x_-.$$

It follows that if r is a positive integer and $-r-1 < \lambda < -r$, then

$$\langle x_+^\lambda \ln^m x_+, \varphi(x) \rangle = \int_0^\infty x^\lambda \ln^m x \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right] dx$$

for arbitrary φ in \mathcal{D} .

In particular, if φ has its support contained in the interval $[-1, 1]$, then

$$\begin{aligned} \langle x_+^\lambda \ln^m x_+, \varphi(x) \rangle &= \int_0^1 x^\lambda \ln^m x \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right] dx \\ &\quad + \sum_{k=0}^{r-1} \frac{(-1)^m m! \varphi^{(k)}(0)}{k! (\lambda + k + 1)^{m+1}} \end{aligned} \quad (1)$$

for $-r-1 < \lambda < -r$, and

$$\begin{aligned} \langle |x|^\lambda \ln^m |x|, \varphi(x) \rangle &= \int_{-1}^1 |x|^\lambda \ln^m |x| \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(2k)!} x^{2k} \right] dx \\ &\quad + \sum_{k=0}^{r-1} \frac{2(-1)^m m! \varphi^{(2k)}(0)}{(2k)! (\lambda + 2k + 1)^{m+1}} \end{aligned} \quad (2)$$

for $-2r-1 < \lambda < -2r+1$ and $\lambda \neq -2r$.

We now let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n : \quad \lambda > 0, r = 1, 2, \dots,$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$,
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in \mathcal{D}' , we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [2], and was originally called the composition of distributions.

Definition 1. Let F be a distribution in \mathcal{D}' and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) , with $-\infty < a < b < \infty$, if

$$\text{N-lim}_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a, b]$ where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a, b]$.

The following two theorems were proved in [2] and [3] respectively:

Theorem 1 *The neutrix compositions $(x_-^\mu)_-^\lambda$ and $(x_+^\mu)_-^\lambda$ exist and*

$$(x_-^\mu)_-^\lambda = (x_+^\mu)_-^\lambda = 0$$

for $\mu > 0$ and $\lambda\mu \neq -1, -2, \dots$ and

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu} (x_+^\mu)_-^\lambda = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu - 1)!} \delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \dots$ and $\lambda\mu = -1, -2, \dots$.

Theorem 2 *The neutrix composition $(x_+^r)_-^{-s}$ exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s} c(\rho)}{r(rs-1)!} \delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \dots$, where $c(\rho) = \int_0^1 \ln t \rho(t) dt$.

In the previous theorem, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \dots$, and not as in Gel'fand and Shilov [6].

The next two theorems were proved in [4] and [5] respectively.

Theorem 3 *The neutrix composition $(x_+^r)_-^{-1}$ exists and*

$$(x_+^r)_-^{-1} = x_+^{-r} + (-1)^r \frac{2c(\rho) - r\phi(r-1)}{r!} \delta^{(r-1)}(x)$$

for $r = 1, 2, \dots$ where

$$c(\rho) = \int_0^1 \ln t \rho(t) dt, \quad \phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

Theorem 4 *The neutrix composition $(x_+^\mu)_+^\lambda$ exists and*

$$(x_+^\mu)_+^\lambda = x_+^{\lambda\mu}$$

for $\lambda < 0$, $\mu > 0$ and $\lambda, \lambda\mu \neq -1, -2, \dots$

To prove the next theorem, we need the following lemma which can easily be proved by induction.

Lemma

$$\int_1^n v^\alpha \ln^r v dv = \frac{(-1)^r r! (n^{\alpha+1} - 1)}{(\alpha+1)^{r+1}} + \sum_{i=0}^{r-1} \frac{(-1)^i r! n^{\alpha+1} \ln^{r-i} n}{(r-i)! (\alpha+1)^{i+1}}$$

for $r = 1, 2, \dots$ and $-1 < \alpha < 0$.

We now prove

Theorem 5 *If $F_{m,\lambda}(x)$ denotes the distribution $x_+^\lambda \ln^m x_+$, then the neutrix composition $F_{m,\lambda}(x_+^\mu)$ exists and*

$$F_{m,\lambda}(x_+^\mu) = \mu^m x_+^{\lambda\mu} \ln^m x_+ \quad (3)$$

for $-1 < \lambda < 0$, $\mu > 0$, $\lambda\mu \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$

Proof For $m = 0$, this is just theorem 4. We then assume that $m \geq 1$. We put

$$[F_{m,\lambda}(x)]_n = (x_+^\lambda \ln^m x_+) * \delta_n(x)$$

and so

$$[F_{m,\lambda}(x)]_n = \begin{cases} \int_{-1/n}^{1/n} (x-t)^\lambda \ln^m(x-t) \delta_n(t) dt, & 1/n < x, \\ \int_{-1/n}^x (x-t)^\lambda \ln^m(x-t) \delta_n(t) dt, & -1/n \leq x \leq 1/n, \\ 0, & x < -1/n. \end{cases}$$

Then

$$[F_{m,\lambda}(x_+^\mu)]_n = \begin{cases} \int_{-1/n}^{1/n} (x^\mu - t)^\lambda \ln^m(x^\mu - t) \delta_n(t) dt, & 1/n < x^\mu, \\ \int_{-1/n}^{x^\mu} (x^\mu - t)^\lambda \ln^m(x^\mu - t) \delta_n(t) dt, & 0 \leq x^\mu \leq 1/n, \\ \int_{-1/n}^0 (-t)^\lambda \ln^m(-t) \delta_n(t) dt, & x < 0. \end{cases} \quad (4)$$

It follows that

$$\begin{aligned} \int_{-1}^1 x^k [F_{m,\lambda}(x_+^\mu)]_n dx &= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^\mu} (x^\mu - t)^\lambda \ln^m(x^\mu - t) \delta_n(t) dt dx \\ &\quad + \int_{n^{-1/\mu}}^1 x^k \int_{-1/n}^{1/n} (x^\mu - t)^\lambda \ln^m(x^\mu - t) \delta_n(t) dt dx \\ &\quad + \int_{-1}^0 x^k \int_{-1/n}^0 (-t)^\lambda \ln^m(-t) \delta_n(t) dt dx \\ &= \frac{n^{-(\lambda\mu+k+1)/\mu}}{\mu} \int_0^1 v^{(k+1)/\mu-1} \int_{-1}^v (v-u)^\lambda [\ln(v-u) - \ln n]^m \rho(u) du dv \\ &\quad + \frac{n^{-(\lambda\mu+k+1)/\mu}}{\mu} \int_{-1}^1 \rho(u) \int_1^n v^{(k+1)/\mu-1} (v-u)^\lambda [\ln(v-u) - \ln n]^m dv du \\ &\quad + n^{-\lambda} \int_{-1}^0 x^k \int_{-1}^0 (-u)^\lambda \ln^m(-u) \rho(u) du dx \\ &= I_1 + I_2 + I_3 \end{aligned} \quad (5)$$

where the substitutions $u = nt$ and $v = nx^\mu$ have been made.

It follows immediately that

$$\text{N-lim}_{n \rightarrow \infty} I_1 = \text{N-lim}_{n \rightarrow \infty} I_3 = 0 \quad (6)$$

for $k = 0, 1, 2, \dots$

Further,

$$\begin{aligned}
& \int_1^n v^{(k+1)/\mu-1} (v-u)^\lambda [\ln(v-u) - \ln n]^m dv \\
&= \sum_{s=0}^m \binom{m}{s} (-1)^{m-s} \ln^{m-s} n \int_1^n v^{(k+1)/\mu-1} (v-u)^\lambda \ln^s (v-u) dv \\
&= \sum_{s=0}^{m-1} \sum_{i=1}^s \binom{m}{s} \binom{s}{i} (-1)^{m-s} \ln^{m-s} n \int_1^n v^{(k+1)/\mu+\lambda-1} (1-u/v)^\lambda \\
&\quad \times \ln^i (1-u/v) \ln^{s-i} v dv \\
&\quad + \sum_{s=0}^{m-1} \binom{m}{s} (-1)^{m-s} \ln^{m-s} n \int_1^n v^{(k+1)/\mu+\lambda-1} (1-u/v)^\lambda \ln^s v dv \\
&\quad + \sum_{i=1}^m \binom{m}{i} \int_1^n v^{(k+1)/\mu+\lambda-1} (1-u/v)^\lambda \ln^i (1-u/v) \ln^{m-i} v dv \\
&\quad + \int_1^n v^{(k+1)/\mu+\lambda-1} (1-u/v)^\lambda \ln^m v dv \\
&= \sum_{s=0}^{m-1} \sum_{i=1}^s (-1)^{m-s+i} \binom{m}{s} \binom{s}{i} \ln^{m-s} n \int_1^n v^{(k+1)/\mu+\lambda-1} \\
&\quad \times \left[\frac{u^i}{v^i} + \left(\frac{i}{2} - \lambda \right) \frac{u^{i+1}}{v^{i+1}} + O(v^{-i-2}) \right] \ln^{s-i} v dv \\
&\quad + \sum_{s=0}^{m-1} \binom{m}{s} (-1)^{m-s} \ln^{m-s} n \int_1^n v^{(k+1)/\mu+\lambda-1} \\
&\quad \times \left[1 - \frac{\lambda u}{v} + O(v^{-2}) \right] \ln^s v dv \\
&\quad + \sum_{i=1}^m (-1)^i \binom{m}{i} \int_1^n v^{(k+1)/\mu+\lambda-1} \left[\frac{u^i}{v^i} + \left(\frac{i}{2} - \lambda \right) \frac{u^{i+1}}{v^{i+1}} + O(v^{-i-2}) \right] \ln^{m-i} v dv \\
&\quad + \int_1^n v^{(k+1)/\mu+\lambda-1} (1-u/v)^\lambda \ln^m v dv. \tag{7}
\end{aligned}$$

Using the lemma, it follows that

$$\begin{aligned}
& n^{-(k+1)/\mu-\lambda} \ln^{m-s} n \int_1^n v^{(k+1)/\mu+\lambda-1} \left[\frac{u^i}{v^i} + \left(\frac{i}{2} - \lambda \right) \frac{u^{i+1}}{v^{i+1}} + \dots \right] \ln^{s-i} v dv \\
&= O(n^{-i} \ln^{m-i} n) + cn^{(k+1)/\mu-\lambda} \ln^{m-s} n \tag{8}
\end{aligned}$$

for some constant c , for $i = 1, \dots, s$ and $s = 0, 1, \dots, m - 1$,

$$\begin{aligned} n^{-(k+1)/\mu-\lambda} \ln^{m-s} n \int_1^n v^{(k+1)/\mu+\lambda-1} \left[1 - \frac{\lambda u}{v} + \dots \right] \ln^s v \, dv \\ = O(n^{-1} \ln^m n) + P(\ln n) + dn^{(k+1)/\mu-\lambda} \ln^{m-s} n \end{aligned} \quad (9)$$

for some constant d , for $s = 0, \dots, m - 1$ where $P(\ln n)$ denotes a polynomial in $\ln n$ with positive powers,

$$\begin{aligned} n^{-(k+1)/\mu-\lambda} \int_1^n v^{(k+1)/\mu+\lambda-1} \left[\frac{u^i}{v^i} + \left(\frac{i}{2} - \lambda \right) \frac{u^{i+1}}{v^{i+1}} + \dots \right] \ln^{m-i} v \, dv \\ = O(n^{-i} \ln^{m-i} n) + en^{(k+1)/\mu-\lambda} \end{aligned} \quad (10)$$

for some constant e , with $i = 1, \dots, m$, and

$$\begin{aligned} n^{-(k+1)/\mu-\lambda} \int_1^n v^{(k+1)/\mu+\lambda-1} (1 - u/v)^\lambda \ln^m v \, dv = \frac{(-1)^m m! (1 - n^{-(k+1)/\mu+\lambda})}{[(k+1)/\mu + \lambda]^{m+1}} \\ + P(\ln n) + O(n^{-2} \ln^m n) \end{aligned} \quad (11)$$

where $P(\ln n)$, once again, denotes a polynomial in $\ln n$ with positive powers.

It now follows from equations (5) and (8) to (11) that

$$\text{N-}\lim_{n \rightarrow \infty} I_2 = \frac{(-1)^m m! \mu^m}{(\lambda \mu + k + 1)^{m+1}} \quad (12)$$

for $k = 0, 1, 2, \dots$

It now follows from equations (5), (6) and (12) that

$$\text{N-}\lim_{n \rightarrow \infty} \int_{-1}^1 x^k [F_{m,\lambda}(x_+^\mu)]_n \, dx = \frac{(-1)^m m! \mu^m}{(\lambda \mu + k + 1)^{m+1}} \quad (13)$$

for $k = 0, 1, 2, \dots$

We now consider the case $k = r$, where r is chosen so that $0 < \lambda \mu + r + 1 < 1$, and let ψ be an arbitrary continuous function. When $0 \leq x^\mu \leq 1/n$, we have

$$\begin{aligned} \int_0^{n^{-1/\mu}} x^r \psi(x) [F_{m,\lambda}(x_+^\mu)]_n \, dx \\ = \frac{n^{-(\lambda \mu + r + 1)/\mu}}{\mu} \int_0^1 \psi\left(\left(\frac{v}{n}\right)^{\frac{1}{\mu}}\right) v^{(r+1)/\mu-1} \int_{-1}^v (v-u)^\lambda [\ln(v-u) - \ln n]^m \rho(u) \, du \, dv \end{aligned}$$

and it follows that

$$\lim_{n \rightarrow \infty} \int_0^{n^{-1/\mu}} x^r \psi(x) [F_{m,\lambda}(x_+^\mu)]_n \, dx = 0. \quad (14)$$

When $x < 0$, we have

$$\int_{-1}^0 x^r \psi(x) [F_{m,\lambda}(x_+^\mu)]_n dx = n^{-\lambda} \int_{-1}^0 x^r \psi(x) \int_{-1}^0 (-u)^\lambda \ln^m(-u) \rho(u) du dx$$

and it follows that

$$\text{N-lim}_{n \rightarrow \infty} \int_{-1}^0 x^r \psi(x) [F_{m,\lambda}(x_+^\mu)]_n dx = 0. \quad (15)$$

When $x^\mu > 1/n$, we have

$$\begin{aligned} [F_{m,\lambda}(x_+^\mu)]_n &= \int_{-1/n}^{1/n} (x^\mu - t)^\lambda \ln^m(x^\mu - t) \delta_n(t) dt \\ &= \int_{-1}^1 (x^\mu - u/n)^\lambda \ln^m(x^\mu - u/n) \rho(u) du \\ &= x^{\lambda\mu} \int_{-1}^1 \left[\ln^m x^\mu - \frac{\lambda u \ln^m x^\mu}{n x^\mu} - \frac{m u \ln^{m-1} x^\mu}{n x^\mu} + O(n^{-2}) \right] \rho(u) du \\ &= \mu^m x^{\lambda\mu} \ln^m x + O(n^{-2}). \end{aligned} \quad (16)$$

Now let $\varphi(x)$ be an arbitrary function in \mathcal{D} with support contained in the interval $[-1, 1]$. By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^r}{r!} \varphi^{(r)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned} \langle [F_{m,\lambda}(x_+^\mu)]_n, \varphi(x) \rangle &= \int_{-1}^1 [F_{m,\lambda}(x_+^\mu)]_n \varphi(x) dx \\ &= \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 x^k [F_{m,\lambda}(x_+^\mu)]_n dx + \int_{n^{-1/\mu}}^1 \frac{x^r}{r!} [F_{m,\lambda}(x_+^\mu)]_n \varphi^{(r)}(\xi x) dx \\ &\quad + \int_0^{n^{-1/\mu}} \frac{x^r}{r!} [F_{m,\lambda}(x_+^\mu)]_n \varphi^{(r)}(\xi x) dx + \int_{-1}^0 \frac{x^r}{r!} [F_{m,\lambda}(x_+^\mu)]_n \varphi^{(r)}(\xi x) dx. \end{aligned}$$

Using equations (13) to (16), it follows that

$$\begin{aligned}
\text{N-}\lim_{n \rightarrow \infty} \langle [F_{m,\lambda}(x_+^\mu)]_n, \varphi(x) \rangle &= \sum_{k=0}^{r-1} \frac{(-1)^m m! \mu^m \varphi^{(k)}(0)}{k! (\lambda \mu + k + 1)^{m+1}} \\
&\quad + \mu^m \int_0^1 \frac{x^{\lambda \mu + r} \ln^m x}{r!} \varphi^{(r)}(\xi x) dx \\
&= \mu^m \int_0^1 x^{\lambda \mu} \ln^m x \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{x^k}{k!} \varphi^{(k)}(0) \right] dx \\
&\quad + \mu^m \sum_{k=0}^{r-1} \frac{(-1)^m m! \varphi^{(k)}(0)}{k! (\lambda \mu + k + 1)^{m+1}} \\
&= \mu^m \langle x_+^{\lambda \mu} \ln^m x_+, \varphi(x) \rangle,
\end{aligned}$$

on using equation (1). This proves equation (3) on the interval $[-1, 1]$. However, equation (3) clearly holds on any interval not containing the origin, and the proof is complete.

Theorem 6. *The neutrix composition $F_{m,\lambda}(|x|^\mu)$ exists and*

$$F_{m,\lambda}(|x|^\mu) = \mu^m |x|^{\lambda \mu} \ln^m |x| \quad (17)$$

for $-1 < \lambda < 0$, $\mu > 0$, $\lambda \mu \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$

Proof It follows from equation (4) that

$$[F_{m,\lambda}(|x|^\mu)]_n = \begin{cases} \int_{-1/n}^{1/n} (|x|^\mu - t)^\lambda \ln^m (|x|^\mu - t) \delta_n(t) dt, & 1/n < |x|^\mu, \\ \int_{-1/n}^{|x|^\mu} (|x|^\mu - t)^\lambda \ln^m (|x|^\mu - t) \delta_n(t) dt, & 0 \leq |x|^\mu \leq 1/n. \end{cases} \quad (18)$$

Since $[F_{m,\lambda}(|x|^\mu)]_n$ is an even function, it follows that

$$\int_{-1}^1 x^k [F_{m,\lambda}(|x|^\mu)]_n dx = 0 \quad (19)$$

for $k = 1, 3, \dots$

In general, we have

$$\begin{aligned}
\int_0^1 x^k [F_{m,\lambda}(|x|^\mu)]_n dx &= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^\mu} (x^\mu - t)^\lambda \ln^m (x^\mu - t) \delta_n(t) dt dx \\
&\quad + \int_{n^{-1/\mu}}^1 x^k \int_{-1/n}^{1/n} (x^\mu - t)^\lambda \ln^m (x^\mu - t) \delta_n(t) dt dx \\
&= I_1 + I_2
\end{aligned}$$

and it follows as above that

$$\text{N-}\lim_{n \rightarrow \infty} \int_{-1}^1 x^k [F_{m,\lambda}(|x|^\mu)]_n dx = \frac{2(-1)^m m! \mu^m}{(\lambda\mu + k + 1)^{m+1}} \quad (20)$$

for $k = 0, 2, 4, \dots$ since the integrand is even.

We now consider the case $k = 2r$, where r is chosen so that $0 < \lambda\mu + 2r + 1 < 2$, and let ψ be an arbitrary continuous function. Then it follows as above that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^{n^{-1/\mu}} x^{2r} \psi(x) [F_{m,\lambda}(|x|^\mu)]_n dx &= \lim_{n \rightarrow \infty} \int_{-n^{-1/\mu}}^0 x^{2r} \psi(x) [F_{m,\lambda}(|x|^\mu)]_n dx \\ &= 0 \end{aligned} \quad (21)$$

and

$$[F_{m,\lambda}(|x|^\mu)]_n = \mu^m |x|^{\lambda\mu} \ln^m |x| + O(n^{-2}) \quad (22)$$

if $|x|^\mu \geq 1/n$.

Again let $\varphi(x)$ be an arbitrary function in \mathcal{D} with support contained in the interval $[-1, 1]$. Then

$$\varphi(x) = \sum_{k=0}^{2r-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^{2r}}{(2r)!} \varphi^{(2r)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned} \langle [F_{m,\lambda}(|x|^\mu)]_n, \varphi(x) \rangle &= \int_{-1}^1 [F_{m,\lambda}(|x|^\mu)]_n \varphi(x) dx \\ &= \sum_{k=0}^{r-1} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \int_{-1}^1 x^{2k+1} [F_{m,\lambda}(|x|^\mu)]_n dx \\ &\quad + \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(2k)!} \int_{-1}^1 x^{2k} [F_{m,\lambda}(|x|^\mu)]_n dx \\ &\quad + \int_{n^{-1/\mu}}^1 \frac{x^{2r}}{(2r)!} [F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) dx \\ &\quad + \int_{-1}^{-n^{-1/\mu}} \frac{x^{2r}}{(2r)!} [F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) dx \\ &\quad + \int_0^{n^{-1/\mu}} \frac{x^{2r}}{(2r)!} [F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) dx \\ &\quad + \int_{-n^{-1/\mu}}^0 \frac{x^{2r}}{(2r)!} [F_{m,\lambda}(|x|^\mu)]_n \varphi^{(2r)}(\xi x) dx. \end{aligned}$$

Using equations (20) to (22), it follows that

$$\begin{aligned}
 \text{N-}\lim_{n \rightarrow \infty} \langle [F_{m,\lambda}(|x|^\mu)]_n, \varphi(x) \rangle &= \sum_{k=0}^{r-1} \frac{2(-1)^m m! \mu^m \varphi^{(2k)}(0)}{(2k)! (\lambda\mu + 2k + 1)^{m+1}} \\
 &\quad + \mu^m \int_{-1}^1 \frac{|x|^{\lambda\mu+2r}}{(2r)!} \ln^m |x| \varphi^{(2r)}(\xi x) dx \\
 &= \mu^m \int_{-1}^1 |x|^{\lambda\mu} \ln^m |x| \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{x^{2k}}{(2k)!} \varphi^{(2k)}(0) \right] dx \\
 &\quad + \mu^m \sum_{k=0}^{r-1} \frac{2(-1)^m m! \varphi^{(2k)}(0)}{(2k)! (\lambda\mu + 2k + 1)^{m+1}} \\
 &= \mu^m \langle |x|^{\lambda\mu} \ln^m |x|, \varphi(x) \rangle,
 \end{aligned}$$

on using equation (2). This proves equation (17) on the interval $[-1, 1]$. However, equation (17) clearly holds on any interval not containing the origin, and the proof is complete.

References

- [1] J.G. van der Corput, *Introduction to the neutrix calculus*, J. Analyse Math., **7**(1959), 291-398.
- [2] B. Fisher, *On defining the change of variable in distributions*, Rostock. Math. Kolloq., **28**(1985), 75-86.
- [3] B. Fisher, *On defining the distribution $(x_+^r)^{-s}$* , Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., **15**(1985), 119-129.
- [4] B. Fisher and J. Nicholas, *Some results on the composition of distributions*, Novi Sad J. Math., **32**(2)(2002), 87-94.
- [5] B. Fisher and K. Taş, *On the composition of the distributions x_+^λ and x_+^μ* , J. Math. Anal. Appl., **318**(2006), 102-111.
- [6] I.M. Gel'fand and G.E. Shilov, "Generalized Functions", Vol. I, Academic Press, 1964.