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DIRECT SUMS OF LIFTING MODULES

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Abstract

This paper is concerned with when a direct sum of lifting modules is lifting. For example, it is proved that for any ring R, the direct sum $M = \oplus_{i\in I} M_i$ is lifting if and only if M is amply supplemented and there exists $i \in I$ such that every coclosed submodule K of M with $M = K+M_i$ or $M = K + M(I - i)$ is a direct summand of M. In addition, we prove that for any right perfect ring R, the right R-module $M = M_1 \oplus M_2$ is lifting if M_1 is a lifting right R-module and M_2 is a semisimple right R-module such that M_2 is N-projective for every proper submodule N of M_1.

1 Introduction

In this paper all rings are associative with identity element and all modules are unital right modules. Let R be a ring and let M be an R-module. $A \leq M$ ($A \ll M$) means that A is a *submodule* (*small submodule*) of M. Let $A \leq B \leq M$. A is called a *coessential submodule* of B in M if $B/A \ll M/A$. A submodule K of M is called *coclosed* in M if K has no proper coessential submodule in M. We will call A an *s-closure* or *coclosure* of B in M if A is a coessential submodule of B in M and A is coclosed in M.

Let M be a module and let N and K be submodules of M. N is called a *supplement* of K in M if it is minimal with respect to $M = N+K$, equivalently, $M = N + K$ and $N \cap K \ll N$. M is called *supplemented* if every submodule of M has a supplement in M and is called *amply supplemented* if for any two submodules A and B of M with $M = A+B$, B contains a supplement of A in

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M. Every homomorphic image of any amply supplemented module is amply supplemented. Let M be a module and let N and K be submodules of M. N is called a *weak supplement* of K in M if $M = N + K$ and $N \cap K \ll M$. M is called *weakly supplemented* if every submodule of M has a weak supplement in M. It is clear that every amply supplemented module is supplemented and every supplemented module is weakly supplemented. A submodule K of any module M is called a *supplement submodule* of M if there exists a submodule N of M such that K is a supplement of N in M. It is well known that if K is a supplement submodule of any module M, then K is a coclosed submodule of M. If M is weakly supplemented, then every coclosed submodule of M is a supplement submodule of M (see [4, Lemma 1.1]). Also we note that any module M is amply supplemented if and only if M is weakly supplemented and every submodule of M has an s-closure in M (see [4, Lemma 1.7]).

Let M be a module. M is called a *lifting* module (or satisfies (D_1)) if for any submodule N of M, there exists a direct summand K of M such that $K \subseteq N$ and $N/K \ll M/K$, equivalently, for every submodule N of M there exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N \cap K' \ll K'$. By [7, Proposition 4.8], the module M is lifting if and only if M is amply supplemented and every supplement (namely coclosed) submodule of M is a direct summand of M. Lifting modules have been extensively studied in recent years (see, for example, [1]-[6] and [8]).

One of the most interesting questions concerning lifting modules is when a (finite or infinite) direct sum of lifting modules is also lifting. Even for the ring $\mathbb{Z}$ of rational integers, the situation is very interesting. For example, for any prime p, the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ is lifting but not the module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ (see [2]).

2 Arbitrary Direct Sums

Let R be any ring. Let M be an R-module such that $M = \oplus_{i \in I} M_i$ is the direct sum of lifting R-modules M_i $(i \in I)$, for some given index set I. We are interested in when M itself is a lifting module. First we recall the following result [4, Lemma 1.4].

Lemma 2.1. *Let M be a weakly supplemented module and $B \subseteq C$ submodules of M such that C/B is coclosed in M/B and B is coclosed in M. Then C is coclosed in M.*

For any set I, $\mid I \mid$ will denote its cardinality. Let $M = \oplus_{i \in I} M_i$. $M(I - i)$ will denote the direct sum $\oplus_{i \neq j \in I} M_j$.

Theorem 2.2. *Let R be any ring and let $M = \oplus_{i \in I} M_i$ be the direct sum of R-modules M_i $(i \in I)$, for some index set I with $\mid I \mid \geq 2$. Assume M is amply supplemented. Then the following are equivalent.*

(i) *M is lifting.*

(ii) *There exists $i \in I$ such that every coclosed submodule K of M with $M = K + M_i$ or $M = K + M(I - i)$ is a direct summand of M.*

(iii) *There exists $i \in I$ such that for every supplement K of M_i or $M(I - i)$ in M, M/K is lifting and K is a direct summand of M.*

(iv) *There exists $i \in I$ such that every coclosed submodule K of M with $(K + M_i)/K \ll M/K$ or $(K + M(I - i))/K \ll M/K$ or $M = K + M_i = K + M(I - i)$ is a direct summand of M.*

Proof $(i) \Leftrightarrow (ii) \Leftrightarrow (iv)$: It is a simple corollary of [4, Theorem 2.1].

$(ii) \Rightarrow (iii)$: Let K be a supplement of M_i in M, namely, $M = K + M_i$ and $K \cap M_i \ll K$. By (ii), K is a direct summand of M. Since M is amply supplemented, M/K is amply supplemented. Let T/K be a coclosed submodule of M/K. Then T is a coclosed submodule of M by Lemma 2.1. By (ii), T is a direct summand of M. Therefore T/K is a direct summand of M/K. Thus M/K is lifting. By the same proof, every supplement K of $M(I-i)$ in M is a direct summand of M and M/K is lifting.

$(iii) \Rightarrow (i)$: Let N be a coclosed submodule of M. Since M/N is amply supplemented, $(N + M_i)/N$ has an s-closure in M/N, namely, there exists a submodule H/N of M/N such that H/N is a coessential submodule of $(N + M_i)/N$ in M/N and H/N is coclosed in M/N. Then $(N+M_i)/H \ll M/H$ and so $M = H + M(I - i)$. By Lemma 2.1, H is coclosed in M. Since M is amply supplemented, there exists a submodule P of M such that $M = P + M(I - i)$ and $P \cap M(I - i) \ll P \leq H$. By (iii), P is a direct summand of M and M/P is lifting. There exists a submodule P' of M such that $M = P \oplus P'$. Since H is coclosed in M, H/P is coclosed in M/P. So, H/P is a direct summand of M/P since M/P is lifting. Then there exists a submodule H' of M with $P \subseteq H'$ such that $M/P = H/P \oplus H'/P$. Then $M = H + H'$ and $P = H \cap H'$. $M = P \oplus P'$ implies that $H' = P \oplus (H' \cap P')$. Then $M = H + P + (H' \cap P') = H \oplus (H' \cap P')$. Let $H'' = H' \cap P'$. So, H is a direct summand of M. Now $((N \oplus H'')/N) \oplus (H/N) = M/N$ because $N = (N \oplus H'') \cap H$. By Lemma 2.1, $N \oplus H''$ is coclosed in M. Clearly, $M = (N \oplus H'') + M_i$. Since M is amply supplemented, there exists a submodule P_1 of M such that $M = P_1 + M_i$ and $P_1 \cap M_i \ll P_1 \subseteq N \oplus H''$. By (iii), M/P_1 is lifting and $M = P_1 \oplus P_1'$ for some submodule P_1' of M. Since $(N \oplus H'')/P_1$ is coclosed in M/P_1, $((N \oplus H'')/P_1) \oplus (P_2/P_1) = M/P_1$ for some submodule P_2 of M with $P_1 \subseteq P_2$. Then $N \oplus H''$ is a direct summand of M and hence N is a direct summand of M. Thus M is lifting. $\square$

Let M_1 and M_2 be modules. The module M_1 is small M_2-projective if every homomorphism $f : M_1 \to M_2/A$, where A is a submodule of M_2 and $Imf \ll M_2/A$, can be lifted to a homomorphism $\varphi : M_1 \to M_2$. The modules

M_1 and M_2 are relatively small projective if M_i is small M_j-projective for every $i, j \in \{1, 2\}, i \neq j$.

Corollary 2.3. *Let $M = \oplus_{i\in I} M_i$ be the direct sum of modules M_i $(i \in I)$, for some index set I with $\mid I \mid \geq 2$. Assume M is amply supplemented. If there exists $i \in I$ such that M_i and $M(I - i)$ are relatively small projective, then M is lifting if and only if M_i and $M(I - i)$ are lifting and any coclosed submodule K of M with $M = K + M_i = K + M(I - i)$ is a direct summand of M.*

Proof Necessity: Clear.

Sufficiency: Let K be a coclosed submodule of M. If $(K + M_i)/K \ll M/K$ or $(K + M(I - i))/K \ll M/K$, by [4, Lemma 2.7], K is a direct summand of M. If $M = K + M_i = K + M(I - i)$, by hypothesis, K is a direct summand of M. Thus by Theorem 2.2(iv), M is lifting. □

We recall the folowing result [3, Proposition 2.10].

Proposition 2.4. *Let $M = M_1 \oplus M_2$ be a direct sum of modules M_1 and M_2.*

(1) *Let K be a supplement of M_1 in M and $f : P \to M_2$ an epimorphism, where P is a projective module. Then $K = \{f(p) - \varphi(p) \mid p \in P\}$ for some homomorphism $\varphi : P \to M_1$.*

(2) *Let $f : P \to M_2$ be a projective cover of M_2. Then $K = \{f(p) - \varphi(p) \mid p \in P\}$ is a supplement of M_1 in M for any homomorphism $\varphi : P \to M_1$.*

We know that every module need not have a projective cover. Let R be a ring. R is called right perfect if every right R-module has a projective cover. By [7, Theorem 4.41], any ring R is right perfect if and only if every right R-module is amply supplemented.

Theorem 2.5. *Let R be a right perfect ring and let $M = \oplus_{i\in I} M_i$ be the direct sum of right R-modules M_i $(i \in I)$, for some index set I with $\mid I \mid \geq 2$. Then the following are equivalent.*

(i) *M is lifting.*

(ii) *There exists $i \in I$ such that for any homomorphisms $\varphi : P \to M_i$ and $\psi : Q \to M(I - i)$, the submodules $L = \{f(p) - \varphi(p) \mid p \in P\}$ and $K = \{g(q) - \psi(q) \mid q \in Q\}$ are direct summands of M and M/L and M/K are lifting, where $f : P \to M(I-i)$ is a projective cover of $M(I-i)$ and $g : Q \to M_i$ is a projective cover of M_i.*

Proof. *(i)* $\Rightarrow$ *(ii)*: By Theorem 2.2, there exists $i \in I$ such that for every supplement K of M_i or $M(I - i)$ in M, M/K is lifting and K is a direct summand of M. Let $\varphi : P \to M_i$ and $\psi : Q \to M(I - i)$ be homomorphisms. Assume $f : P \to M(I - i)$ is a projective cover of $M(I - i)$ and $g : Q \to M_i$

is a projective cover of M_i. By Proposition 2.4, $L = \{f(p) - \varphi(p) \mid p \in P\}$ is a supplement of M_i in M and $K = \{g(q) - \psi(q) \mid q \in Q\}$ is a supplement of $M(I - i)$ in M. Therefore M/K and M/L are lifting and K and L are direct summands of M.

$(ii) \Rightarrow (i)$: Let K be a supplement of $M(I - i)$ in M. Then $K = \{g(q) - \psi(q) \mid q \in Q\}$ for some homomorphism $\psi : Q \to M(I-i)$ by Proposition 2.4. By (ii), K is a direct summand of M and M/K is lifting. Let L be a supplement of M_i in M. Then $L = \{f(p)-\varphi(p) \mid p \in P\}$ for some homomorphism $\varphi : P \to M_i$ by Proposition 2.4. By (ii), L is a direct summand of M and M/L is lifting. Thus by Theorem 2.2(iii), M is lifting. □

3 UCC-Modules

Ganesan and Vanaja define the UCC-modules in [1] as follows.

Definition 3.1. A module M is called a *unique coclosure module* (denoted by UCC-module) if every submodule of M has a unique coclosure in M. Hollow modules and semisimple modules are UCC-modules. By [4, Lemma 1.7], every weakly supplemented UCC-module is amply supplemented.

Theorem 3.2. *Let $M = \oplus_{i\in I} M_i$ be a weakly supplemented UCC-module with $\mid I \mid\geq 2$. Then the folowing are equivalent.*

(i) *M is lifting.*

(ii) *There exists $i \in I$ such that $M(I - i)$ is lifting and every coclosed submodule K of M with $M = K + M_i$ is a direct summand of M.*

(iii) *There exists $i \in I$ such that $M(I - i)$ is lifting and for every supplement K of M_i in M, M/K is lifting and K is a direct summand of M.*

(iv) *There exist $i \neq j$ in I such that $M(I - i)$ is lifting and every coclosed submodule K of M with $(K + M_j)/K \ll M/K$ or $M = K + M_j$ is a direct summand of M.*

(v) *There exist $i \neq j$ in I such that $M(I - i)$ is lifting and every coclosed submodule K of M with $(K+M_j)/K \ll M/K$ or $M = K+M_j = K+M_i$ is a direct summand of M.*

Proof $(i) \Rightarrow (ii)$, $(i) \Rightarrow (iv)$, and $(iv) \Rightarrow (v)$ are clear.

$(ii) \Rightarrow (iii)$: By (ii), there exists $i \in I$ such that every coclosed submodule K of M with $M = K+M_i$ is a direct summand of M and $M(I-i)$ is lifting. Let K be a supplement of M_i in M. By (ii), K is a direct summand of M. By [4, Lemma 1.7], M is amply supplemented. Hence M/K is amply supplemented. Let N/K be coclosed in M/K. By Lemma 2.1, N is coclosed in M, and clearly

$M = N + M_i$. By (ii), N is a direct summand of M. There exists N' of M such that $M = N \oplus N'$. Now $M/K = N/K \oplus (N' \oplus K)/K$ since $K = N \cap (N' \oplus K)$. Therefore M/K is lifting.

$(iii) \Rightarrow (i)$: Let H be a coclosed submodule of M. By [1, Proposition 3.14 and Theorem 3.16], $(H + M_i)/M_i$ is coclosed in M/M_i. Since M/M_i is lifting, $(H + M_i)/M_i$ is a direct summand of M/M_i and hence $H + M_i$ is a direct summand of M. Then $M = (H + M_i) \oplus H'$ for some submodule H' of M. Now $M/H = ((H + M_i)/H) \oplus ((H' \oplus H)/H)$ since $H = (H + M_i) \cap (H' \oplus H)$. By Lemma 2.1, $H \oplus H'$ is coclosed in M. Since $M = (H' \oplus H) + M_i$ and M is amply supplemented, there exists a submodule P of M such that $M = P + M_i$ and $P \cap M_i \ll P \subseteq H' \oplus H$. By hypothesis, P is a direct summand of M and M/P is lifting. Since $H \oplus H'$ is coclosed in M, $(H \oplus H')/P$ is coclosed in M/P. Therefore $(H \oplus H')/P$ is a direct summand of M/P. Thus M is lifting because H is a direct summand of M.

$(v) \Rightarrow (ii)$: Let K be a coclosed submodule of M with $M = K + M_i$. If $M = K + M_j$, then by (v), K is a direct summand of M. Assume that $M \neq K + M_j$. Now by [4, Lemma 1.7], $(K + M_j)/K$ has an s-closure in M/K, namely, there exists a submodule H/K of M/K such that H/K is a coessential submodule of $(K + M_j)/K$ in M/K and H/K is coclosed in M/K. By Lemma 2.1, H is coclosed in M. Since $H + M_j = K + M_j$, $(H + M_j)/H \ll M/H$. Then by (v), H is a direct summand of M. Now $M = H \oplus H'$ for some submodule H' of M. Since $K = (K \oplus H') \cap H$, $M/K = ((K \oplus H')/K) \oplus (H/K)$. Then $K \oplus H'$ is coclosed in M by Lemma 2.1. Since $M = (K \oplus H') + M_i = (K \oplus H') + M_j$, $K \oplus H'$ is a direct summand of M by (v). Thus K is a direct summand of M. □

Corollary 3.3. *Let $M = \oplus_{i \in I} M_i$ be a weakly supplemented UCC-module with $| I | \geq 2$. If there exist $i \neq j \in I$ such that M_j is small $M(I - j)$-projective, then M is lifting if and only if $M(I - j)$ and $M(I - i)$ are lifting and any coclosed submodule K of M with $M = K + M_i = K + M_j$ is a direct summand of M.*

Proof Necessity: Clear.

Sufficiency: Let K be a coclosed submodule of M. If $(K + M_j)/K \ll M/K$, K is a direct summand of M by [4, Lemma 2.7]. If $M = K + M_i = K + M_j$, K is a direct summand of M by hypothesis. Thus M is lifting by Theorem 3.2(v). □

Theorem 3.4. *Let R be a right perfect ring. Let M be a UCC right R-module such that $M = \oplus_{i \in I} M_i$ is the direct sum of right R-modules M_i $(i \in I)$ with $| I | \geq 2$. Then the following are equivalent.*

(i) *M is lifting.*

(ii) *There exists $i \in I$ such that $M(I-i)$ is lifting and for each homomorphism $\varphi : P \to M_i$ the submodule $K = \{f(p)-\varphi(p) \mid p \in P\}$ is a direct summand of M and M/K is lifting, where $f : P \to M(I-i)$ is a projective cover of $M(I-i)$.*

Proof $(i) \Rightarrow (ii)$: By Theorem 3.2, there exists $i \in I$ such that $M(I-i)$ is lifting and for every supplement K of M_i in M, M/K is lifting and K is a direct summand of M. Let $\varphi : P \to M_i$ be a homomorphism and $f : P \to M(I-i)$ be a projective cover of $M(I-i)$. Then $K = \{f(p)-\varphi(p) \mid p \in P\}$ is a supplement of M_i by Proposition 2.4. So, K is a direct summand of M and M/K is lifting.

$(ii) \Rightarrow (i)$: Let K be a supplement of M_i in M. Then $K = \{f(p) - \varphi(p) \mid p \in P\}$ for some homomorphism $\varphi : P \to M_i$. By (ii), K is a direct summand of M and M/K is lifting. Thus M is lifting by Theorem 3.2(iii). □

4 Modules With Semisimple Summands

As we remarked before in this note, for any prime number p, the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ is lifting, but the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not lifting. Note that $\mathbb{Z}/p\mathbb{Z}$ is a simple $\mathbb{Z}$-module. From this it is natural to ask for which modules M_1 and semisimple modules M_2 is the module $M_1 \oplus M_2$ lifting. First we recall the following theorem.

Theorem 4.1. ([2, Theorem 6]) *Let the module $M = M_1 \oplus M_2$ be a direct sum of relatively projective modules M_1 and M_2 such that M_1 is semisimple and M_2 is lifting. Then M is lifting.*

Proposition 4.2. *Let M be a UCC-module such that $M = M_1 \oplus M_2$ is the direct sum of any module M_1 and a semisimple module M_2. Then M is lifting if and only if M_1 is lifting.*

Proof Necessity: Clear.

Sufficiency: Assume M_1 is lifting. Then by [9, 41.2], M is weakly supplemented. Let K be a supplement of M_2 in M. Then it is easy to check that $K \cap M_2 = 0$ and so, K is a direct summand of M. Since $M/K \cong M_2$ and M_2 is lifting, M/K is lifting. Thus M is lifting by Theorem 3.2(iii). □

Lemma 4.3. *Let M_1 and M_2 be modules with M_2 semisimple. Then $M = M_1 \oplus M_2$ is lifting if and only if M is amply supplemented and for every supplement K of M_1 in M, M/K is lifting and K is a direct summand of M.*

Proof By Theorem 2.2(iii) and the proof of Proposition 4.2. □

Theorem 4.4. *Let R be a right perfect ring, let M_1 be a lifting right R-module and let M_2 be a semisimple right R-module such that M_2 is N-projective for every proper submodule N of M_1. Then the right R-module $M = M_1 \oplus M_2$ is lifting.*

Proof Assume that $f : P \to M_2$ is a projective cover of M_2. Let K be a supplement of M_1 in M. Then by Proposition 2.4, $K = \{f(p) - \varphi(p) \mid p \in P\}$ for some homomorphism $\varphi : P \to M_1$. Let $\pi_1 : M_1 \to M/K$, $\pi_1(m_1) = m_1 + K$ and $\pi_2 : M_2 \to M/K$, $\pi_2(m_2) = m_2 + K$. If we check the proof of [3, Proposition 2.10(1)], we see that $\pi_2 f = \pi_1 \varphi$. Therefore $f^{-1}(Ker\pi_2) = f^{-1}(K \cap M_2) = \varphi^{-1}(Ker\pi_1) = \varphi^{-1}(K \cap M_1)$, namely, $Ker\varphi \subseteq f^{-1}(K \cap M_2)$. Suppose $Im\varphi = M_1$. Then $M = K + M_2$. Since M_2 is semisimple, $K \cap M_2$ is a direct summand of M_2. There exists a submodule T of M such that $M_2 = (K \cap M_2) \oplus T$. Then $M = K + (K \cap M_2) + T = K \oplus T$. Therefore K is a direct summand of M. Now $M/K \cong T$ and T is a semisimple module. Thus M/K is also lifting.

Now suppose that $Im\varphi \neq M_1$. By hypothesis, M_2 is $Im\varphi$-projective. Then $(M_2 + K)/K$ is $(P/Ker\varphi)$-projective. Define the epimorphism $\alpha : P/Ker\varphi \to (M_2 + K)/K$, $\alpha(p + Ker\varphi) = f(p) + K$ $(p \in P)$. Therefore $Ker\alpha = f^{-1}(K \cap M_2)/Ker\varphi$ is a direct summand of $P/Ker\varphi$. There exists a submodule H of P with $Ker\varphi \subseteq H$ such that $P/Ker\varphi = (f^{-1}(K \cap M_2)/Ker\varphi) \oplus (H/Ker\varphi)$. So $P = f^{-1}(K \cap M_2) + H$ and $Ker\varphi = H \cap f^{-1}(K \cap M_2)$. Define $\theta : P \to M_1$, $\theta(x + h) = \varphi(h)$, where $x \in f^{-1}(K \cap M_2)$ and $h \in H$. Clearly θ is well defined. Let $K' = \{f(p) - \theta(p) \mid p \in P\}$. It is easy to check that $M = K' \oplus M_1$. We claim that $K' \subseteq K$. Let $f(p) - \theta(p) \in K'$, where $p \in P$ and $p = x + h$, where $x \in f^{-1}(K \cap M_2)$ and $h \in H$. Then $f(p) - \theta(p) = f(x) + f(h) - \varphi(h) \in K$, since $f(h) - \varphi(h) \in K$ and $f(x) \in K \cap M_2$. Since K is a supplement of M_1 in M, $K = K'$ and hence K is a direct summand of M. Since $M/K \cong M_1$ and M_1 is lifting, M/K is lifting. By Lemma 4.3, M is lifting. □

Note that we can give the following alternative proof to Theorem 4.4:

Proof Note that M is amply supplemented because R is a right perfect ring. Since M_2 is N-projective for evey proper submodule N of M_1, M_2 is X-projective for every small submodule X of M_1. Now we claim that M_2 is small M_1-projective. For, let $A \leq M_1$ and $f : M_2 \longrightarrow M_1/A$ be a homomorphism such that $\mathrm{Im} f = T/A \ll M_1/A$. Since M is amply supplemented, A has a supplement B in M_1. Let $B/(A \cap B) = (T \cap B)/(A \cap B) + L/(A \cap B)$ for any submodule $L/(A \cap B)$ of $B/(A \cap B)$. Then $M_1 = A + B = A + (T \cap B) + L = T + L$. Since $T/A \ll M_1/A$, $M_1/A = (L + A)/A$ and hence $M_1 = L + A$. By the minimality of B in M_1, $L = B$. Therefore $(T \cap B)/(A \cap B) \ll B/(A \cap B)$. Now $T \cap B \ll B$. Let $X = T \cap B$. Since $T = A + X$, we define the epimorphism $\varphi : X \longrightarrow T/A$ such that $\varphi(x) = x + A$, where $x \in X$. So there exists a homomorphism $\alpha : M_2 \longrightarrow X$ such that $\varphi\alpha = f$. Hence f can be lifted to the homomorphism $i\alpha$, where $i : X \longrightarrow M_1$ is the inclusion map. Therefore M_2 is small M_1-projective. By [4, Theorem 2.8], M is lifting. $\square$

Let R be any ring and let M be an R-module. The module M is called *small* if $M \ll E(M)$, where $E(M)$ is the injective hull of M. In [8], Talebi and Vanaja define $\overline{Z}(M)$=$\bigcap\{Kerg \mid g : M \to N$ and N is small$\}$. They call M *cosingular* if $\overline{Z}(M) = 0$ and *noncosingular* if $\overline{Z}(M) = M$.

Let M be a module. Talebi and Vanaja define $\overline{Z}^0(M) = M$, $\overline{Z}^1(M) = \overline{Z}(M)$ and define inductively $\overline{Z}^\alpha(M)$ for any ordinal α. Thus, if α is not a limit ordinal they set $\overline{Z}^\alpha(M) = \overline{Z}(\overline{Z}^{\alpha-1}(M))$, while if α is a limit ordinal they set $\overline{Z}^\alpha(M) = \bigcap_{\beta<\alpha} \overline{Z}^\beta(M)$. This gives the descending sequence $M = \overline{Z}^0(M) \supseteq \overline{Z}(M) \supseteq \overline{Z}^2(M) \supseteq ...$ of submodules of M (see [8]).

Let M be a module. Talebi and Vanaja prove in [8, Theorem 4.1] that M is lifting if and only if $M = \overline{Z}^2(M) \oplus N$ for some submodule N of M such that N and $\overline{Z}^2(M)$ are lifting, N is $\overline{Z}^2(M)$-projective and M is amply supplemented.

Lemma 4.5. *Let M_1 be a lifting module and let M_2 be a semisimple module. Suppose that $M_1 = M_{11} \oplus M_{12}$ and that $M_2 = M_{21} \oplus M_{22}$, where M_{11} and M_{21} are noncosingular and M_{12} and M_{22} are cosingular. Then the module $M = M_1 \oplus M_2$ is lifting if and only if M is amply supplemented, $M_{12} \oplus M_{22}$ is lifting and M_{11}-projective.*

Proof Assume that $M = M_1 \oplus M_2$ is lifting. Then M is amply supplemented and $M_{12} \oplus M_{22}$ is lifting. By [8, Corollary 2.2 and Proposition 2.4], $M_{11} \oplus M_{21}$ is noncosingular and $M_{12} \oplus M_{22}$ is cosingular. Then $\overline{Z}^2(M) = \overline{Z}^2(M_{11} \oplus M_{21}) \oplus \overline{Z}^2(M_{12} \oplus M_{22}) = M_{11} \oplus M_{21}$ implies that $M = \overline{Z}^2(M) \oplus M_{12} \oplus M_{22}$. By [8, Theorem 4.1], $M_{12} \oplus M_{22}$ is $\overline{Z}^2(M) = M_{11} \oplus M_{21}$-projective and so $M_{12} \oplus M_{22}$ is M_{11}-projective.

Conversely, assume that M is amply supplemented, $M_{12} \oplus M_{22}$ is lifting and M_{11}-projective. $M_{11} \oplus M_{21}$ is UCC by [1, Proposition 4.2]. Now by Proposition 4.2, since M_{21} is semisimple and M_{11} is lifting, $M_{11} \oplus M_{21}$ is lifting. Also, $M_{12} \oplus M_{22}$ is $M_{11} \oplus M_{21}$-projective. Therefore $M = \overline{Z}^2(M) \oplus M_{12} \oplus M_{22} = M_{11} \oplus M_{21} \oplus M_{12} \oplus M_{22} = M_1 \oplus M_2$ is lifting by [8, Theorem 4.1]. $\square$

Let R be a ring. Consider the following (*) property:

(*) Every cosingular right R-module is projective.

Corollary 4.6. *Let R satisfy (*). Let M_1 be a lifting right R-module and M_2 be a semisimple right R-module. Suppose that $M = M_1 \oplus M_2$ is amply supplemented. Then M is lifting.*

Proof By [8, Theorem 3.8(4)], $M_1 = M_{11} \oplus M_{12}$ and $M_2 = M_{21} \oplus M_{22}$, where M_{11} and M_{21} are noncosingular and M_{12} and M_{22} are cosingular. Since

$M_{12} \oplus M_{22}$ is cosingular, by hypothesis, $M_{12} \oplus M_{22}$ is projective and so it is M_{11}-projective. Since $M_{12} \oplus M_{22}$ is projective and amply supplemented, $M_{12} \oplus M_{22}$ is lifting by [3, Proposition 2.3]. Hence M is lifting by Lemma 4.5 □

Let M be a module. M is called *quasi-discrete* if M is lifting and for any direct summands M_1 and M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of M.

Corollary 4.7. *Let R satisfy (*). Let M_1 be a quasi-discrete right R-module and M_2 be a semisimple right R-module. Suppose that $M = M_1 \oplus M_2$ is amply supplemented. Then M is lifting.*

Proof By [8, Theorem 3.8(4)], $M_1 = M_{11} \oplus M_{12}$ and $M_2 = M_{21} \oplus M_{22}$, where M_{11} and M_{21} are noncosingular and M_{12} and M_{22} are cosingular. Since M_1 is quasi-discrete, by [7, Lemma 4.23], M_{12} is M_{11}-projective. By assumption, M_{22} is projective, and so it is M_{11}-projective and M_{12}-projective. Then $M_{12} \oplus M_{22}$ is M_{11}-projective and by Theorem 4.1, $M_{12} \oplus M_{22}$ is lifting. Thus by Lemma 4.5, M is lifting. □

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