

## JORDAN DERIVATIONS ON LIE IDEALS OF PRIME AND SEMIPRIME RINGS

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### Abstract

We prove the following theorem: Let  $R$  be a 2-torsion free semiprime ring and  $U$  a Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $\prime$  is an additive mapping of  $R$  into itself satisfying  $(u^2)' = u'u + uu'$  and  $u' \in U$  for all  $u \in U$  then  $(uv)' = u'v + uv'$  for all  $u, v \in U$ . We also give a short and elementary proof of a theorem of Awtar [1] which extends a well known result of Herstein [5] on Lie ideals.

Throughout this paper  $R$  will denote an associative ring and  $U$  a Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . So  $uv + vu = (u + v)^2 - (u^2 + v^2)$  and  $2uv = (uv + vu) + [u, v]$  are in  $U$  for all  $u, v \in U$ . Let  $\prime$  be an additive mapping of  $R$  into itself such that  $(u^2)' = u'u + uu'$  for all  $u \in U$ . From this we obtain.

**Lemma 1.** *If  $U$  is Lie ideal of a ring  $R$  then*

$$(uv + vu)' = u'v + uv' + v'u + vu' \text{ for all } u, v \in U.$$

Let  $u^v = (uv)' - u'v - uv'$ . Then  $u^{v+w} = u^v + u^w$ ,  $(u + v)^w = u^w + v^w$  and  $u^v + v^u = 0$  for all  $u, v, w \in U$ .

If  $R$  is a 2-torsion free semiprime ring,  $U$  a commutative Lie ideal of  $R$  and  $u \in U$  then  $[u, [u, r]] = 0$  for all  $r \in R$ . By sublemma on p.5 of [5],  $u \in Z$ , the

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centre of  $R$ . Hence  $U \subseteq Z$ . By Lemma 1,  $(uv)' = u'v + uv'$  for all  $u, v \in U$ . Thus we assume that  $U \not\subseteq Z$ .

**Lemma 2.** *If  $U$  is a Lie ideal of a 2-torsion free ring  $R$  then*

1.  $(uvu)' = u'vu + uv'u + uvu'$
2.  $(uvw+vvu)' = u'vw+uv'w+uvw'+w'vu+uv'u+vvu'$  for all  $u, v, w \in U$ .

**Proof** (i) Using Lemma 1, evaluate  $(u(uv + vu) + (uv + vu)u)'$   
 $= (u^2v + vu^2 + 2uvu)'$ .

(ii) Linearize  $u$  in (i).

**Lemma 3.** *If  $U$  is a Lie ideal of a 2-torsion free ring  $R$  then  $u^v x[u, v] + [u, v]xu^v = 0$  for all  $u, v, x \in U$ .*

**Proof** If  $u, v \in U$  then  $2uv \in U$ . Since  $R$  is 2-torsion free, without loss of generality we can assume that  $uv \in U$ . Now we evaluate  $(uvxvu + vuxuv)'$  in two ways. By using Lemma 2, we have

$$\begin{aligned} (u(vxv)u + v(uxu)v)' &= u'(vxv)u + u(v'xv + vx'v + vxv')u + u(vxv)u' \\ &\quad + v'(uxu)v + v(u'xu + ux'u + uxu')v + v(uxu)v' \\ &= (uv)'xvu + uvx'vu + uvx(vu)' + (vu)'xuv \\ &\quad + vux'uv + vux(uv)'. \end{aligned}$$

By comparing and using  $u^v + v^u = 0$ , we obtain the result.

**Lemma 4** ([5], p. 5). *If  $U$  is a Lie ideal of a ring  $R$  then  $T(U) = \{a \in R : [a, R] \subseteq U\}$  is both a subring and a Lie ideal of  $R$  containing  $U$ .*

Now we prove the following Lemma which is given in [2] for prime rings.

**Lemma 5.** *If  $R$  is a 2-torsion free semiprime ring and  $U$  a Lie ideal of  $R$  then there exists a non zero ideal  $M = R[U, U]R$  of  $R$  generated by  $[U, U]$  such that  $[M, R] \subseteq U$ .*

**Proof** We see that  $[U, U] \neq 0$ . Suppose  $[U, U] = 0$ . Let  $u \in U$ . Then  $[u, [u, r]] = 0$  for all  $r \in R$ . By sublemma on p. 5 of [5],  $u \in Z$ . So  $U \subseteq Z$ , a contradiction. Let  $M = R[U, U]R$  be the ideal of  $R$  generated by  $[U, U]$ . Clearly  $M \neq 0$ . Let  $u, v \in U$  and  $r \in R$ . We have  $[u, vr] = v[u, r] + [u, v]r$ . Since  $[u, vr], v, [u, r] \in U \subseteq T(U)$ , so by Lemma 4,  $[u, v]r \in T(U)$ . Similarly  $r[u, v] \in T(U)$ . Since  $[[U, U], R] \subseteq U$ ,  $[[[[u, v], r], s], t] \in U$  for all  $r, s, t \in R$ . Therefore  $[[u, v]rs - r[u, v]s + [s, r][u, v] - [s[u, v], r], t] \in U$ . Since  $[u, v]rs, [s, r][u, v], s[u, v] \in T(U)$ ; so  $[r[u, v]s, t] \in U$  for all  $r, s, t \in R$ . Hence  $[M, R] \subseteq U$ .

**Lemma 6.** *Let  $R$  be a 2-torsion free ring,  $U$  a Lie idea of  $R$  and  $u, v \in U$ . If  $uxv + vxu = 0$  for all  $x \in U$  then  $uxvUxv = 0$ .*

**Proof** Let  $y \in U$ . Then  $(uxv)y(uxv) = -(vXu)yuXv = -v(xuy)uxv = u(xuy)vXv = ux(uyv)Xv = -uxvyuxv$ . Hence the result.

**Lemma 7.** *Let  $R$  be a 2-torsion free semiprime ring,  $U$  a Lie ideal of  $R$  and  $v \in U$ . If  $vUv = 0$  then  $v^2 = 0$  and there exists a non zero ideal  $M = R[U, U]R$  of  $R$  generated by  $[U, U]$  such that  $[M, R] \subseteq U$  and  $Mv = vM = 0$ .*

**Proof** Let  $vUv = 0$ . Then  $v[v, vr]v = 0$  for all  $r \in R$ . Hence  $v^2Rv^2 = 0$ . Now  $v^2 = 0$ . We have  $v[mv, r]Xv = 0$  for all  $m \in M$ ,  $r \in R$  and  $x \in U$ . Hence  $vmvrxv = 0$ . So  $vmvr[m, v]v = 0$ . Now  $vmvRvmv = 0$ . Hence  $vMv = 0$ . It implies that  $vRMv = 0$ . So  $Mv = 0$ . Similarly  $vM = 0$ .

The following theorem extends the main theorem of Bresar [4] on Lie ideals.

**Theorem 8.** *Let  $R$  be a 2-torsion free semiprime ring and  $U$  a Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $'$  is an additive mapping of  $R$  into itself satisfying  $(u^2)' = u'u + uu'$  and  $u' \in U$  for all  $u \in U$  then  $(uv)' = u'v + uv'$  for all  $u, v \in U$ .*

**Proof** By Lemma 3, we have  $u^vX[u, v] + [u, v]Xu^v = 0$  for all  $u, v, x \in U$ . Since  $u' \in U$  for all  $u \in U$  and  $R$  is 2-torsion free, without loss of generality we can assume that  $u^v \in U$  for all  $u, v \in U$ . By Lemmas 6 and 7, there exists a non zero ideal  $M = R[U, U]R$  of  $R$  generated by  $[U, U]$  such that  $[M, R] \subseteq U$  and  $Mu^vX[u, v] = 0$ . Therefore  $u^vX[u, v]Ru^vX[u, v] = 0$ . Now

$$u^vX[u, v] = [u, v]Xu^v = 0 \text{ for all } u, v, x \in U. \quad (1)$$

Linearizing  $v$ , we get  $u^vX[u, w] + u^wX[u, v] = 0$  for all  $u, v, w, x \in U$ . Using this relation and (1), we get  $u^vX[u, w]yu^vX[u, w] = -u^vX[u, w]yu^wX[u, v] = 0$  for all  $y \in U$ . By Lemma 7, we have  $Mu^vX[u, w] = 0$ . So  $u^vX[u, w]Ru^vX[u, w] = 0$ . Now  $u^vX[u, w] = 0$ . Similarly  $[u, w]Xu^v = 0$ . By linearizing  $u$ , we get  $u^vX[z, w] + z^vX[u, w] = 0$  for all  $z \in U$ . Hence  $u^vX[z, w]yu^vX[z, w] = -u^vX[z, w]yz^vX[u, w] = 0$  for all  $y \in U$ . Therefore  $Mu^vX[z, w] = 0$ . Now  $u^vX[z, w]Ru^vX[z, w] = 0$ . Hence

$$u^vX[z, w] = 0. \text{ Similarly } [z, w]Xu^v = 0. \quad (2)$$

Now  $[u^v, w]X[u^v, w] = (u^v w - w u^v)X[u^v, w] = u^v(wX)[u^v, w] - w u^vX[u^v, w] = 0$ . By Lemma 7,  $M[u^v, w] = 0$ . So  $[u^v, w]R[u^v, w] = 0$ . Now  $u^v \in C_R(U)$ , the centralizer of  $U$ . Therefore  $[u^v, [u^v, r]] = 0$  for all  $r \in R$ . By sublemma on p. 5 of [5],  $u^v \in Z$  for all  $u, v \in U$ . From (2), we have  $u^v[z, w]Xu^v[z, w] = 0$  for all  $x \in U$ . By Lemma 7,  $Mu^v[z, w] = 0$ . Now  $u^v[z, w]Ru^v[z, w] = 0$ . So

$$u^v[z, w] = 0. \quad (3)$$

We have  $u^v = -v^u$ . So  $2(u^v)^2 = u^v(u^v - v^u) = u^v([u, v]' + [v', u] + [v, u'])$ . From (3), we have  $u^v[v', u] = u^v[v, u'] = 0$ . Hence

$$2(u^v)^2 = u^v[u, v]'. \quad (4)$$

We have  $u^v[u, v] + [u, v]u^v = 0$ . So  $u^v[u, v]' + (u^v)'[u, v] + [u, v](u^v)' + [u, v]'u^v = 0$ . From (4), we get  $4(u^v)^2 + (u^v)'[u, v] + [u, v](u^v)' = 0$ . Multiplying it by  $u^v$  and using  $R$  is 2-torsion free, we get  $(u^v)^3 = 0$ . Now  $u^v = 0$ .

**Lemma 9 ([2]).** *If  $R$  is a 2-torsion free prime ring,  $U$  a Lie ideal of  $R$  and  $a, b \in R$  such that  $aUb = 0$  then  $a = 0$  or  $b = 0$ .*

**Lemma 10.** *Let  $R$  be a 2-torsion free prime ring,  $U$  a Lie ideal of  $R$  and  $a, b \in R$  such that one of  $a, b$  is in  $U$ . If  $axb + bxa = 0$  for all  $x \in U$  then  $axb = bxa = 0$ . So  $a = 0$  or  $b = 0$*

**Proof** Let  $y \in U$ . Suppose  $a \in U$ . Then

$$\begin{aligned} (axb)y(axb) &= -(bxa)y(axb) = -b(xay)axb = a(xay)bx b \\ &= ax(ayb)xb = -(axb)y(axb). \end{aligned}$$

By Lemma 9, we get the result.

Now we give a short and elementary proof of the following theorem of Awtar [1] which extends a well known result of Herstein [5] on Lie ideals.

**Theorem 11.** *Let  $R$  be a 2-torsion free prime ring and  $U$  a Lie ideal of  $R$  such that  $u^2 \in U$  for all  $u \in U$ . If  $'$  is an additive mapping of  $R$  into itself satisfying  $(u^2)' = u'u + uu'$  for all  $u \in U$  then  $(uv)' = u'v + uv'$  for all  $u, v \in U$ .*

**Proof** By Lemmas 3 and 10, we get  $u^v x[u, v] = 0$  for all  $u, v, x \in U$ . Suppose  $[u, v] \neq 0$ . Then by Lemma 9,  $u^v = 0$ . Now assume that  $[u, v] = 0$ . Let  $u, v \in C_R(U)$ . Since  $u, v \in U$ , we have  $[u, [u, r]] = 0$  for all  $r \in R$ . By sublemma on p.5 of [5],  $u \in Z$ . Similarly  $v \in Z$ . By Lemma 1,  $u^v = 0$ . Let  $u \notin C_R(U)$ . Then there exists  $w \in U$  such that  $[u, w] \neq 0$ . Hence  $u^w = 0$ . Clearly  $[u, v+w] \neq 0$ . Therefore  $u^{v+w} = u^v + u^w = u^v = 0$ . If  $v \notin C_R(U)$  then  $v^u = 0$ . So  $u^v = 0$ .

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