East-West J. of Mathematics: Vol. 9, No 1 (2007) pp. 47-51

JORDAN DERIVATIONS ON LIE IDEALS OF PRIME AND SEMIPRIME RINGS

Vishnu Gupta

Department of Mathematics
University of Delhi, Delhi 110 007, India
e-mail: vishnu_gupta2k3@yahoo.co.in

Abstract

We prove the following theorem: Let R ba a 2-torsion free semiprime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If is an additive mapping of R into itself satisfying $(u^2)' = u'u + uu'$ and $u' \in U$ for all $u \in U$ then $(uv)' = u'v + uv'$ for all $u, v \in U$. We also give a short and elementary proof of a theorem of Awtar [1] which extends a well known result of Herstein [5] on Lie ideals.

Throughout this paper R will denote an associative ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. So $uv + vu = (u+v)^2 - (u^2 + v^2)$ and $2uv = (uv + vu) + [u, v]$ are in U for all $u, v \in U$. Let $'$ be an additive mapping of R into itself such that $(u^2)' = u'u + uu'$ for all $u \in U$. From this we obtain.

Lemma 1. *If U is Lie ideal of a ring R then*

$$(uv + vu)' = u'v + uv' + v'u + vu' \text{ for all } u, v \in U.$$

Let $u^v = (uv)' - u'v - uv'$. Then $u^{v+w} = u^v + u^w$, $(u+v)^w = u^w + v^w$ and $u^v + v^u = 0$ for all $u, v, w \in U$.

If R is a 2-torsion free semiprime ring , U a commutative Lie ideal of R and $u \in U$ then $[u, [u, r]] = 0$ for all $r \in R$. By sublemma on p.5 of [5], $u \in Z$, the

Key words: Jordan derivation, derivation, prime ring, semiprime ring, Lie ideal, 2-torsion free ring.
2000 AMS Mathematics Subject Classification: 16W25, 16N60

centre of R. Hence $U \subseteq Z$. By Lemma 1, $(uv)' = u'v + uv'$ for all $u, v \in U$. Thus we assume that $U \nsubseteq Z$.

Lemma 2. *If U is a Lie ideal of a 2-torsion free ring R then*

1. $(uvu)' = u'vu + uv'u + uvu'$
2. $(uvw+wvu)' = u'vw+uv'w+uvw'+w'vu+wv'u+wvu'$ *for all* $u, v, w \in U$.

Proof (i) Using Lemma 1, evaluate $(u(uv + vu) + (uv + vu)u)'$
$= (u^2v + vu^2 + 2uvu)'$.
(ii) Linearize u in (i).

Lemma 3. *If U is a Lie ideal of a 2-torsion free ring R then $u^v x[u, v] + [u, v]xu^v = 0$ for all $u, v, x \in U$.*

Proof If $u, v \in U$ then $2uv \in U$. Since R is 2-torsion free, without loss of generality we can assume that $uv \in U$. Now we evaluate $(uvx\ vu + vuxuv)'$ in two ways. By using Lemma 2, we have

$$\begin{aligned}
(u(vxv)u + v(uxu)v)' &= u'(vxv)u + u(v'xv + vx'v + vxv')u + u(vxv)u' \\
&\quad + v'(uxu)v + v(u'xu + ux'u + uxu')v + v(uxu)v' \\
&= (uv)'xvu + uvx'vu + uvx(vu)' + (vu)'xuv \\
&\quad + vux'uv + vux(uv)'.
\end{aligned}$$

By comparing and using $u^v + v^u = 0$, we obtain the result.

Lemma 4 ([5], p. 5). *If U is a Lie ideal of a ring R then $T(U) = \{a \in R : [a, R] \subseteq U\}$ is both a subring and a Lie ideal of R containing U.*

Now we prove the following Lemma which is given in [2] for prime rings.

Lemma 5. *If R is a 2-torsion free semiprime ring and U a Lie ideal of R then there exists a non zero ideal $M = R[U, U]R$ of R generated by $[U, U]$ such that $[M, R] \subseteq U$.*

Proof We see that $[U, U] \neq 0$. Suppose $[U, U] = 0$. Let $u \in U$. Then $[u, [u, r]] = 0$ for all $r \in R$. By sublemma on p. 5 of [5], $u \in Z$. So $U \subseteq Z$, a contradiction. Let $M = R[U, U]R$ be the ideal of R generated by $[U, U]$, Clearly $M \neq 0$. Let $u, v \in U$ and $r \in R$. We have $[u, vr] = v[u, r] + [u, v]r$. Since $[u, vr], v, [u, r] \in U \subseteq T(U)$, so by Lemma 4, $[u, v]r \in T(U)$. Similarly $r[u, v] \in T(U)$. Since $[[U, U], R] \subseteq U$, $[[[[u, v], r], s], t] \in U$ for all $r, s, t \in R$. Therefore $[[u, v]rs - r[u, v]s + [s, r][u, v] - [s[u, v], r], t] \in U$. Since $[u, v]rs, [s, r][u, v], s[u, v] \in T(U)$; so $[r[u, v]s, t] \in U$ for all $r, s, t \in R$. Hence $[M, R] \subseteq U$.

Lemma 6. *Let R be a 2-torsion free ring, U a Lie idea of R and $u, v \in U$. If $uxv + vxu = 0$ for all $x \in U$ then $uxvUuxv = 0$.*

Proof Let $y \in U$. Then $(uxv)y(uxv) = -(vxu)yuxv = -v(xuy)uxv = u(xuy)vxv = ux(uyv)xv = -ux\ vy\ uxv$. Hence the result.

Lemma 7. *Let R be a 2-torsion free semiprime ring, U a Lie ideal of R and $v \in U$. If $vUv = 0$ then $v^2 = 0$ and there exists a non zero ideal $M = R[U, U]R$ of R generated by $[U, U]$ such that $[M, R] \subseteq U$ and $Mv = vM = 0$.*

Proof Let $vUv = 0$. Then $v[v, vr]v = 0$ for all $r \in R$. Hence $v^2Rv^2 = 0$. Now $v^2 = 0$. We have $v[mv, r]xv = 0$ for all $m \in M$, $r \in R$ and $x \in U$. Hence $vmvrxv = 0$. So $vmvr[m, v]v = 0$. Now $vmvRvmv = 0$. Hence $vMv = 0$. It implies that $vRMv = 0$. So $Mv = 0$. Similarly $vM = 0$.

The following theorem extends the main theorem of Bresar [4] on Lie ideals.

Theorem 8. *Let R be a 2-torsion free semiprime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If $'$ is an additive mapping of R into itself satisfying $(u^2)' = u'u + uu'$ and $u' \in U$ for all $u \in U$ then $(uv)' = u'v + uv'$ for all $u, v \in U$.*

Proof By Lemma 3, we have $u^v x[u, v] + [u, v]xu^v = 0$ for all $u, v, x \in U$. Since $u' \in U$ for all $u \in U$ and R is 2-torsion free, without loss of generality we can assume that $u^v \in U$ for all $u, v \in U$. By Lemmas 6 and 7, there exists a non zero ideal $M = R[U, U]R$ of R generated by $[U, U]$ such that $[M, R] \subseteq U$ and $Mu^v x[u, v] = 0$. Therefore $u^v x[u, v]Ru^v x[u, v] = 0$. Now

$$u^v x[u, v] = [u, v]xu^v = 0 \text{ for all } u, v, x \in U. \tag{1}$$

Linearizing v, we get $u^v x[u, w] + u^w x[u, v] = 0$ for all $u, v, w, x \in U$. Using this relation and (1), we get $u^v x[u, w]yu^v x[u, w] = -u^v x[u, w]yu^w x[u, v] = 0$ for all $y \in U$. By Lemma 7, we have $Mu^v x[u, w] = 0$. So $u^v x[u, w]Ru^v x[u, w] = 0$. Now $u^v x[u, w] = 0$. Similarly $[u, w]xu^v = 0$. By linearizing u, we get $u^v x[z, w] + z^v x[u, w] = 0$ for all $z \in U$. Hence $u^v x[z, w]yu^v x[z, w] = -u^v x[z, w]yz^v x[u, w] = 0$ for all $y \in U$. Therefore $Mu^v x[z, w] = 0$. Now $u^v x[z, w]Ru^v x[z, w] = 0$. Hence

$$u^v x[z, w] = 0. \text{ Similarly } [z, w]xu^v = 0. \tag{2}$$

Now $[u^v, w]x[u^v, w] = (u^v w - wu^v)x[u^v, w] = u^v(wx)[u^v, w] - wu^v x[u^v, w] = 0$. By Lemma 7, $M[u^v, w] = 0$. So $[u^v, w]R[u^v, w] = 0$. Now $u^v \in C_R(U)$, the centralizer of U. Therefore $[u^v, [u^v, r]] = 0$ for all $r \in R$. By sublemma on p. 5 of [5], $u^v \in Z$ for all $u, v \in U$. From (2), we have $u^v[z, w]xu^v[z, w] = 0$ for all $x \in U$. By Lemma 7, $Mu^v[z, w] = 0$. Now $u^v[z, w]Ru^v[z, w] = 0$. So

$$u^v[z, w] = 0. \tag{3}$$

We have $u^v = -v^u$. So $2(u^v)^2 = u^v(u^v - v^u) = u^v([u, v]' + [v', u] + [v, u'])$. From (3), we have $u^v[v', u] = u^v[v, u'] = 0$. Hence

$$2(u^v)^2 = u^v[u, v]'. \tag{4}$$

We have $u^v[u,v]+[u,v]u^v = 0$. So $u^v[u,v]'+(u^v)'[u,v]+[u,v](u^v)'+[u,v]'u^v = 0$. From (4), we get $4(u^v)^2+(u^v)'[u,v]+[u,v](u^v)' = 0$. Multiplying it by u^v and using R is 2- torsion free, we get $(u^v)^3 = 0$. Now $u^v = 0$.

Lemma 9 ([2]). *If R is a 2-torsion free prime ring, U a Lie ideal of R and $a,\ b \in R$ such that $aUb = 0$ then $a = 0$ or $b = 0$.*

Lemma 10. *Let R be a 2-torsion free prime ring, U a Lie ideal of R and $a,\ b \in R$ such that one of a, b is in U. If $axb + bxa = 0$ for all $x \in U$ then $axb = bxa = 0$. So $a = 0$ or $b = 0$*

Proof Let $y \in U$. Suppose $a \in U$. Then

$$\begin{aligned}(axb)y(axb) &= -(bxa)y(axb) = -b(xay)axb = a(xay)bxb \\ &= ax(ayb)xb = -(axb)y(axb).\end{aligned}$$

By Lemma 9, we get the result.

Now we give a short and elementary proof of the following theorem of Awtar [1] which extends a well known result of Herstien [5] on Lie ideals.

Theorem 11. *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If $'$ is an additive mapping of R into itself satisfying $(u^2)' = u'u+uu'$ for all $u \in U$ then $(uv)' = u'v+uv'$ for all $u, v \in U$.*

Proof By Lemmas 3 and 10, we get $u^v x[u,v] = 0$ for all $u, v, x \in U$. Suppose $[u,v] \neq 0$. Then by Lemma 9, $u^v = 0$. Now assume that $[u,v] = 0$. Let $u,\ v \in C_R(U)$. Since $u, v \in U$, we have $[u,[u,r]] = 0$ for all $r \in R$. By sublemma on p.5 of [5], $u \in Z$. Similarly $v \in Z$. By Lemma 1, $u^v = 0$. Let $u \notin C_R(U)$. Then there exists $w \in U$ such that $[u,w] \neq 0$. Hence $u^w = 0$. Clearly $[u, v+w] \neq 0$. Therefore $u^{v+w} = u^v + u^w = u^v = 0$. If $v \notin C_R(U)$ then $v^u = 0$. So $u^v = 0$.

Acknowledgement The author is thankful to the referee for his helpful suggestions.

References

[1] R. Awtar, *Lie ideals and Jordan derivations of prime rings*, Proc. Amer. Math Soc. 90(1984), 9-14.

[2] J. Bergen, I. N. Herstein and J. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra 71(1981), 259-267.

[3] M. Bresar and J. Vukman, *Jordan derivations of prime rings*, Bull. Aust. Math. Soc. 37(1988), 321-322.

[4] M. Bresar, *Jordan derivations of semi prime rings*, Proc. Amer. Math. Soc. 104(1988), 1003-1006.

[5] I. N. Herstein, "Topics is ring theory", University of Chicago Press, Chicago, 1969.

[6] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. 8(1957), 1104-1110.