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ON DEFECT AND TRUNCATED RELATIONS FOR HOLOMORPHIC CURVES INTO LINEAR SUBSPACES

Ha Tran Phuong* and **Mai Van Tu**†

**Department of Mathematics,*
Thai Nguyen University of education
Luong ngoc quyen street, Thai Nguyen City, VietNam
e-mail: hatranphuong@yahoo.com

†*Department of Mathematics*
Vinh University, Nguyen Du street, Vinh City, VietNam
e-mail: maivantu110@yahoo.com

Abstract

In 2004, M.Ru established a defect relation for algebraically non-degenerate holomorphic maps. Recently, T.T.H. An and H.T. Phuong proved an inequality of the second main theorem type, with ramification for holomorphic curves. In this paper we will establish a truncated defect relation for holomorphic curves into linear subspaces.

1 Introduction

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic map and $D_j, 1 \leqslant j \leqslant q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. In 1933 Cartan [3] and in 1983 Nochka [5] established a truncated defect relation for a linearly non-degenerate holomorphic map f intersecting hyperplanes. In 2004, M.Ru [6] established a defect relation for algebraically non-degenerate holomorphic map f intersecting hypersurfaces $D_j, 1 \leqslant j \leqslant q$. In this paper we will give a truncated defect relation for holomorphic curves into linear subspaces. To state our result, we first introduce some standard notations in Nevanlinna theory.

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Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $f = (f_0 : \cdots : f_n)$ be a reduced representative of f, where $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. Let P be the homogeneous polynomial of degree d defining D. The proximity function of f is defined by

$$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d}{|P(f)(re^{i\theta})|} d\theta.$$

The above definitions are independent, up to an additive constant, of the choice of the reduced representation of f and the choice of the defining polynomial P. Let $n_f(r, D)$ be the number of zeros of $P \circ f$ in the disk $|z| < r$, counting multiplicity, and $n_f^{\Delta}(r, D)$ be the number of zeros of $P \circ f$ in the disk $|z| < r$, truncated multiplicity by a positive integer Δ. The counting function and truncated counting function are defined by

$$N_f(r, D) = \int_0^r \frac{n_f(t, D) - n_f(0, D)}{t} dt + n_f(0, D) \log r;$$

$$N_f^{\Delta}(r, D) = \int_0^r \frac{n_f^{\Delta}(t, D) - n_f^{\Delta}(0, D)}{t} dt + n_f^{\Delta}(0, D) \log r.$$

In this paper, we write $N_f(r, D)$ as $N_f(r, P)$ and $N_f^{\Delta}(r, D)$ as $N_f^{\Delta}(r, P)$ sometimes.

Since Poisson-Jensen's formula, we have

First Main Theorem. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. If $f(\mathbb{C}) \not\subset D$, then for every real number r with $0 < r < \infty$*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

For a hypersurface D we define the defect

$$\delta_f(D) = 1 - \limsup_{r \to +\infty} \frac{N_f(r, D)}{(\deg D) T_f(r)},$$

and the truncated defect

$$\delta_f^{\Delta}(D) = 1 - \limsup_{r \to +\infty} \frac{N_f^{\Delta}(r, D)}{(\deg D) T_f(r)},$$

where Δ is a positive integer. It is easy to see that

$$0 \leqslant \delta_f(D) \leqslant \delta_f^{\Delta}(D) \leqslant 1$$

for any positive integer Δ and hypersurface D.

Let X be a k-dimensional projective subvariety of $\mathbb{P}^n(\mathbb{C}), 1 \leqslant k \leqslant n$. A collection of hypersurfaces $D_1, \ldots, D_q$ $(q \geq k+1)$ in $\mathbb{P}^n(\mathbb{C})$, which are defined by homogeneous polynomials $P_j,\ 1 \leqslant j \leqslant q$, is said to be *in general position with* X if for any subset $\{i_0, \ldots, i_k\}$ of $\{1, \ldots, q\}$ of cardinality $k+1$,

$$\{x \in X : P_{i_j}(x) = 0,\ j = 0, \ldots, k\} = \emptyset.$$

In [3], H. Cartan showed the following

Theorem A. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an linearly non-degenerate holomorphic map, and let $H_j, 1 \leqslant j \leqslant q$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have*

$$\sum_{j=1}^{q} \delta_f^n(H_j) \leqslant n+1.$$

And in [6], M.Ru showed the following theorem

Theorem B. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic map, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Then we have*

$$\sum_{j=1}^{q} \delta_f(D_j) \leqslant n+1.$$

The following results are obtained in this paper.

Theorem 1. *Let X be a k-dimension linear subspace of $\mathbb{P}^n(\mathbb{C})$ and let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position with X, and let d be the least common multiple of d_j's. Then for any $0 < \varepsilon < 1$, there exists a positive integer $\Delta = 2d[2^k(k+1)k(d+1)\varepsilon^{-1}]^k$ such that*

$$\sum_{j=1}^{q} \delta_f^{\Delta}(D_j) \leqslant k+1+\varepsilon. \tag{1.1}$$

Furthermore,

$$\sum_{j=1}^{q} \delta_f(D_j) \leqslant k+1. \tag{1.2}$$

Note that, when $k = n$ then $X = \mathbb{P}^n(\mathbb{C})$ and $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is an algebraically non-degenerate holomorphic map and hypersurfaces $D_j,\ j = 1, \ldots, q$ are in general position in $\mathbb{P}^n(\mathbb{C})$. Hence Theorem B is a special case of Theorem 2 when $k = n$. Our proofs of Theorem 1 and Theorem 2 base on results of An-Phuong [2] and Ru[6].

Obviously, we can choose Δ for any $\varepsilon > 0$ in Theorem 1, but it is large and depends on ε. It would be interesting if one can find a Δ term independing on ε. It is very important, because we can obmit term ε in the right side in (1.1) in that case.

Notice, our results are also true for curves from a complete non-Archimedean field of characteristic zero. And by the standard process of averaging over the complex lines in the complex space $\mathbb{C}^m$, one can easily extend these results to holomorphic map $f : \mathbb{C}^m \to X$.

2 Proof of Theorem 1.

To prove Theorem 1 we first recall the following Second Main Theorem for holomorphic curves intersecting hypersurfaces with ramification. The theorem is stated and proved by An and Phuong in [2].

Theorem 2.1. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic map, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d be the least common multiple of d_j. Let $0 < \varepsilon < 1$ and*

$$\Delta \geqslant 2d[2^n(n+1)n(d+1)\varepsilon^{-1}]^n.$$

Then

$$(q - (n+1) - \varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} d_j^{-1} N_f^{\Delta}(r, D_j)$$

where the inequality holds for all large r outside a set of finite Lebesgue measure.

Proof of Theorem 1. Now let X be a k-dimension linear subspace of $\mathbb{P}^n(\mathbb{C})$ and let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Let $f = (f_0 : \cdots : f_n)$ be a reduced representative of f. Since f maps into a $k-$dimension linear subspace, there are $(k+1)$ functions $f_{s_0}, \ldots, f_{s_k}$, which are algebraically independent, and $f_s,\ s \notin \{s_0, \ldots, s_k\}$, can be written as a linear form of $f_{s_0}, \ldots, f_{s_k}$.

Without loss of generality, we may assume (by rearranging the indices {0,...,n}) that $f_0, \ldots, f_k$ are algebraically independent, and

$$f_s = \sum_{i=0}^{k} b_{s,i} f_i,\ s = k+1, \ldots, n.$$

Set $f^* = (f_0 : \cdots : f_k) : \mathbb{C} \to \mathbb{P}^k(\mathbb{C})$. Since f is an algebraically non-degenerate holomorphic map on X we have f^* is an algebraically non-degenerate holomorphic map on $\mathbb{P}^k(\mathbb{C})$.

For $z \in \mathbb{C}$ and for any $s = k+1, \ldots, n$ we have

$$\begin{aligned}|f_s(z)| = |\sum_{i=0}^{k} b_{s,i} f_i(z)| &\leqslant \sum_{i=0}^{k} |b_{s,i} f_i(z)| \leqslant \sum_{i=0}^{k} |b_{s,i}|.|f_i(z)| \\ &\leqslant \max\{|f_0(z)|, \ldots, |f_k(z)|\}. \sum_{i=0}^{k} |b_{s,i}| = c_s. \max\{|f_0(z)|, \ldots, |f_k(z)|\}.\end{aligned}$$

where c_s is a positive constant, depends only on the $b_{s,i}$ and not on z and f^*. Set

$$c = \max\{1, c_{k+1}, \ldots, c_n\},$$

then we have, for any $z \in \mathbb{C}$,

$$|f_s(z)| \leqslant c. \max\{|f_0(z)|, \ldots, |f_k(z)|\} \text{ for any } s = (k+1), \ldots, n.$$

Hence

$$\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\} \leqslant c \max\{|f_0(z)|, \ldots, |f_k(z)|\} = c\|f^*(z)\|,$$

where c is a positive constant, depends only on the $b_{s,i}$ and not on z and f^*. This implies

$$\begin{aligned}T_f(r) &= \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \|f^*(re^{i\theta})\| d\theta + O(1) \\ &= T_{f^*}(r) + O(1).\end{aligned} \tag{2.1}$$

Now let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position with X. Let $Q_j,\ j = 1, \ldots, q$ be the homogeneous polynomials of degree d_j in $\mathbb{C}[z_0, \ldots, z_n]$ defining D_j. For any $j = 1, \ldots, q$, we set

$$Q_j^* = Q_j^*(z_0, \ldots, z_k) = Q_j\Bigg(z_0, \ldots, z_k, \sum_{i=0}^{k} b_{k+1,i} z_i, \ldots, \sum_{i=0}^{k} b_{n,i} z_i\Bigg).$$

Then Q_j^* is a homogeneous polynomial of degree d_j in $\mathbb{C}[z_0, \ldots, z_k]$. Obviously, by the construction of f^* and homogeneous polynomials Q_j^*, we have

$$Q_j \circ f(z) = Q_j^* \circ f^*(z)$$

for any $z \in \mathbb{C}$. Hence if $z \in \mathbb{C}$ is a zero of $Q_j \circ f$ with multiplicity α, then z will be a zero of $Q_j^* \circ f^*$ with multiplicity α. This implies

$$\begin{aligned} &N_f(r, D_j) = N_{f^*}(r, D_j^*); \\ &N_f^{\Delta}(r, D_j) = N_{f^*}^{\Delta}(r, D_j^*) \text{ for any positive integer } \Delta. \end{aligned} \tag{2.2}$$

For any $j = 1, \dots, q$, let D_j^* be the hypersurface in $\mathbb{P}^k(\mathbb{C})$ which is defined by the homogeneous polynomial Q_j^*. Next we will prove that the hypersurfaces $D_j^*,\ j = 1, \dots, q$ are in general position with $\mathbb{P}^k(\mathbb{C})$. Assume for the sake contradiction that there are $(k+1)$ hypersurfaces $D_{i_0}^*, \dots, D_{i_k}^* \in \{D_1^*, \dots, D_q^*\}$ and $\mathbf{a}^* = (a_0, \dots, a_k) \in \mathbb{P}^k(\mathbb{C})$ such that

$$Q_{i_0}^*(\mathbf{a}^*) = \cdots = Q_{i_k}^*(\mathbf{a}^*) = 0.$$

Set

$$\mathbf{a} = \left(a_0, \dots, a_k, \sum_{i=0}^{k} b_{k+1,i} a_i, \dots, \sum_{i=0}^{k} b_{n,i} a_i\right),$$

then $\mathbf{a} \in X$ and

$$Q_{i_0}(\mathbf{a}) = \cdots = Q_{i_k}(\mathbf{a}) = 0.$$

This is a contradiction with the assumption "in general position with X" of hypersurfaces $D_j,\ j = 1, \dots, q$.

For ε and Δ as in Theorem 1, applying Theorem 2.1 to the algebraically non-degenerate holomorphic map $f^* : \mathbb{C} \to \mathbb{P}^k(\mathbb{C})$ and hypersurfaces $D_j^*,\ j = 1, \dots, q$ we have

$$(q - (k+1) - \varepsilon) T_{f^*}(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_{f^*}^{\Delta}(r, D_j^*), \tag{2.3}$$

where inequality (2.3) holds for all large positive real number r. Combining formulas (2.1), (2.2), and (2.3) together, we have

$$(q - (k+1) - \varepsilon) T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^{\Delta}(r, D_j) + O(1),$$

so

$$\sum_{j=1}^{q} \left(1 - \frac{N_f^{\Delta}(r, D_j)}{d_j T_f(r)}\right) \leqslant (k + 1 + \varepsilon) + \frac{O(1)}{T_f(r)}.$$

This implies

$$\sum_{j=1}^{q} \delta_f^{\Delta}(D_j) \leqslant (k + 1 + \varepsilon).$$

This completes the inequality (1.1) of Theorem 1.

To prove the inequality (1.2) we need the following Second Main Theorem for holomorphic curve intersecting hypersurfaces, without ramification (see [6]).

Theorem 2.2. *Let $f : \mathbb{C} \to \mathbb{P}^k(\mathbb{C})$ be an algebraically non-degenerate holomorphic map. Let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^k(\mathbb{C})$ in general position with degree d_j. Then for every $\varepsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leqslant (k + 1 + \varepsilon) T_f(r). \tag{2.4}$$

where the inequality holds for all $r \in (0, +\infty)$ except for a possible set E with finite Lebesgue measure.

By First Main Theorem, (2.4) can be reformuled as follow

$$(q - (k + 1) - \varepsilon) T_f(r) \leqslant \sum_{j=1}^{q} d_j^{-1} N_f(r, D_j). \tag{2.5}$$

For every $\varepsilon > 0$, applying Theorem 2.2 with formula (2.5) to the holomorphic map $f^* : \mathbb{C} \to \mathbb{P}^k(\mathbb{C})$ and hypersurfaces $D_j^*,\ j = 1, \ldots, q$, we have

$$(q - (k + 1) - \varepsilon) T_{f^*}(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_{f^*}(r, D_j^*). \tag{2.6}$$

Combining formulas (2.1), (2.2) and (2.6) together, we have

$$(q - (k + 1) - \varepsilon) T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f(r, D_j) + O(1),$$

so

$$\sum_{j=1}^{q} \left(1 - \frac{N_f(r, D_j)}{d_j T_f(r)}\right) \leqslant (k + 1 + \varepsilon) + \frac{O(1)}{T_f(r)}.$$

This implies

$$\sum_{j=1}^{q} \delta_f(D_j) \leqslant (k + 1 + \varepsilon)$$

for every $\varepsilon > 0$. Hence

$$\sum_{j=1}^{q} \delta_f(D_j) \leqslant (k + 1).$$

This finishes the proof of Theorem 1. $\square$

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