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RING EXTENSIONS AND THEIR QUOTIENT RINGS

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Abstract

In this article we introduce a ring extension $E(R) = R[x, \sigma, \tau, \delta]$ of a ring R and show that if R is an order in an artinian ring, then $R[x, \sigma, \tau, \delta]$ is an order in an artinian ring, where σ and τ are automorphisms of R and δ is a (σ, τ)-derivation of R.

1 Introduction and Preliminaries

Throughout this article R is an associative ring with identity and any R-module is unitary. Let R be a ring. Spec(R) denotes the set of prime ideals of R. MinSpec(R) denotes the set of minimal prime ideals of R. The set of associated prime ideals of R is denoted by Ass(R). C(0) denotes the set of regular elements of R. C(I) denotes the set of elements of R regular modulo I, where I is an ideal of R. N(R) denotes the nil radical of R. $\mid M \mid_r$ denotes the right Krull dimension of a right R-module M. $\mid M \mid$ denotes the Krull dimension of an R-module M. Q denotes the field of rational numbers.

Now let R be a ring, which is an order in an artinian ring S. Let σ and τ be automorphisms of R and δ be a (σ, τ)-derivation of R. We first show that σ and τ and can be extended respectively to automorphisms (say) α and β of S and δ can be extended to an (α, β)-derivation (say) ρ of S. Let now $E(S) = S[x, \alpha, \beta, \rho]$ and M be the set of monic polynomials of E(S). Then we show that $E(R) = R[x, \sigma, \tau, \delta]$ is an order in $E(S)_M$ which is artinian. We also show that E(S) has an exhaustive set.

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We begin with the following definition:

Definition 1.1. Let R be a ring. Let σ and τ be automorphisms of R. A (σ, τ)-derivation of R is an additive mapping $\delta : R \to R$ satisfying $\delta(ab) = \delta(a)\sigma(b) + \tau(a)\delta(b)$.

Definition 1.2. Let R be a ring. Let σ and τ be automorphisms of R and δ be a (σ, τ)-derivation of R. Then the extension $E(R) = R[x, \sigma, \tau, \delta] = \{f = \sum x^i a_i, a_i \in R\}$, $0 \leq i \leq n$ subject to the relation $ax = x\sigma(\tau(a)) + \delta(a)$ for all $a \in R$.

We note that σ and τ can be extended to automorphism of E(R) such that $\sigma(x) = x$ and $\tau(x) = x$; i.e. $\sigma(xa) = x\sigma(a)$ and $\tau(xa) = x\tau(a)$ for all $a \in R$. Also δ can be extended to a (σ, τ)-derivation of E(R) such that $\delta(x) = 0$; i.e. $\delta(xa) = x\delta(a)$ for all $a \in R$.

We now state some well known results, which are crucial in studying skew-polynomial rings and their properties.

Theorem 1.3. *Let R be a right/left Noetherian ring. Then E(R) is right/left Noetherian.*

Proof The proof is on the same lines as in Theorem (1.12) of [3]. □

Theorem 1.4. *Let R be a commutative Noetherian Q-algebra and D be a derivation of R. Then $D(P) \subseteq P$, for all $P \in Ass(R)$.*

Proof See Theorem (1) of [5]. □

Theorem 1.5. *Let R be a Noetherian Q-algebra and ∂ be a derivation of R. Then $\partial(P) \subseteq P$ for all $P \in MinSpec(R)$.*

Proof See Lemma (3.4) of [2]. □

Proposition 1.6. *Let R be a Noetherian Q-algebra. Let σ be an automorphism of R and ∂ be a σ-derivation of R such that $\sigma(\partial(a)) = \partial(\sigma(a))$, for all $a \in R$. Then:*

1. *$\sigma(N(R)) = N(R)$*
2. *If $P \in MinSpec(R)$ is such that $\sigma(P) = P$, then $\partial(P) \subseteq P$.*

Proof **(1)** Denote N(R) by N. We have $\sigma(N) \subseteq N$ as $\sigma(N)$ is a nilpotent ideal of R. Now let $n \in N$. Then there exists $a \in R$ such that $n = \sigma(a)$. Therefore $I = \sigma^{-1}(N) = \{a \in R$ such that $\sigma(a) = n \in N\}$ is an ideal of R. Now I is nilpotent, so $I \subseteq \sigma(N)$, which implies that $N \subseteq \sigma(N)$. Hence $\sigma(N) = N$.

(2) Let T = $\{a \in P$ such that $\partial^k(a) \in P$ for all integers $k \geq 1\}$. Then T is a ∂-invariant ideal of R. Now it can be seen that $T \in Spec(R)$. Now $T \subseteq P$, so T = P as $P \in MinSpec(R)$. Hence $\partial(P) \subseteq P$. □

Proposition 1.7. *Let R be a semi prime Noetherian ring and $f \in E(R)$ be a regular element. Then there exists $g \in E(R)$ such that gf has leading coefficient regular in R.*

Proof Let K $= \{a_m \in R$, such that $x^m a_m + ... + a_0 \in E(R)f$, some m$\} \cup \{0\}$. Then it is easy to see that K is a left ideal of R. We will show K is essential. Let $0 \neq I \subseteq R$ be a left ideal of R. Then E(I) is a left ideal of E(R). Now E(R)f is an essential left ideal of R, therefore $E(I) \cap E(R)f \neq 0$. Let $\sum x^j a_j \in E(I) \cap E(R)f$, $0 \leq j \leq k$. Then $a_k \in I \cap K$, which implies that K is essential as a left ideal and therefore, K contains a left regular element. Now R being semi prime implies using Proposition (3.2.13) of Rowen [4] that, K contains a regular element. Therefore there exists $g \in E(R)$ such that gf has leading coefficient regular in R. □

2 Main Result

In this section, we prove the main result in the form of 2.11, and before that we prove some elementary results and give an analogue of some known results. The following Proposition though trivial plays an important role in proving the main Theorem.

Proposition 2.1. *Now let R be a ring, which is an order in an artinian ring S. Let σ and τ be automorphisms of R and δ be a (σ, τ)-derivation of R. Then σ and τ and can be extended respectively to automorphisms (say) α and β of S and δ can be extended to an (α, β)-derivation (say) ρ of S.*

Proof Define α, β on S as for any $as^{-1} \in S$; $\alpha(as^{-1}) = \alpha(a)\alpha(s)^{-1}$, $\beta(as^{-1}) = \beta(a)\beta(s)^{-1}$. It can be easily verified that α and β are automorphisms of S. Now define ρ on S as for any $as^{-1} \in S$; $\rho(as^{-1}) = (\delta(a) - \tau(a)\tau(s)^{-1}\delta(s))\sigma(s)^{-1}$. Now it can be seen that ρ is an (α, β)-derivation of S. Here we have used the fact that $\delta(ss^{-1}) = 0$ implies that $\delta(s^{-1}) = \tau(s)^{-1} - \delta(s)\sigma(s)^{-1}$. The other details are left to the reader. □

We now have some definitions, Lemmas and analogue of some results of [1], that lead us to the main result. The corresponding results and other details can be seen in [1].

Definition 2.2. Let R be a ring and U be a right Ore set in R. Let M be a right R-module. The set $T_U(M) = \{m \in M$ such that $mu = 0$ for some $u \in U\}$ is called the U-torsion submodule of M. M is said to be U-torsion if and only if $T_U(M) = M$ and is said to be torsion free if $T_U(M) = 0$.

Definition 2.3. Let R be a ring. Let σ and δ be automorphisms of R and δ be a (σ, τ)-derivation of R. Let M $= \{f \in E(R)$, such that f is monic$\}$.

Lemma 2.4. *Let R be a ring. A right E(R)-module W is M-torsion if and only if every finitely generated E(R)-submodule of W is finitely generated as an R-module where M is the set of monic polynomials of E(R).*

Proof Suppose W is M-torsion, then it is easy to verify that every cyclic E(R)-submodule of W is finitely generated over R, and therefore every finitely generated E(R)-submodule of W is finitely generated over R. Conversely W(E(R)) is finitely generated over R and it can be generated by w, wx, ..., wx^n for some positive integer n. Thus W is M-torsion. □

Lemma 2.5. *Let R be a right Noetherian ring. Let σ, τ and δ be as usual. Let M be the set of monic polynomials of E(R). Then M is a right denominator set in E(R).*

Proof It is sufficient to show that for any $g \in E(R)$ and $f \in M$ we have $E(R)/(fE(R))$ is right M-torsion. Let J = $\{h \in E(R)$ such that $gh \in fE(R)\}$. Now $\varphi : (E(R)/J) \to (E(R)/fE(R))$ defined by $\varphi(s) = gs + fE(R)$ is an embedding. Now $E(R)/fE(R)$ is finitely generated over R, and therefore is M-torsion as in [1]. □

Theorem 2.6. *Let R be a right Noetherian ring and M be the set of monic polynomials in E(R). Then* $\mid E(R)_M \mid_r = \mid R \mid_r$.

Proof Let L(A) denotes the lattice of right ideals in a ring A. As outlined in [1], we construct a strictly increasing function $\phi : L(R) \to L(E(R)_M)$ and $\psi : L(E(R)_M) \to L(R)$ to prove the Theorem. Now $\phi : L(R) \to L(E(R)_M)$ defined by $\phi(I) = I(E(R)_M)$ for any ideal I of R is the extension map. Now as in [1], consider $\eta : L(E(R)) \to L(R)$ the leading coefficient map by $\eta_i(J)$ to be the right ideal of R generated by the leading coefficient of elements of J with degree i for any right ideal J of E(R). Let $J \subseteq K$ be a right ideal of E(R). Then $\eta(J) = \eta(K)$ if and only if K/J is finitely generated as an R-module. Therefore for any right ideal T of $E(R)_M$, if we put $\psi(T) = \eta(E(R) \cap T)$, then by 2.5 ψ is strictly increasing. □

Lemma 2.7. *Let R be a ring which is an order in a right artinian ring S. Let σ, τ and δ be as usual. Then:*

1. *Every regular element of R is regular in E(R).*
2. *Set of regular elements of R satisfies the right Ore-condition in E(R).*
3. *Any element of E(S) has the form $f(x)c^{-1}$ for some $f(x) \in E(R)$ and some c regular in R.*
4. *If g(x) = $f(x)c^{-1}$ is regular in E(S), then f(x) is regular in E(R).*

5. *Let M be the set of monic polynomials of E(S). Then every regular element of E(R) is right regular as an element of $E(S)_M$.*

Proof The proof of this Lemma is obvious. The details are left to the reader. □

Definition 2.8. Let R be a ring, which has a right quotient ring Q(R). A multiplicative closed subset I of regular elements of R is said to be exhaustive if any $q \in Q(R)$ is such that q = ra^{-1} for some $r \in R$ and some $a \in I$. Same thing with some modifications when Q(R) is left quotient ring of R.

Definition 2.9. Let R be a ring. Define M(E(R)) = $\{f \in E(R)$ such that leading coefficient of f is regular in R$\}$.

Proposition 2.10. *Let R be a semiprime Noetherian ring then M(E(R)) is an exhaustive set.*

Proof The proof is obvious. □

We are now in a position to prove the main result in the form of the following Theorem:

Theorem 2.11. *Let R be a ring which is an order in a right artinian ring S. Then E(R) is an order in a right artinian ring and E(R) has an exhaustive set of elements which have leading coefficients regular in R.*

Proof We know that σ and τ and can be extended respectively to automorphisms (say) α and β of S and δ can be extended to an (α, β)-derivation (say) ρ of S by 2.1. Let M be the set of monic polynomials of E(S). Then since E(S) is a right Noetherian ring, therefore by 2.5, M is a right denominator set in E(S). Therefore 2.6 implies that $\mid E(S)_M \mid = \mid S \mid = 0$, and thus $E(S)_M$ is right artinian ring. Now any regular element of R is regular in E(R) by 2.7 and the set of regular elements of R forms a right denominator set in E(R) by 2.7. Also any element of E(S) is of the form $f(x)c^{-1}$ for some $f(x) \in R$ and some $c \in C(0)$ by 2.7. Moreover if $f(x)c^{-1}$ is a regular element of E(S); then f(x) is a regular element of E(R). Therefore any element of $E(S)_M$ is of the form $g(x)h(x)^{-1}$, where $g(x) \in E(R)$ and h(x) regular in E(R). Also every element of E(R) is right regular as an element of $E(S)_M$ by 2.7, and so is invertible in $E(S)_M$. Thus E(R) is a right order in $E(S)_M$. Now let $s \in E(S)_M$. Then s = $f(x)c^{-1}g(x)^{-1}$, where $g(x) \in M$, which implies that s = $f(x)(g(x)c)^{-1}$. Therefore any element s of $E(S)_M$ has the form s = $f(x).h(x)^{-1}$, where $f(x) \in E(R)$ and h(x) is regular in E(R) with leading coefficient regular in R. Hence E(R) has an exhaustive set consisting of those elements whose leading coefficients are regular in R. □

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