East-West J. of Mathematics: Vol. 9, No 1 (2007) pp. 11-24

ON THE PERIODIC SOLUTIONS OF COMPLETE SECOND ORDER DIFFERENTIAL EQUATIONS ON HILBERT SPACES

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Abstract

For the complete, second order differential equation $u''(t) = Au'(t) + Bu(t) + f(t)$ (*), $0 \leq t \leq T$ on a Hilbert space E, we find the necessary and sufficient conditions so that (*) has a unique T-periodic solution. Some applications are also given.

1 Introduction

In this paper, of concern is the second order differential equation:

$$u''(t) = Au'(t) + Bu(t) + f(t), \quad 0 \leq t \leq T, \tag{1.1}$$

related to the first order equation:

$$u'(t) = Au(t) + f(t), \quad 0 \leq t \leq T, \tag{1.2}$$

where A and B are linear, closed operators in a Hilbert space E and f is a function from $[0, T]$ with values in E. The periodicity of solutions of Equation (1.2) has been intensively studied, when A generates a strongly continuous semigroups (see e.g. [2], [5], [8] and [10]), and when A is closed operator (see

Key words: finitely annihilated, T-nilpotent, weak generator.
2000 AMS Mathematics Subject Classification: Primary 34G10, 34K06, Secondary 47D06.

[6], [7], [11]). However, for the complete higher differential equations, we have little consideration about the regularity of their solutions, mainly because of the complexity of the structure of the equation.

In this paper, we first investigate the periodicity of the solutions of Equation (1.2) by using the method of Fourier series. Next, we study the periodicity of solutions of Equation (1.1) by reducing the second order equation to a first order one. Namely, if we define new variables u_1 and u_2 as follows:

$$\begin{cases} u_1(t) := Au(t) \\ u_2(t) := u'(t), \end{cases}$$

then, from (1.1), we obtain the differential equations:

$$\begin{cases} u_1'(t) := Au_2(t) \\ u_2'(t) := u''(t) = Au'(t) + Bu + f = Au_2 + BA^{-1}u_1 + f(t) \end{cases}$$

or, equivalently,

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & A \\ BA^{-1} & A \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \tag{1.3}$$

as a single first order differential equation on $E \times E$. In Lemma 3.5 we show the equivalent relation between solution of (1.1) and that of (1.3). That lemma allows us to determine the existence and uniqueness, and the periodicity of solutions to the complete second order differential equation by studying those to the corresponding first order differential equation. As the main result, we show the different equivalent conditions so that (1.1) has a unique periodic solution. The main tool we use here is the Fourier series method. For an integrable function $f(t)$ from $[0, T]$ to E, the k^{th} Fourier coefficient of $f(t)$ is defined as

$$f_k = \frac{1}{T}\int_0^T f(s)e^{-2k\pi is/T}ds, \ k \in Z.$$

Then $f(t)$ can be represented by Fourier series

$$f(t) \approx \sum_{k=-\infty}^{\infty} e^{2k\pi it/T} f_k.$$

Let us first fix some notations. From now on E is a Hilbert space. For a closed operator A in E, the resolvent set and spectrum of A are denoted by $\varrho(A)$ and $\sigma(A)$. A continuous function on $[0, T]$ is said to be T-periodic if $u(0) = u(T)$. For the sake of simplicity (and without loss of generality) we assume $T = 1$ and put $J := [0, 1]$. $L_2(J)$ denotes the space of E-valued functions on J with $\int_0^1 \|f(t)\|^2 dt < \infty$ and $C(J)$ the space of continuous functions on J with

$\|f\| = \sup_J \|f(t)\| < \infty$. Moreover, for $m > 0$, the function space $W_2^m(J)$ is defined by

$$W_2^m(J) := \{f \in L_p(J) : f', f'', ..., f^{(m)} \in L_2(J)\}.$$

Recall, $L_2(J)$ and $W_2^m(J)$ are Hilbert spaces with the norms

$$\|f\|_{L_2(J)} := (\int_0^1 \|f(t)\|^2 dt)^{1/2} \text{ and } \|f\|_{W_2^m} := \sum_{k=0}^{m} \|f^{(k)}\|_{L_2(J)}.$$

We will use the following lemma, which can be found in [4].

Lemma 1.1. *If F is a continuous function on J such that $f = F' \in L_2(J)$, then for $k \neq 0$ we have*

$$F_k = \frac{1}{2k\pi i} f_k + \frac{F(0) - F(1)}{2k\pi i},$$

where f_k and F_k are the k^{th} Fourier coefficients of f and F, respectively.

More details about the Fourier series can be found in e.g. [4].

2 Periodic mild solutions of first order differential equations

In this section, we consider the first order differential equation

$$u'(t) = Au(t) + f(t), \quad 0 \leq t \leq T, \tag{2.1}$$

where A is a linear, closed operator in a Hilbert space E.

Definition 2.1. *(i) A function u is called a classical solution of* (2.1) *if it is continuously differentiable and satisfies* (2.1) *for all $t \in [0, T]$.*

(ii) A function u is called a mild solution of (2.1) *if $\int_0^t u(s)ds \in D(A)$ and*

$$u(t) = u(0) + A \int_0^t u(s)ds + \int_0^t f(s)ds \tag{2.2}$$

for all $t \in [0, T]$.

It is not hard to see that a mild solution will be a classical one, if it is continuously differentiable. The following proposition gives the conditions for the existence of periodic mild solutions of (2.1), when the inhomogeneity f is given.

Proposition 2.2. *Let E be a Hilbert space and f be a given function in $L_2(J)$. The following are equivalent:*

(i) Equation (2.1) has a 1-periodic mild solution contained in $W_2^1(J)$.

(ii) For each $k \in \mathbb{Z}$, $f_k \in Range(2k\pi i - A)$ and there exists a sequence $\{u_k\}_{k\in\mathbb{Z}} \subset E$ such that

$$(2k\pi i - A)u_k = f_k \text{ and } \sum_{-\infty}^{\infty} k^2\|u_k\|^2 < \infty. \tag{2.3}$$

Proof $(i) \Rightarrow (ii)$: Let $u(t)$ be a 1-periodic solution to (2.1) of the form

$$u(t) = u(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds. \tag{2.4}$$

Take the k^{th} Fourier coefficient of u from (2.4), we obtain

$$\begin{aligned} u_k &= \int_0^1 e^{-2k\pi it}u(0)dt + A\int_0^1 e^{-2k\pi it}\int_0^t u(s)dsdt + \int_0^1 e^{-2k\pi it}\int_0^t f(s)dsdt \\ &= \frac{Au_k - A\int_0^1 u(t)dt}{2k\pi i} + \frac{f_k - \int_0^1 f(t)dt}{2k\pi i}. \end{aligned}$$

Thus,

$$(2k\pi i - A)u_k = f_k - \frac{A\int_0^1 u(t)dt + \int_0^1 f(t)dt}{2k\pi i} = f_k. \tag{2.5}$$

Here we used the fact that $A\int_0^1 u(t)dt + \int_0^1 f(t)dt = 0$, which is obtained by putting $t = 1$ in formula (2.4). Hence, $f_k \in Range(2k\pi i - A)$ for every $k \in \mathbb{Z}$. Moreover, since $u \in W_{1,2}(J)$, we have

$$\sum_{-\infty}^{\infty} k^2\|u_k\|^2 = \frac{1}{4\pi^2}\|u'\|^2_{L_2(J)} < \infty.$$

$(ii) \Rightarrow (i)$: Let $\{u_k\}$ be a sequence satisfying $(2k\pi i - A)u_k = f_k$ and $\sum_{-\infty}^{\infty} k^2\|u_k\|^2 < \infty$. Define

$$f_N(t) := \sum_{k=-N}^{N} e^{2k\pi it}f_k \text{ and } u_N(t) := \sum_{k=-N}^{N} e^{2k\pi it}u_k.$$

Then f_N and u_N are continuous, 1-periodic functions satisfying

$$u_N(t) = u_N(0) + A\int_0^t u_N(s)ds + \int_0^t f_N(s)ds. \tag{2.6}$$

Moreover, f_N converges to f, and by assumption (2.3), $u_N \to u$ and $u'_N \to v$ in $L_2(J)$ for some certain function u and v in $L_2(J)$ as $N \to \infty$. Since the differential operator $\frac{d}{dt}$ is closed in $L_2(J)$, we have $u \in W_2^1(J)$ and $u' = v$. As a consequence, u is a continuous function and $u_N \overset{N\to\infty}{\longrightarrow} u$ in $C(J)$. Hence, u a 1-periodic function.

Put now $x_N := \int_0^t u_N(s)ds$, then $\lim\limits_{N\to\infty} x_N = \int_0^t u(s)ds$. Moreover, from (2.6) we have

$$\begin{aligned} Ax_N &= u_N(t) - u_N(0) - \int_0^t f_N(s)ds \\ &\overset{N\to\infty}{\longrightarrow} u(t) - u(0) - \int_0^t f(s)ds. \end{aligned}$$

Since A is a closed operator, we obtain that $\int_0^t u(s)ds \in D(A)$ and

$$A\int_0^t u(s)ds = u(t) - u(0) - \int_0^t f(s)ds$$

for $0 \le t \le T$, which means that u is a mild solution of (2.1). □

From the structure of its 1-periodic solution, Equation (2.1) can have more than one 1-periodic solutions. However, if $(2k\pi i - A)$ are injective for all $k \in \mathbb{Z}$, then (2.1) has at most one 1-periodic solution. This is contained in the following main theorem of this section, which has a similar version in [5], when A generates a C_0-semigroup, with a different proof.

Theorem 2.3. *Suppose A is a closed operator on a Hilbert space E, then the following are equivalent*

(i) For each function $f \in W_2^1(J)$, equation (2.1) has a unique 1-periodic mild solution contained in $W_2^1(J)$.

(ii) For each $k \in \mathbb{Z}$, $2k\pi i \in \varrho(A)$ and

$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| \le \infty. \tag{2.7}$$

Proof $(i) \Rightarrow (ii)$: By Proposition 2.2, $f_k \in Range(2k\pi i - A)$ for each function f and $k \in \mathbb{Z}$, hence $(2k\pi i - A)$ is surjective. On the other hand, $(2k\pi i - A)$ is injective, otherwise, $u(t) \equiv 0$ and $u(t) = e^{2k\pi i t}x$, where x is a nonzero vector in E satisfying $(2k\pi i - A)x = 0$, would be two distinct 1-periodic mild solutions to (2.1) with $f(t) \equiv 0$. Hence $(2k\pi i - A)$ is bijective and $2k\pi i \in \varrho(A)$ for all $k \in \mathbb{Z}$.

We now define the operator $L : W_2^1(J) \mapsto W_2^1(J)$ by follows: $L(f)$ is the unique 1-periodic mild solution to (2.1) corresponding to f. Then, by the assumption, L is everywhere defined. We will prove that L is a bounded operator by showing that L is closed in $W_2^1(J)$. Let $f_n \to f$ and $Lf_n \to u$ in $W_2^1(J)$, where

$$(Lf_n)(t) = (Lf_n)(0) + A\int_0^t (Lf_n)(s)ds + \int_0^t f_n(s)ds. \tag{2.8}$$

Since the convergence in $W_2^1(J)$ is stronger than that in $C(J)$, it is $Lf_n \overset{n\to\infty}{\longrightarrow} u$ in $C(J)$. Hence, u is 1-periodic. Moreover, from (2.8) we obtain

$$A\int_0^t (Lf_n)(s)ds = (Lf_n)(t) - (Lf_n)(0) - \int_0^t f_n(s)ds \overset{n\to\infty}{\longrightarrow} u(t) - u(0) - \int_0^t f(s)ds$$

for each $t \in J$. Since A is a closed operator, $\int_0^t u(s)ds \in D(A)$ and

$$A\int_0^t u(s)ds = u(t) - u(0) - \int_0^t f(s)ds,$$

which means $f \in D(L)$ and $Lf = u$ and hence, L is closed.

Next, for any $x \in E$, put $f(t) = e^{2k\pi it}x$, then $u(t) = e^{2k\pi it}(2k\pi i - A)^{-1}x$ is the unique 1-periodic solution to (2.1), i.e., $u = Lf$. Using the boundedness of operator L, we obtain

$$(2|k|\pi + 1)\|(2k\pi i - A)^{-1}x\| = \|u\|_{W_2^1(J)} \le \|L\|\|f\|_{W_2^1(J)} = \|L\|(2|k|\pi + 1)\|x\|,$$

which implies

$$\|(2k\pi i - A)^{-1}x\| \le \|L\| \cdot \|x\|$$

for any $x \in E$ and any $k \in \mathbb{Z}$. Thus, (2.7) holds.

$(ii) \Rightarrow (i)$: For any function f in $W_2^1(J)$, put $u_k := (2k\pi i - A)^{-1}f_k$, where f_k is the k^{th} Fourier coefficient of f. Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} k^2\|u_k\|^2 &\le \sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\|^2 \sum_{k\in\mathbb{Z}} k^2\|f_k\|^2 \\ &= \sup_{k\in\mathbb{Z}} \frac{\|(2k\pi i - A)^{-1}\|^2}{4\pi^2}\|f'\|^2_{L_2(J)} < \infty.\end{aligned}$$

By Proposition 2.2, Equation (2.1) has an 1-periodic mild solution in $W_2^1(J)$. Finally, if (2.1) had another 1-periodic solution f', then, from (2.5), its Fourier coefficients would be determined by $u'_k = (2k\pi i - A)^{-1}f_k$, i.e. $u'_k = u_k$ for all $k \in \mathbb{Z}$, and thus, $f' = f$. □

In particular, if A generates a C_0-semigroup $(T(t))_{t\ge 0}$, then (see [1, Theorem 2.5]), mild solutions of the first order differential equation (2.1) can be explicitly expressed by

$$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)ds. \tag{2.9}$$

In this case, we obtain the following results, in which we show the Gearhart's Theorem (the equivalence $(iv) \Leftrightarrow (v)$) with a short proof.

Corollary 2.4. *Let A generate a C_0-semigroup $(T(t))$ on a Hilbert E, then the following are equivalent*

(i) *For each function $f \in L_2(J)$, equation* (2.1) *has a unique 1-periodic mild solution.*

(ii) *For each function $f \in W_2^1(J)$, equation* (2.1) *has a unique 1-periodic classical solution.*

(iii) *For each function $f \in W_2^1(J)$, equation* (2.1) *has a unique 1-periodic solution contained in $W_2^1(J)$.*

(iv) *For each $k \in \mathbb{Z}$, $2k\pi i \in \varrho(A)$ and*

$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| \leq \infty. \tag{2.10}$$

(v) $1 \in \varrho(T(1))$.

Proof The equivalence $(iii) \Leftrightarrow (iv)$ is shown Theorem 2.3, $(i) \Leftrightarrow (ii)$ can be easily proved by using standard arguments, $(i) \Leftrightarrow (v)$ has been shown in [8] and $(ii) \Rightarrow (iii)$ is obvious. So, it remains to show the inclusion $(iii) \Rightarrow (ii)$.

Let f be any function in $W_2^1(J)$ and $u(t)$ be the unique mild solution of (2.1), which is 1-periodic and which is in $W_2^1(J)$. Since for each $f \in W_2^1(J)$ it is that $\int_0^t T(t-s)f(s)ds \in D(A)$ and $t \to \int_0^t T(t-s)f(s)ds$ is continuously differentiable (see [9]), to show u is a classical solution, it suffices to show $u(0) \in D(A)$.

By formula (2.9), the function $t \mapsto T(t)u(0) = u(t) - \int_0^t T(t-s)f(s)ds$ belongs to $W_2^1(J)$. It follows that $T(t)u(0) \in D(A)$ for $t > 0$ (since $t \mapsto T(t)x$ is differentiable at t_0 if and only if $T(t_0)x \in D(A)$). Hence, $u(1)$, and thus, $u(0) = u(1)$ belongs to $D(A)$. The uniqueness of this 1-periodic classical solution is obvious and the proof is complete. $\square$

3 Periodic mild solutions of complete second order differential equations

We now turn to the complete differential equation

$$u''(t) = Au'(t) + Bu(t) + f(t), \qquad 0 \leq t \leq T \tag{3.1}$$

and the related first order equation

$$\left\{ \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}' = \mathcal{A} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \right. ,0 \leq t \leq T, \tag{3.2}$$

where $\mathcal{A} = \begin{pmatrix} 0 & A \\ BA^{-1} & A \end{pmatrix}$ on $E \times E$ with $D(\mathcal{A}) = E \times D(A)$. Recall that a function u is a classical solution of Equation (3.1) if it is twice continuously differentiable, $Au'(\cdot)$ and $Bu(\cdot)$ are continuous and (3.1) is satisfied. We first state the relationship between the classical solutions of (3.1) and those of (3.2).

Lemma 3.1. *Let A and B be closed operators on E such that $D(B) \supseteq D(A)$ and A is invertible. Then the following statements hold:*

(1) If u is a classical solution of (3.1) *then $(Au, u')^T$ is a classical solution of* (3.2)*.*

(2) If $(u_1, u_2)^T$ is a classical solution of (3.2)*, then $u(t) = A^{-1}u_1(t)$ is a classical solution of* (3.1)*.*

Proof Let u be a classical solution of (3.1), then $t \mapsto u'(t)$ and $t \mapsto Au'(t)$ are continuous and

$$\int_0^t Au'(s)ds = A \int_0^t u'(s) = Au(t) - Au(0),$$

which implies $Au(t) = Au(0) + \int_0^t Au'(s)ds$. Hence, $t \mapsto Au(t)$ is continuously differentiable and

$$\frac{d}{dt} Au(t) = Au'(t). \tag{3.3}$$

Let now $(u_1, u_2) = (Au, u')$, then, by (3.3), we have

$$u_1' = Au_2$$

and

$$u_2' = u'' = Au' + Bu + f = Au_2 + BA^{-1}u_1 + f.$$

Thus, $(u_1, u_2)^T$ is a classical solution of (3.2).

Conversely, let $(u_1, u_2)^T$ be a classical solution of (3.2) and let $u(t) := A^{-1}u_1(t)$, then it is easy to see that

$$u'(t) = u_2(t)$$

and

$$u''(t) = u_2'(t) = BA^{-1}u_1(t) + Au_2(t) + f(t) = Au(t) + Au'(t) + f(t),$$

which indicates that $u(t)$ is a classical solution of (3.1). □

From the above lemma, it suggests us to define mild solutions of Equation (3.1) as follows

Definition 3.2. *A function $u : [0,T] \mapsto E$ is a mild solution of the complete second order differential equation* (3.1)*, if $u(t) := A^{-1}u_1(t)$, where $u_1(t)$ is the first component of a mild solution of equation* (3.2)*.*

It is also not hard to see that a mild solution is a classical solution if and only if it is twice continuously differentiable. We now study the periodicity of the mild solutions of Equation (3.1). First, we define the following sets

$\varrho(A,B) := \{\lambda \in \mathbb{C} : (\lambda^2 - \lambda A - B)$ is bijective and has bounded inverse$\}$, where $(\lambda^2 - \lambda A - B) : D(A) \cap D(B) \mapsto E$ is defined by

$$(\lambda^2 - \lambda A - B)x = \lambda^2 x - \lambda A x - Bx;$$

$\sigma(A,B) := \mathbb{C}/\varrho(A,B);$

and

$\sigma_p(A,B) := \{\lambda \in \mathbb{C} : \exists x \in E : x \neq 0$ and $(\lambda^2 - \lambda A - B)x = 0\}$.

Moreover, for $\lambda \in \varrho(A,B)$, the bounded inverse of $(\lambda^2 - \lambda A - B)$ is denoted by $R(\lambda, A, B)$. We are now in a position to state the main theorem of this section.

Theorem 3.3. *Let A and B be two closed operators on a Hilbert space E, such that $D(A) \subseteq D(B)$ and A is invertible. Then the following statements are equivalent.*

(i) For each function $f \in W_2^1(J)$, Equation (3.1) *admits a unique 1-periodic mild solution u in $W_2^2(J)$.*

(ii) For every $k \in \mathbb{Z}$, $2k\pi i \in \varrho(A,B)$ and

$$\sup_{k\in\mathbb{Z}} \|kR(2k\pi i, A, B)^{-1}\| < \infty \text{ and } \sup_{k\in\mathbb{Z}} \|AR(2k\pi i, A, B)^{-1}\| < \infty. \quad (3.4)$$

We need the following lemma:

Lemma 3.4. *The following statements hold:*

(i) *λ is in the point spectrum of $\mathcal{A}$ if and only if $\lambda \in \sigma_p(A,B)$;*

(ii) *$(\lambda - \mathcal{A})$ is injective if and only if $(\lambda^2 - \lambda A - B)$ is injective.*

In particular, $\lambda \in \varrho(A,B)$ if and only if $\lambda \in \varrho(\mathcal{A})$ and in this case,

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \frac{I + AR(\lambda,A,B)BA^{-1}}{\lambda} & AR(\lambda, A, B) \\ R(\lambda, A, B)BA^{-1} & \lambda R(\lambda, A, B) \end{pmatrix} \quad (3.5)$$

for $\lambda \neq 0$. For $\lambda = 0$, then $R(0, A, B) = B^{-1}$ and

$$\mathcal{A}^{-1} = \begin{pmatrix} AB^{-1} & -AB^{-1} \\ -A^{-1} & 0 \end{pmatrix}. \quad (3.6)$$

Proof It is not hard to show (i) and (ii) by using the standard arguments. If now $\lambda \in \varrho(A,B)$, then $\lambda \in \varrho(\mathcal{A})$ by (3.5) for $\lambda \neq 0$ and (3.6) for $\lambda = 0$.

Conversely, if $\lambda \in \varrho(\mathcal{A})$, then there exist bounded operators X_1, X_2, X_3 and X_4 on E such that

$$\begin{pmatrix} \lambda & -A \\ -BA^{-1} & (\lambda - A) \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} \tag{3.7}$$

and

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} \lambda & -A \\ -BA^{-1} & (\lambda - A) \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}. \tag{3.8}$$

From identity (3.7) we obtain

$$\lambda X_2 - AX_4 = 0$$

and

$$-BA^{-1}X_2 + (\lambda - A)X_4 = Id,$$

which imply

$$(\lambda^2 - \lambda A - B) \cdot (X_4/\lambda) = Id. \tag{3.9}$$

Similarly, using identity (3.8) we obtain

$$(X_4/\lambda) \cdot (\lambda^2 - \lambda A - B) = Id. \tag{3.10}$$

Hence, $\lambda \in \varrho(A,B)$ for $\lambda \neq 0$. With the same manner of arguing we also obtain that $\lambda \in \varrho(A,B)$ when $\lambda = 0$, and the lemma is proved. □

Proof of Theorem 3.3: Suppose (i) holds, then for each $f \in W_2^1(J)$, Equation (3.2) has a unique 1-periodic mild solution $\mathcal{U}(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ in $W_2^1(J, E \times E)$. Define the operator $G : W_2^1(J) \mapsto W_2^1(J, E \times E)$ by $Gf = \mathcal{U}$. It is obvious that G is linear and everywhere defined. Using the same arguments as in the proof of Theorem 2.3, we can show G is a closed and hence, a bounded operator.

Let now $f(t) = e^{2k\pi it}x$ for any $k \in \mathbb{Z}$ and $x \in E$. Using the fact that $(2k\pi i - \mathcal{A})\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$, where $\begin{pmatrix} x_k \\ y_k \end{pmatrix}$ is the k^{th} Fourier coefficient of $Gf(t)$, we have

$$(2k\pi i - A)x_k = 0$$

and

$$-BA_{-1}x_k + (2k\pi i - A)y_k = x,$$

from which it implies $((2k\pi i)^2 - 2k\pi iA - B)y_k = (2k\pi i)x$. Thus $(2k\pi i)^2 - 2k\pi iA - B$ is surjective. On the other hand, if there is a non-zero vector $y \in E$ satisfying $((2k\pi i)^2 - 2k\pi iA - B)y = 0$, then the functions $u_1(t) \equiv 0$ and $u_2(t) = e^{2k\pi it}y$ are two distinct mild solutions of (3.1) corresponding to $f(t) \equiv 0$. It would be contradicted to the assumption, hence $((2k\pi i)^2 - 2k\pi iA - B)$ is injective. By Lemma 3.4, it implies that $2k\pi i \in \varrho(A, B)$.

Finally, for $f(t) = e^{2k\pi it}x$ $(k \neq 0)$ we have

$$\begin{aligned}
\mathcal{U}(t) = Gf(t) &= e^{2k\pi it}(2k\pi i - \mathcal{A})^{-1}\begin{pmatrix} 0 \\ x \end{pmatrix} \\
&= e^{2k\pi it}\begin{pmatrix} \frac{I + AR(2k\pi i, A, B)BA^{-1}}{2k\pi i} & AR(2k\pi i, A, B) \\ R(2k\pi i, A, B)BA^{-1} & 2k\pi iR(2k\pi i, A, B) \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} \\
&= \begin{pmatrix} AR(2k\pi i, A, B)e^{2k\pi it}x \\ 2k\pi iR(2k\pi i, A, B)e^{2k\pi it}x \end{pmatrix}.
\end{aligned}$$

Here we used formula (3.5). Hence,

$$\|Gf\|_{W_2^1(J, E\times E)} = (1 + 2|k|\pi)(\|AR(\lambda, A, B))x\| + \|2kR(\lambda, A, B))x\|.$$

Using now the boundedness of G, we obtain

$$\|AR(\lambda, A, B)x\| + \|2kR(\lambda, A, B)x\| \leq \|G\|\|x\|$$

for each $x \in E$, which implies (3.4).

Conversely, suppose (ii) holds, then $2k\pi i \in \varrho(\mathcal{A})$ for all $k \in \mathbb{Z}$ and, by formula (3.5),

$$\sup_{k\in\mathbb{Z}} \|R(2k\pi i, \mathcal{A})^{-1}\| < \infty.$$

By Theorem 2.3, for each function $f \in W_{1,2}(J)$, equation (3.2) has a unique 1-periodic mild solution $\begin{pmatrix} x(\cdot) \\ y(\cdot) \end{pmatrix} \in W_2^1(J, E \times E)$. It is then easy to see that

$$u(t) = A^{-1}x(t) = u(0) + \int_0^t y(s)ds$$

is a 1-periodic mild solution to (3.1), which is contained in $W_2^2(J)$. The uniqueness of this solution is from its definition, and the proof is complete. □

Remark: We actually have proved a stronger statement: if for each $f \in W_2^1(J)$, Equation (3.2) has a unique 1-periodic mild solution in $W_2^1(J, E \times E)$, then equation

$$\left\{ \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & A \\ BA^{-1} & A \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + F(t) \right. \tag{3.11}$$

has a unique 1-periodic mild solution in $W_2^1(J, E \times E)$ for each function $F \in W_2^1(J, E \times E)$.

If operator A generates a C_0-semigroup in E, then operator $\mathcal{A}$ also generates a C_0-semigroup on $E \times E$, as the lemma below states. Hence, we can make use of Corollary 2.4 and obtain interesting results about the periodicity of mild and classical solutions of (3.1).

Lemma 3.5. *Operator $\mathcal{A}$ generates a C_0 semigroup on $E \times E$ if and only if operator A generates a C_0 semigroup on E.*

To prove Lemma 3.5, we need the following result, which can be found in [3].

Lemma 3.6. *Let H be a closed operator on a Banach space X. The following hold*

(i) (Similar semigroups) If S is an isomorphism on X, then H generates a C_0-semigroup on X if and only if $S^{-1}HS$ generates a C_0-semigroup on X.

(ii) (Bounded perturbation) If C is a bounded operator on X, then H generates a C_0-semigroup on X if and only if $H+C$ generates a C_0-semigroup on X.

Proof of Lemma 3.5. We have

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 0 & 0 \\ BA^{-1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & A \\ 0 & A \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ BA^{-1} & 0 \end{pmatrix} + \begin{pmatrix} Id & Id \\ 0 & Id \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} Id & -Id \\ 0 & Id \end{pmatrix} \\ &= C + S^{-1} \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} S \end{aligned}$$

where C and S are bounded operators on $E \times E$. By Lemma 3.6, $\mathcal{A}$ generates a C_0-semigroup on $E \times E$ if and only if operator $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$ generates a C_0-semigroup on $E \times E$, which in turn does if and only if A generates a C_0-semigroup on E, and the lemma is proved. $\square$

Corollary 3.7. *If A generates a C_0 semigroup, then the following statements are equivalent.*

(i) For each function $f \in L_2(J)$, Equation (3.1) admits a unique 1-periodic mild solution which is one-time continuously differentiable;

(ii) For each periodic function $f \in W_2^1(J)$, Equation (3.1) admits a unique 1-periodic classical solution.

(iii) For each periodic function $f \in W_2^1(J)$*, Equation* (3.1) *admits a unique 1-periodic mild solution in* $W_2^2(J)$*.*

(iv) For every $k \in \mathbb{Z}$*,* $2k\pi i \in \varrho(A, B)$ *and*

$$\sup_{k\in\mathbb{Z}} \|kR(2k\pi i, A, B)\| < \infty \text{ and } \sup_{k\in\mathbb{Z}} \|AR(2k\pi i, A, B)\| < \infty.$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ can be shown by standard arguments, the equivalence $(iii) \Leftrightarrow (iv)$ holds by Theorem 3.3 and the implication $(ii) \Rightarrow (iii)$ is obvious. So, it remains to show that (iii) implies (ii). But this follows from that the fact that $\mathcal{A}$ generates a C_0-semigroup on $E \times E$ and the equivalence $(i) \Leftrightarrow (ii)$ in Corollary 2.4. □

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