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# A REMARK ON SOME SEMIGROUPS OF HYPERGROUP HOMOMORPHISMS

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**Abstract**

Let $\mathbb{Z}$ be the set of integers, $n$ a positive integer and $(\mathbb{Z}, \circ_n)$ the hypergroup where $x \circ_n y = x + y + n\mathbb{Z}$ for all $x, y \in \mathbb{Z}$. Denote by $\operatorname{Hom}(\mathbb{Z}, \circ_n)$ the semigroup, under composition, of all homomorphisms of $(\mathbb{Z}, \circ_n)$. It has been shown that for $f : \mathbb{Z} \to \mathbb{Z}, f \in \operatorname{Hom}(\mathbb{Z}, \circ_n)$ if and only if $f(x + n\mathbb{Z}) = xf(1) + n\mathbb{Z}$ for all $x \in \mathbb{Z}$ and $|\operatorname{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$. Using this characterization, we show in this paper that the relation $\delta$ on $\operatorname{Hom}(\mathbb{Z}, \circ_n)$ defined by $f\delta g \Leftrightarrow f(1) \equiv g(1) \mod n$ is a congruence on $\operatorname{Hom}(\mathbb{Z}, \circ_n)$ and $\operatorname{Hom}(\mathbb{Z}, \circ_n)/\delta \cong (\mathbb{Z}_n, \cdot)$.

## 1 Introduction

A *hyperoperation* on a nonempty set $H$ is a function $\circ : H \times H \to \mathcal{P}^*(H)$ where $\mathcal{P}(H)$ is the power set of $H$ and $\mathcal{P}^*(H) = \mathcal{P}(H) \smallsetminus \{\emptyset\}$. For $x, y \in H, x \circ y$ denotes the value of $(x, y)$. For $A, B \subseteq H$, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

If $x \in H$ and $A \subseteq H$, let $x \circ A$ and $A \circ x$ stand for $\{x\} \circ A$ and $A \circ \{x\}$, respectively. The system $(H, \circ)$ is called a *hypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \text{ and } H \circ x = x \circ H = H \text{ for all } x, y, z \in H.$$

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Then every group is a hypergroup. By a *homomorphism* of the hypergroup $(H, \circ)$ we mean a function $f : H \to H$ such that

$$f(x \circ y) = f(x) \circ f(y) \text{ for all } x, y \in H.$$

We note here that our homomorphisms are *good homomorphisms* in [1]. Denote by $\mathrm{Hom}(H, \circ)$ the set of all homomorphisms of $(H, \circ)$. Then $\mathrm{Hom}(H, \circ)$ is closed under composition. To show this, let $f, g \in \mathrm{Hom}(H, \circ)$ and $x, y \in H$. Then

$$\begin{aligned}(gf)(x \circ y) = g(f(x \circ y)) &= g(f(x) \circ f(y)) \\ &= g(f(x)) \circ g(f(y)) = (gf)(x) \circ (gf)(y).\end{aligned}$$

It follows that $\mathrm{Hom}(H, \circ)$ is a semigroup under composition. Notice that the identity mapping on $H, 1_H$, is the identity of the semigroup $\mathrm{Hom}(H, \circ)$.

If $G$ is a group, $N$ is a normal subgroup of $G$ and $\circ_N$ is the hyperoperation on $G$ defined by

$$x \circ_N y = xyN \text{ for all } x, y \in G,$$

then $(G, \circ)$ is a hypergroup ([1], page 11). Observe that if $N = \{e\}$ where $e$ is the identity of $G$, then $(G, \circ_N) = G$. If $\mathbb{Z}$ is the set of integers and $n$ is a positive integer, let $(\mathbb{Z}, \circ_n) = (\mathbb{Z}, \circ_{n\mathbb{Z}})$ under usual addition, that is,

$$x \circ_n y = x + y + n\mathbb{Z} \text{ for all } x, y \in \mathbb{Z}.$$

In [3], the authors characterized the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ as follows:

**Theorem 1.1.** ([3]) *If $f : \mathbb{Z} \to \mathbb{Z}$, then the following statements are equivalent.*

(i) $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$.

(ii) $f(x + n\mathbb{Z}) = xf(1) + n\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer $a$ such that*

$$f(x + n\mathbb{Z}) = xa + n\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

The cardinality of a set $X$ is denoted by $|X|$.

The following fact was also provided in [3].

**Theorem 1.2.** ([3]) *The following statements hold.*

(i) *For each* $a \in \mathbb{Z}$, $|\{f \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid f(1) = a\}| = 2^{\aleph_0}$.

(ii) $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

We note here that multi-valued homomorphisms between groups were defined naturally in [6]. Some studies of this notation can be found in [6],[4], [5] and [7].

Let $\mathbb{Z}_n$ be the set of integers modulo $n$. For $x \in \mathbb{Z}$, let $\overline{x}$ be the congruence class of $x$ modulo $n$. Recall that $\mathbb{Z}_n = \{\overline{x} \mid x \in \mathbb{Z}\} = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$. If $\overline{x} \cdot \overline{y} = \overline{xy}$ for all $x, y \in \mathbb{Z}$, then $(\mathbb{Z}_n, \cdot)$ is a semigroup of order $n$ having $\overline{0}$ and $\overline{1}$ as its zero and identity, respectively.

Recall that an equivalence relation $\rho$ on a semigroup $S$ is called a *congruence* on $S$ if

$$\text{for all } x, y, z \in S,\ \ x \,\rho\, y \ \ \Rightarrow\ \ zx \,\rho\, zy \text{ and } xz \,\rho\, yz.$$

If $\rho$ is a congruence on a semigroup $S$, then $S/\rho$ under the operation

$$(x\rho)(y\rho) = (xy)\rho \ \text{ for all } x, y \in S$$

is a semigroup where $x\rho$ is the $\rho$-class of $S$ containing $x$. Moreover, $x \mapsto x\rho$ is an epimorphism from $S$ onto $S/\rho$.

Let $\delta$ be the relation defined on the semigroup $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ as follows:

$$\text{for } f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n),\ \ (f, g) \in \delta \ \ \Leftrightarrow\ \ f(1) \equiv g(1) \mod n.$$

The purpose of this paper is to show that $\delta$ is a congruence on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ and $\mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta \cong (\mathbb{Z}_n, \cdot)$.

## 2 Main Result

First, we give as a lemma how $gf$ and $fg$ map on each coset $x + n\mathbb{Z}$ where $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$.

**Lemma 2.1.** *If $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, then*

$$(gf)(x + n\mathbb{Z}) = (fg)(x + n\mathbb{Z}) = xf(1)g(1) + n\mathbb{Z} \ \textit{ for all } x \in \mathbb{Z}.$$

*Proof.* By Theorem 1.1, we have that for $x \in \mathbb{Z}$,

$$\begin{aligned}(gf)(x + n\mathbb{Z}) = g(f(x + n\mathbb{Z})) &= g(xf(1) + n\mathbb{Z}) \\ &= xf(1)g(1) + n\mathbb{Z} \\ &= xg(1)f(1) + n\mathbb{Z} \\ &= f(xg(1) + n\mathbb{Z}) \\ &= f(g(x + n\mathbb{Z})) \\ &= (fg)(x + n\mathbb{Z}).\end{aligned}$$

□

**Corollary 2.2.** *If $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, then*

$$(gf)(1) \equiv f(1)g(1) \equiv (fg)(1) \mod n.$$

*Proof.* By Lemma 2.1,

$$(gf)(1+n\mathbb{Z}) = (fg)(1+n\mathbb{Z}) = f(1)g(1) + n\mathbb{Z}.$$

Then $(gf)(1) \in (gf)(1+n\mathbb{Z}) = f(1)g(1)+n\mathbb{Z}$ and $(fg)(1) \in (fg)(1+n\mathbb{Z}) = f(1)g(1)+n\mathbb{Z}$, so

$$(gf)(1) \equiv f(1)g(1) \mod n \;\text{ and }\; (fg)(1) \equiv f(1)g(1) \mod n.$$

Since $f(1)g(1) = g(1)f(1)$, the desired result follows. □

**Proposition 2.3.** *The relation $\delta$ is a congruence on the semigroup* $\mathrm{Hom}(\mathbb{Z}, \circ_n)$.

*Proof.* The relation $\delta$ is clearly an equivalence relation on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$. If $f, g, h \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ are such that $f\,\delta\,g$, then $f(1) \equiv g(1) \mod n$. Thus

$$h(1)f(1) \equiv h(1)g(1) \mod n. \tag{1}$$

By Corollary 2.2,

$$\begin{aligned}(hf)(1) &\equiv h(1)f(1) \equiv (fh)(1) \mod n,\\ (hg)(1) &\equiv h(1)g(1) \equiv (gh)(1) \mod n.\end{aligned} \tag{2}$$

From (1) and (2), we have

$$(hf)(1) \equiv (fh)(1) \equiv (hg)(1) \equiv (gh)(1) \mod n.$$

Hence $hf\,\delta\,hg$ and $fh\,\delta\,gh$. This shows that $\delta$ is a congruence on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$. □

**Theorem 2.4.** *If $\varphi : \mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta \to \mathbb{Z}_n$ is defined by*

$$\varphi(f\delta) = \overline{f(1)} \;\; \textit{for all } f \in \mathrm{Hom}(\mathbb{Z}, \circ_n),$$

*then $\varphi$ is an isomorphism from $\mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta$ onto $(\mathbb{Z}_n, \cdot)$.*

*Proof.* To show that $\varphi$ is well-defined, let $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ be such that $f\delta = g\delta$, that is, $f\,\delta\,g$. Then $f(1) \equiv g(1) \mod n$. Hence $\overline{f(1)} = \overline{g(1)}$.

If $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ are such that $\overline{f(1)} = \overline{g(1)}$, then $f(1) \equiv g(1) \mod n$. Thus $f\,\delta\,g$, so $f\delta = g\delta$. Hence $\varphi$ is 1-1. For $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$,

$$\begin{aligned}\varphi\big((f\delta)(g\delta)\big) &= \varphi\big((fg)\delta\big)\\ &= \overline{(fg)(1)}\\ &= \overline{f(1)g(1)} \quad \text{from Corollary 2.2}\\ &= \overline{f(1)}\;\overline{g(1)}\\ &= \varphi(f\delta)\varphi(g\delta).\end{aligned}$$

Therefore $\varphi$ is a semigroup homomorphism. If $k \in \mathbb{Z}$, by Theorem 1.2(i), there is $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f(1) = k$. Thus $\varphi(f\delta) = \overline{f(1)} = \overline{k}$. This shows that $\varphi$ is onto.

This shows that $\varphi$ is an isomorphism from $\mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta$ onto $(\mathbb{Z}_n, \cdot)$, as desired. □

**Remark 2.5.** For $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, by the definition of $\delta$,

$$f\delta = \{g \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid g(1) \equiv f(1) \mod n\}.$$

Thus

$$f\delta = \{g \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid g(1) \in f(1) + n\mathbb{Z}\}.$$

From this fact and Theorem 1.2, the following results are clearly seen.

(1) The cardinality of each $\delta$-class is $2^{\aleph_0}$, that is, for each $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, $|\{g \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid g \in f\delta\}| = 2^{\aleph_0}$.

(2) For each $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n), \{g(1) \mid g \in f\delta\} = f(1) + n\mathbb{Z}$.

**Remark 2.6.** A semigroup $S$ is called *regular* if for every $x \in S, x = xyx$ for some $y \in S$. Ehrlich [2] has shown that $(\mathbb{Z}_n, \cdot)$ is a regular semigroup if and only if $n$ is square-free. From this fact, Theorem 2.4 and Remark 2.5, we deduce that the following statements are equivalent.

(i) For every $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, there is an element $g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f\delta = (fgf)\delta$.

(ii) For every $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, there is an element $g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f(1) \equiv f(1)^2 g(1) \mod n$.

(iii) For every $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, there is an element $g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f(1)^2 g(1) \in f(1) + n\mathbb{Z}$

(iv) $n$ is square-free.

For example, $\mathrm{Hom}(\mathbb{Z}, \circ_6)/\delta$ is a regular semigroup but $\mathrm{Hom}(\mathbb{Z}, \circ_4)/\delta$ is not.

# References

[1] P. Corsini, "Prolegomena of Hypergroup Theory", Aviani Editore, Udine (1993).

[2] G. Ehrlich, *Unit-regular rings*, Portugaliae Mathematica, **27** (1968), 209-212.

[3] W. Mora, W. Hemakul and Y. Kemprasit, *On homomorphisms of certain hypergroups*, East-West J., Spec. Issue for ICDMA 2008 (2008), 137-144.

[4] S. Nenthein and P. Lertwichitsilp, *Surjective multihomomorphisms between cyclic groups*, Thai J. Math., **4**(1) (2006), 35-42.

[5] W. Teparos and Y. Kemprasit, *Regularity of semigroups of multihomomorphisms of* $(\mathbb{Z}_n, +)$, Thai J. Math., Spec. Issue for Annual Meetings in Math. (2006), 25-30.

[6] N. Triphop, A. Harnchoowong and Y. Kemprasit, *Multihomomorphisms between cyclic groups*, Set-valued Math. and Appl., **1**(1) (2007), 9-18.

[7] P. Youngkhong and K. Savettaseranee, *Multihomomorphisms from groups into groups of real numbers*, Thai J. Math., **4**(1) (2006), 43-48.