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A REMARK ON SOME SEMIGROUPS OF HYPERGROUP HOMOMORPHISMS

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Abstract

Let $\mathbb{Z}$ be the set of integers, n a positive integer and $(\mathbb{Z}, \circ_n)$ the hypergroup where $x \circ_n y = x + y + n\mathbb{Z}$ for all $x, y \in \mathbb{Z}$. Denote by $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ the semigroup, under composition, of all homomorphisms of $(\mathbb{Z}, \circ_n)$. It has been shown that for $f : \mathbb{Z} \to \mathbb{Z}, f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ if and only if $f(x + n\mathbb{Z}) = xf(1) + n\mathbb{Z}$ for all $x \in \mathbb{Z}$ and $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$. Using this characterization, we show in this paper that the relation δ on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ defined by $f\delta g \Leftrightarrow f(1) \equiv g(1) \mod n$ is a congruence on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ and $\mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta \cong (\mathbb{Z}_n, \cdot)$.

1 Introduction

A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \to \mathcal{P}^*(H)$ where $\mathcal{P}(H)$ is the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For $x, y \in H, x \circ y$ denotes the value of (x, y). For $A, B \subseteq H$, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

If $x \in H$ and $A \subseteq H$, let $x \circ A$ and $A \circ x$ stand for $\{x\} \circ A$ and $A \circ \{x\}$, respectively. The system $(H, \circ)$ is called a *hypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \text{ and } H \circ x = x \circ H = H \text{ for all } x, y, z \in H.$$

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Then every group is a hypergroup. By a *homomorphism* of the hypergroup $(H, \circ)$ we mean a function $f : H \to H$ such that

$$f(x \circ y) = f(x) \circ f(y) \text{ for all } x, y \in H.$$

We note here that our homomorphisms are *good homomorphisms* in [1]. Denote by $\mathrm{Hom}(H, \circ)$ the set of all homomorphisms of $(H, \circ)$. Then $\mathrm{Hom}(H, \circ)$ is closed under composition. To show this, let $f, g \in \mathrm{Hom}(H, \circ)$ and $x, y \in H$. Then

$$\begin{aligned}(gf)(x \circ y) &= g(f(x \circ y)) = g(f(x) \circ f(y)) \\ &= g(f(x)) \circ g(f(y)) = (gf)(x) \circ (gf)(y).\end{aligned}$$

It follows that $\mathrm{Hom}(H, \circ)$ is a semigroup under composition. Notice that the identity mapping on $H, 1_H$, is the identity of the semigroup $\mathrm{Hom}(H, \circ)$.

If G is a group, N is a normal subgroup of G and $\circ_N$ is the hyperoperation on G defined by

$$x \circ_N y = xyN \text{ for all } x, y \in G,$$

then $(G, \circ)$ is a hypergroup ([1], page 11). Observe that if $N = \{e\}$ where e is the identity of G, then $(G, \circ_N) = G$. If $\mathbb{Z}$ is the set of integers and n is a positive integer, let $(\mathbb{Z}, \circ_n) = (\mathbb{Z}, \circ_{n\mathbb{Z}})$ under usual addition, that is,

$$x \circ_n y = x + y + n\mathbb{Z} \text{ for all } x, y \in \mathbb{Z}.$$

In [3], the authors characterized the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ as follows:

Theorem 1.1. ([3]) *If $f : \mathbb{Z} \to \mathbb{Z}$, then the following statements are equivalent.*

(i) $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$.

(ii) $f(x + n\mathbb{Z}) = xf(1) + n\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer a such that*

$$f(x + n\mathbb{Z}) = xa + n\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

The cardinality of a set X is denoted by $|X|$.

The following fact was also provided in [3].

Theorem 1.2. ([3]) *The following statements hold.*

(i) *For each* $a \in \mathbb{Z}$, $|\{f \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid f(1) = a\}| = 2^{\aleph_0}$.

(ii) $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

We note here that multi-valued homomorphisms between groups were defined naturally in [6]. Some studies of this notation can be found in [6],[4], [5] and [7].

Let $\mathbb{Z}_n$ be the set of integers modulo n. For $x \in \mathbb{Z}$, let $\overline{x}$ be the congruence class of x modulo n. Recall that $\mathbb{Z}_n = \{\overline{x} \mid x \in \mathbb{Z}\} = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$. If $\overline{x} \cdot \overline{y} = \overline{xy}$ for all $x, y \in \mathbb{Z}$, then $(\mathbb{Z}_n, \cdot)$ is a semigroup of order n having $\overline{0}$ and $\overline{1}$ as its zero and identity, respectively.

Recall that an equivalence relation ρ on a semigroup S is called a *congruence* on S if

$$\text{for all } x, y, z \in S, \;\; x \,\rho\, y \;\; \Rightarrow \;\; zx \,\rho\, zy \text{ and } xz \,\rho\, yz.$$

If ρ is a congruence on a semigroup S, then S/ρ under the operation

$$(x\rho)(y\rho) = (xy)\rho \;\; \text{for all } x, y \in S$$

is a semigroup where $x\rho$ is the ρ-class of S containing x. Moreover, $x \mapsto x\rho$ is an epimorphism from S onto S/ρ.

Let δ be the relation defined on the semigroup $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ as follows:

$$\text{for } f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n), \;\; (f, g) \in \delta \;\; \Leftrightarrow \;\; f(1) \equiv g(1) \mod n.$$

The purpose of this paper is to show that δ is a congruence on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ and $\mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta \cong (\mathbb{Z}_n, \cdot)$.

2 Main Result

First, we give as a lemma how gf and fg map on each coset $x + n\mathbb{Z}$ where $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$.

Lemma 2.1. *If $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, then*

$$(gf)(x + n\mathbb{Z}) = (fg)(x + n\mathbb{Z}) = xf(1)g(1) + n\mathbb{Z} \;\; \textit{for all } x \in \mathbb{Z}.$$

Proof. By Theorem 1.1, we have that for $x \in \mathbb{Z}$,

$$\begin{aligned}(gf)(x + n\mathbb{Z}) = g(f(x + n\mathbb{Z})) &= g(xf(1) + n\mathbb{Z}) \\ &= xf(1)g(1) + n\mathbb{Z} \\ &= xg(1)f(1) + n\mathbb{Z} \\ &= f(xg(1) + n\mathbb{Z}) \\ &= f(g(x + n\mathbb{Z})) \\ &= (fg)(x + n\mathbb{Z}).\end{aligned}$$

□

Corollary 2.2. *If $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, then*

$$(gf)(1) \equiv f(1)g(1) \equiv (fg)(1) \mod n.$$

Proof. By Lemma 2.1,

$$(gf)(1+n\mathbb{Z}) = (fg)(1+n\mathbb{Z}) = f(1)g(1) + n\mathbb{Z}.$$

Then $(gf)(1) \in (gf)(1+n\mathbb{Z}) = f(1)g(1) + n\mathbb{Z}$ and $(fg)(1) \in (fg)(1+n\mathbb{Z}) = f(1)g(1) + n\mathbb{Z}$, so

$$(gf)(1) \equiv f(1)g(1) \mod n \ \text{ and } \ (fg)(1) \equiv f(1)g(1) \mod n.$$

Since $f(1)g(1) = g(1)f(1)$, the desired result follows. □

Proposition 2.3. *The relation δ is a congruence on the semigroup* $\mathrm{Hom}(\mathbb{Z}, \circ_n)$.

Proof. The relation δ is clearly an equivalence relation on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$. If $f, g, h \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ are such that $f \,\delta\, g$, then $f(1) \equiv g(1) \mod n$. Thus

$$h(1)f(1) \equiv h(1)g(1) \mod n. \tag{1}$$

By Corollary 2.2,

$$\begin{aligned} (hf)(1) &\equiv h(1)f(1) \equiv (fh)(1) \mod n, \\ (hg)(1) &\equiv h(1)g(1) \equiv (gh)(1) \mod n. \end{aligned} \tag{2}$$

From (1) and (2), we have

$$(hf)(1) \equiv (fh)(1) \equiv (hg)(1) \equiv (gh)(1) \mod n.$$

Hence $hf \,\delta\, hg$ and $fh \,\delta\, gh$. This shows that δ is a congruence on $\mathrm{Hom}(\mathbb{Z}, \circ_n)$. □

Theorem 2.4. *If $\varphi : \mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta \to \mathbb{Z}_n$ is defined by*

$$\varphi(f\delta) = \overline{f(1)} \ \textit{ for all } f \in \mathrm{Hom}(\mathbb{Z}, \circ_n),$$

then φ is an isomorphism from $\mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta$ onto $(\mathbb{Z}_n, \cdot)$.

Proof. To show that φ is well-defined, let $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ be such that $f\delta = g\delta$, that is, $f \,\delta\, g$. Then $f(1) \equiv g(1) \mod n$. Hence $\overline{f(1)} = \overline{g(1)}$.

If $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ are such that $\overline{f(1)} = \overline{g(1)}$, then $f(1) \equiv g(1) \mod n$. Thus $f \,\delta\, g$, so $f\delta = g\delta$. Hence φ is 1-1. For $f, g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$,

$$\begin{aligned} \varphi\big((f\delta)(g\delta)\big) &= \varphi\big((fg)\delta\big) \\ &= \overline{(fg)(1)} \\ &= \overline{f(1)g(1)} \quad \text{from Corollary 2.2} \\ &= \overline{f(1)}\ \overline{g(1)} \\ &= \varphi(f\delta)\varphi(g\delta). \end{aligned}$$

Therefore φ is a semigroup homomorphism. If $k \in \mathbb{Z}$, by Theorem 1.2(i), there is $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f(1) = k$. Thus $\varphi(f\delta) = \overline{f(1)} = \overline{k}$. This shows that φ is onto.

This shows that φ is an isomorphism from $\mathrm{Hom}(\mathbb{Z}, \circ_n)/\delta$ onto $(\mathbb{Z}_n, \cdot)$, as desired. $\square$

Remark 2.5. For $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, by the definition of δ,

$$f\delta = \{g \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid g(1) \equiv f(1) \mod n\}.$$

Thus

$$f\delta = \{g \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid g(1) \in f(1) + n\mathbb{Z}\}.$$

From this fact and Theorem 1.2, the following results are clearly seen.

(1) The cardinality of each δ-class is $2^{\aleph_0}$, that is, for each $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, $|\{g \in \mathrm{Hom}(\mathbb{Z}, \circ_n) \mid g \in f\delta\}| = 2^{\aleph_0}$.

(2) For each $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, $\{g(1) \mid g \in f\delta\} = f(1) + n\mathbb{Z}$.

Remark 2.6. A semigroup S is called *regular* if for every $x \in S, x = xyx$ for some $y \in S$. Ehrlich [2] has shown that $(\mathbb{Z}_n, \cdot)$ is a regular semigroup if and only if n is square-free. From this fact, Theorem 2.4 and Remark 2.5, we deduce that the following statements are equivalent.

(i) For every $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, there is an element $g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f\delta = (fgf)\delta$.

(ii) For every $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, there is an element $g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f(1) \equiv f(1)^2 g(1) \mod n$.

(iii) For every $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, there is an element $g \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ such that $f(1)^2 g(1) \in f(1) + n\mathbb{Z}$

(iv) n is square-free.

For example, $\mathrm{Hom}(\mathbb{Z}, \circ_6)/\delta$ is a regular semigroup but $\mathrm{Hom}(\mathbb{Z}, \circ_4)/\delta$ is not.

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