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NON-DETERMINISTIC HYPERIDENTITIES IN MANY-SORTED ALGEBRAS

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Abstract

Many-sorted algebras are used in Computer Science for abstract data type specifications. It is widely believed that many-sorted algebras are the appropriate mathematical tools to explain what abstract data types are ([6]). In this paper we extend the approach to non-deterministic hypersubstitutions and non-deterministic hyperidentities given in [4] to the many-sorted case. The main result is the characterization of non-deterministic solid varieties. This will be done by showing that on the basis of non-deterministic hypersubstitutions one obtains a conjugate pair of additive closure operators which allows to apply the theory of conjugate pairs of additive closure operators also to this case (see [7]). Our results form a universal-algebraic background of the theory of many-sorted tree languages (see [8]).

1 Introduction

We follow the definition of terms for many-sorted algebras given in [1] and the superposition of many-sorted terms from [3].

To describe terms over many-sorted algebras we need the following notation.

Let I be a non-empty set and $n \in \mathbb{N}^+ := \mathbb{N}\setminus\{0\}$, let $I^* := \bigcup_{n\geq 1} I^n$, $\Sigma \subseteq I^* \times I$. Then we define $\Sigma_n := \Sigma \cap (I^n \times I)$. Let $\Sigma_m(i) := \{\gamma \in \Sigma_m \mid \gamma(m+1) = i\}$,

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$i \in I$, $m \in \mathbb{N}^+$. We set $\Sigma(i) := \bigcup\limits_{m \in \mathbb{N}^+} \Sigma_m(i)$. Let K_γ be a set of indices of each $\gamma \in \Sigma$. If $|K_\gamma| = 1$, we will drop the index.

Definition 1.1 ([1]) Let $X^{(n)} := (X_i^{(n)})_{i \in I}$ be an I-sorted set of variables, also called an n-element I-sorted alphabet, with $X_i^{(n)} := \{x_{i1}, \ldots, x_{in}\}, i \in I$, and let $((f_\gamma)_k)_{k \in K_\gamma, \gamma \in \Sigma}$ be an indexed set of Σ-sorted operation symbols. Then a set $W_n(i)$ which is called the set of all n-ary Σ-terms of sort i, is inductively defined as follows: For all $i \in I$ we set

(i) $W_0^n(i) := X_i^{(n)}$.

(ii) $W_{l+1}^n(i) := W_l^n(i) \cup \{(f_\gamma)_k(t_1, \ldots, t_m) \mid k \in K_\gamma, \gamma \in \Sigma_m(i)\}, l \in \mathbb{N}$, $t_j \in W_l^n(i_j), 1 \leq j \leq m, m \in \mathbb{N}$ whenever $\gamma = (i_1, \ldots, i_m, i)$.

Then $W_n(i) := \bigcup\limits_{l=0}^{\infty} W_l^n(i)$ and we set $W(i) := \bigcup\limits_{n \in \mathbb{N}^+} W_n(i)$. Let $X_i := \bigcup\limits_{n \in \mathbb{N}^+} X_i^{(n)}$ and $X := (X_i)_{i \in I}$. Let $W_\Sigma(X) := (W(i))_{i \in I}$. The set $W_\Sigma(X)$ is called I-sorted set of all Σ-terms.

For $\alpha \in \Sigma_m$ let $\alpha(j)$ be the j-th component of α for $1 \leq j \leq m$. Then for any $n \in \mathbb{N}^+, i \in I$ we set

$\Lambda_n(i) := \{(w, i) \in I^n \times I \mid \exists m \in \mathbb{N}^+, \exists \alpha \in \Sigma_m, \exists j\ (1 \leq j \leq m)(\alpha(j) = i)\}$.

Let $\Lambda(i) := \bigcup\limits_{n=1}^{\infty} \Lambda_n(i)$ and we set $\Lambda := \bigcup\limits_{i \in I} \Lambda(i)$.

Let $\mathcal{P}(W(i))$ be the power set of $W(i)$. The elements of $\mathcal{P}(W(i))$ are called tree languages of sort i. Now we define superposition operations on many-sorted sets of tree languages.

Definition 1.2 ([3]) Let $T \in \mathcal{P}(W(i))$, $T_j \in \mathcal{P}(W(k_j))$, $1 \leq j \leq n, n \in \mathbb{N}$, such that T, T_j are non-empty. Then the superposition operation

$$S_\alpha^{nd} : \mathcal{P}(W(i)) \times \mathcal{P}(W(k_1)) \times \cdots \times \mathcal{P}(W(k_n)) \to \mathcal{P}(W(i))$$

with $\alpha = (k_1, \ldots, k_n; i) \in \Lambda$, is inductively defined in the following way:

1) If $T = \{x_{ij}\}$ where $x_{ij} \in X_i$, then

  1.1) for $i \neq k_j$,
  $S_\alpha^{nd}(\{x_{ij}\}, T_1, \ldots, T_n) := \{x_{ij}\}$,

  1.2) for $i = k_j$,
  $S_\alpha^{nd}(\{x_{ij}\}, T_1, \ldots, T_n) := T_j$.

2) If $T = \{(f_\gamma)_k(s_1, \ldots, s_m)\} \in \mathcal{P}(W(i))$ where $k \in K_\gamma, \gamma = (i_1, \ldots, i_m; i) \in \Sigma$, $s_q \in W(i_q), 1 \leq q \leq m, m \in \mathbb{N}$, and if we assume that $S_{\alpha_q}^{nd}(\{s_q\}, T_1, \ldots, T_n)$ with $\alpha_q = (k_1, \ldots, k_n; i_q) \in \Lambda$, are already satisfied, then

$$S_\alpha^{nd}(\{(f_\gamma)_k(s_1, \ldots, s_m)\}, T_1, \ldots, T_n) := \{(f_\gamma)_k(r_1, \ldots, r_m) \mid r_q \in S_{\alpha_q}^{nd}(\{s_q\}, T_1, \ldots, T_n)\}.$$

3) If T is an arbitrary subset of $W(i)$, then
$S^{nd}_{\alpha}(T, T_1, \ldots, T_n) := \bigcup_{t \in T} S^{nd}_{\alpha}(\{t\}, T_1, \ldots, T_n)$.

If one of the sets $T, T_1, \ldots, T_n$ is empty, then we define $S^{nd}_{\alpha}(T, T_1, \ldots, T_n) := \emptyset$.

Non-deterministic many-sorted hypersubstitutions map many-sorted operation symbols to sets of many-sorted terms and are defined as follows.

Definition 1.3 ([3]) Let $((f_\gamma)_k)_{k \in K_\gamma, \gamma \in \Sigma}$ be an indexed set of Σ-sorted operation symbols and $\mathcal{P}(W_\Sigma(X)) := (\mathcal{P}(W(i)))_{i \in I}$. Any mapping

$$\sigma^{nd}_i : \{(f_\gamma)_k \mid k \in K_\gamma, \gamma \in \Sigma(i)\} \to \mathcal{P}(W(i)), i \in I$$

with $\sigma^{nd}_i((f_\gamma)_k) \subseteq W_i \subseteq W(i)$ such that W_i is the set of all Σ-terms of sort i which have arity $|\gamma|$-1, is said to be a non-deterministic Σ-hypersubstitution of sort i. Let $nd\Sigma(i)$-Hyp be the set of all non-deterministic Σ-hypersubstitutions of sort i. The I-sorted mapping $\sigma^{nd} := (\sigma^{nd}_i)_{i \in I}$ is called an I-sorted non-deterministic Σ-hypersubstitution. Let $nd\Sigma$-Hyp be the set of all I-sorted non-deterministic Σ-hypersubstitutions. Any I-sorted non-deterministic Σ-hypersubstitution σ^{nd} can inductively be extended to an I-sorted mapping $\hat{\sigma}^{nd} := (\hat{\sigma}^{nd}_i)_{i \in I}$. The I-sorted mapping

$$\hat{\sigma}^{nd} : \mathcal{P}(W_\Sigma(X)) \to \mathcal{P}(W_\Sigma(X))$$

is defined in the following way: For all $i \in I$, for every $T \subseteq W(i)$,

(1) if $T = \emptyset$, then $\hat{\sigma}^{nd}_i[T] := \emptyset$,

(2) if $T = \{x_{ij}\}, x_{ij} \in X_i$, then $\hat{\sigma}^{nd}_i[T] := \{x_{ij}\}$,

(3) if $T = \{(f_\gamma)_k(t_1, \ldots, t_n)\}$, with $k \in K_\gamma, \gamma \in \Sigma_n(i)$ and $t_j \in W(k_j), 1 \leq j \leq n, n \in \mathbb{N}$ whenever $\gamma = (k_1, \ldots, k_n, i)$, and if we assume that $\hat{\sigma}^{nd}_{k_j}[\{t_j\}]$ are already defined, then

$$\hat{\sigma}^{nd}_i[T] := S^{nd}_{\gamma}(\sigma^{nd}_i((f_\gamma)_k), \hat{\sigma}^{nd}_{k_1}[\{t_1\}], \ldots, \hat{\sigma}^{nd}_{k_n}[\{t_n\}]),$$

(4) if T is an arbitrary subset of $W(i)$, then $\hat{\sigma}^{nd}_i[T] := \bigcup_{t \in T} \hat{\sigma}^{nd}_i[\{t\}]$.

A many-sorted Σ-algebra is a pair $\mathcal{A} := ((A_i)_{i \in I}; (f^{\mathcal{A}}_\gamma)_{\gamma \in \Sigma})$ consisting of an I-sorted set and a Σ-sorted set of I-sorted fundamental operations. Important examples for I-sorted Σ-algebras are vector spaces over a field $\mathcal{F}$ and deterministic automata. Let $Alg(\Sigma)$ be the class of all many-sorted Σ-algebras. The connection between many-sorted terms and term operations of many-sorted algebras of the same type is given by inducing term operations by terms.

Definition 1.4 ([3]) Let $X^{(n)}$ be an n-element I-sorted alphabet and let A be an I-sorted set. Let $\mathcal{A} \in Alg(\Sigma)$ be a Σ-algebra, and $t \in W_n(i)$ be an n-ary Σ-term of sort $i \in I$. Let $f := (f_i)_{i \in I}$ where $f_i : X_i^{(n)} \to A_i$ be an I-sorted evaluation mapping of variables from $X^{(n)}$ by elements in A. Each mapping f_i can be extended in a canonical way to a mapping $\bar{f_i} : W_n(i) \to A_i$. Then $t^{\mathcal{A}} : A^{X^{(n)}} \to A_i$ defined by

$$t^{\mathcal{A}}(f) := \bar{f_i}(t) \text{ for all } f \in A^{X^{(n)}},$$

where $\bar{f_i}$ is the extension of the evaluation mapping $f_i : X_i^{(n)} \to A_i$. $t^{\mathcal{A}}$ is called the n-ary Σ-term operation on $\mathcal{A}$ induced by the n-ary Σ-term t of sort i.

Let $W^{\mathcal{A}}(i)$ be the set of all Σ-term operations on $\mathcal{A}$ induced by all Σ-terms of sort i. Then we set $W_{\Sigma}^{\mathcal{A}}(X) := (W^{\mathcal{A}}(i))_{i \in I}$ and call this set I-sorted set of Σ-term operations induced on $\mathcal{A}$ by the Σ-terms. This can be extended to sets of terms.

Definition 1.5 Let $\mathcal{A}$ be a Σ-algebra, and $B \in \mathcal{P}(W(i)), i \in I$. Then we define the set $B^{\mathcal{A}}$ of Σ-term operations on $\mathcal{A}$ induced by Σ-terms of sort i as follows:

(1) If $B = \{x_{ij}\}$, then $B^{\mathcal{A}} := \{x_{ij}^{\mathcal{A}}\}$.

(2) If $B = \{(f_\gamma)_k(t_1, \ldots, t_n)\}$ where $k \in K_\gamma, \gamma \in \Sigma_n(i)$ and $t_j \in W(i_j), 1 \leq j \leq n, n \in \mathbb{N}$ whenever $\gamma = (i_1, \ldots, i_n, i)$, then $B^{\mathcal{A}} := \{((f_\gamma)_k)^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})\}$ where $((f_\gamma)_k)^{\mathcal{A}}$ is the fundamental operation of $\mathcal{A}$ corresponding to the operation symbol $(f_\gamma)_k$ and where $t_j^{\mathcal{A}}$ are the Σ-term operations on $\mathcal{A}$ which are induced in the usual way by the t_j $'s$.

(3) If B is an arbitrary non-empty subset of $W(i)$, then we define $B^{\mathcal{A}} := \bigcup\limits_{b \in B} \{b\}^{\mathcal{A}}$.

If B is empty, then we define $B^{\mathcal{A}} := \emptyset$.

A superposition operation for sets of Σ-term operations on the many-sorted algebra $\mathcal{A}$ can be defined in the following way:

Definition 1.6 Let $\mathcal{A}$ be a Σ-algebra and let $T \in \mathcal{P}(W(i))$, $T_j \in \mathcal{P}(W(k_j))$, $1 \leq j \leq n, n \in \mathbb{N}$, such that T, T_j are non-empty. Then the superposition operation

$$S_\alpha^{ndA} : \mathcal{P}(W^{\mathcal{A}}(i)) \times \mathcal{P}(W^{\mathcal{A}}(k_1)) \times \cdots \times \mathcal{P}(W^{\mathcal{A}}(k_n)) \to \mathcal{P}(W^{\mathcal{A}}(i))$$

where $\alpha = (k_1, \ldots, k_n; i) \in \Lambda$, is inductively defined in the following way:

1) If $T = \{x_{ij}\}$ where $x_{ij} \in X_i$, then

1.1) for $i \neq k_j$,
$S_\alpha^{ndA}(\{x_{ij}\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) := \{x_{ij}\}^{\mathcal{A}}$,

1.2) for $i = k_j$,
$S_\alpha^{ndA}(\{x_{ij}\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) := T_j^{\mathcal{A}}$.

2) If $T = \{(f_\gamma)_k(s_1, \ldots, s_m)\} \in \mathcal{P}(W(i))$ where $k \in K_\gamma, \gamma = (i_1, \ldots, i_m; i) \in \Sigma,\ s_q \in W(i_q), 1 \leq q \leq m, m \in \mathbb{N}$, and if we assume that

$$S_{\alpha_q}^{ndA}(\{s_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})$$

with $\alpha_q = (k_1, \ldots, k_n; i_q) \in \Lambda$, are already defined, then
$S_\alpha^{ndA}(\{(f_\gamma)_k(s_1, \ldots, s_m)\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})$
$:= \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_m^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in S_{\alpha_q}^{ndA}(\{s_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})\}$.

3) If T is an arbitrary subset of $W(i)$, then
$S_\alpha^{ndA}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) := \bigcup_{t \in T} S_\alpha^{ndA}(\{t\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})$.

If one of the sets $T, T_1, \ldots, T_n$ is empty, then we define $S_\alpha^{ndA}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) := \emptyset$.

For illustration we consider the following example.

Example 1.7 Let $I = \{1, 2\}$, $\Sigma = \{(1, 2, 1), (2, 1, 1)\}$ and $\mathcal{A}$ be a Σ-algebra. Let $T = \{x_{12}, f_{(1,2,1)}(x_{11}, x_{21})\}, T_1 = \{f_{(2,1,1)}(x_{21}, x_{11})\}$ and $T_2 = \{x_{22}\}$. Then

$$\begin{aligned}
S_{(1,2,1)}^{ndA}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, T_2^{\mathcal{A}}) &= S_{(1,2,1)}^{ndA}(\{x_{12}, f_{(1,2,1)}(x_{11}, x_{21})\}^{\mathcal{A}}, \{f_{(2,1,1)}(x_{21}, x_{11})\}^{\mathcal{A}}, \\
&\qquad \{x_{22}\}^{\mathcal{A}}) \\
&= S_{(1,2,1)}^{ndA}(\{x_{12}\}^{\mathcal{A}}, \{f_{(2,1,1)}(x_{21}, x_{11})\}^{\mathcal{A}}, \{x_{22}\}^{\mathcal{A}}) \cup \\
&\qquad S_{(1,2,1)}^{ndA}(\{f_{(1,2,1)}(x_{11}, x_{21})\}^{\mathcal{A}}, \{f_{(2,1,1)}(x_{21}, x_{11})\}^{\mathcal{A}}, \{x_{22}\}^{\mathcal{A}}) \\
&= \{x_{12}\}^{\mathcal{A}} \cup \{f_{(1,2,1)}^{\mathcal{A}}(r_{11}^{\mathcal{A}}, r_{21}^{\mathcal{A}}) \mid r_{11}^{\mathcal{A}} \in S_{(1,2,1)}^{ndA}(\{x_{11}\}^{\mathcal{A}}, \\
&\qquad \{f_{(2,1,1)}(x_{21}, x_{11})\}^{\mathcal{A}}, \{x_{22}\}^{\mathcal{A}}), r_{21}^{\mathcal{A}} \in S_{(1,2,1)}^{ndA}(\{x_{21}\}^{\mathcal{A}}, \\
&\qquad \{f_{(2,1,1)}(x_{21}, x_{11})\}^{\mathcal{A}}, \{x_{22}\}^{\mathcal{A}})\} \\
&= \{x_{12}\}^{\mathcal{A}} \cup \{f_{(1,2,1)}^{\mathcal{A}}(r_{11}^{\mathcal{A}}, r_{21}^{\mathcal{A}}) \mid r_{11}^{\mathcal{A}} \in \{f_{(2,1,1)}(x_{21}, x_{11})\}^{\mathcal{A}}, \\
&\qquad r_{21}^{\mathcal{A}} \in \{x_{21}\}^{\mathcal{A}}\} \\
&= \{x_{12}\}^{\mathcal{A}} \cup \{f_{(1,2,1)}^{\mathcal{A}}(r_{11}^{\mathcal{A}}, r_{21}^{\mathcal{A}}) \mid r_{11}^{\mathcal{A}} \in \{(f_{(2,1,1)}(x_{21}, x_{11}))^{\mathcal{A}}\}, \\
&\qquad r_{21}^{\mathcal{A}} \in \{x_{21}^{\mathcal{A}}\}\} \\
&= \{x_{12}\}^{\mathcal{A}} \cup \{f_{(1,2,1)}^{\mathcal{A}}((f_{(2,1,1)}(x_{21}, x_{11}))^{\mathcal{A}}, x_{21}^{\mathcal{A}})\} \\
&= \{x_{12}\}^{\mathcal{A}} \cup \{(f_{(1,2,1)}(f_{(2,1,1)}(x_{21}, x_{11}), x_{21})^{\mathcal{A}}\} \\
&= \{x_{12}\}^{\mathcal{A}} \cup \{f_{(1,2,1)}(f_{(2,1,1)}(x_{21}, x_{11}), x_{21})\}^{\mathcal{A}} \\
&= \{x_{12}, f_{(1,2,1)}(f_{(2,1,1)}(x_{21}, x_{11}), x_{21})\}^{\mathcal{A}}
\end{aligned}$$

Proposition 1.8 *Let $\mathcal{A}$ be a Σ-algebra and let $\alpha = (i_1, \ldots, i_m; i)$, $\beta = (k_1, \ldots, k_n; i)$, $\beta_j = (i_1, \ldots, i_m; k_j) \in \Lambda$ with $m \leq n, 1 \leq j \leq n$ such that $m, n \in \mathbb{N}^+$. Assume that $i \neq i_q, 1 \leq q \leq m$ if $i \neq k_j$. Let $S \in \mathcal{P}(W(i)), L_j \in \mathcal{P}(W(k_j)), T_q \in \mathcal{P}(W(i_q))$ such that L_j, T_q are non-empty. Then we have*

$$S_\alpha^{ndA}(S_\beta^{ndA}(S^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_\beta^{ndA}(S^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})).$$

Proof If S is empty, then all is clear. If S is non-empty, then we will give a proof by induction on the complexity of the Σ-term which is the only element of the one-element set S.

1) If $S = \{x_{ij}\}$ where $x_{ij} \in X_i$, then

1.1) for $i \neq k_j$,

$$S_\alpha^{ndA}(S_\beta^{ndA}(S^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_\alpha^{ndA}(S_\beta^{ndA}(\{x_{ij}\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_\alpha^{ndA}(\{x_{ij}\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = \{x_{ij}\}^{\mathcal{A}} \\ = S_\beta^{ndA}(\{x_{ij}\}^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})) \\ = S_\beta^{ndA}(S^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})),$$

1.2) for $i = k_j$,

$$S_\alpha^{ndA}(S_\beta^{ndA}(S^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_\alpha^{ndA}(S_\beta^{ndA}(\{x_{ij}\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_\alpha^{ndA}(L_j^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_{\beta_j}^{ndA}(L_j^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_\beta^{ndA}(\{x_{ij}\}^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})) \\ = S_\beta^{ndA}(S^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})).$$

2) If $S = \{(f_\gamma)_k(s_1, \ldots, s_p)\} \in \mathcal{P}(W(i))$ with $k \in K_\gamma, \gamma = (h_1, \ldots, h_p; i)$ $\in \Sigma, s_t \in W(h_t), 1 \leq t \leq p, p \in \mathbb{N}$ and if we assume that the equations $S_{\alpha_t}^{ndA}(S_{\lambda_t}^{ndA}(\{s_t\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) = S_{\lambda_t}^{ndA}(\{s_t\}^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}},$ $T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}))$ with $\lambda_t = (k_1, \ldots, k_n; h_t), \alpha_t =$ $(i_1, \ldots, i_m; h_t) \in \Lambda$, are satisfied, then $S_\alpha^{ndA}(S_\beta^{ndA}(S^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})$

$$= S_\alpha^{ndA}(S_\beta^{ndA}((\{(f_\gamma)_k(s_1, \ldots, s_p)\})^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = S_\alpha^{ndA}(\{((f_\gamma)_k)^{\mathcal{A}}(u_1^{\mathcal{A}}, \ldots, u_p^{\mathcal{A}}) \mid u_t^{\mathcal{A}} \in S_{\lambda_t}^{ndA}(\{s_t\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}})\}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\ = \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_p^{\mathcal{A}}) \mid r_t^{\mathcal{A}} \in S_{\alpha_t}^{ndA}(\{u_t^{\mathcal{A}} \mid u_t^{\mathcal{A}} \in S_{\lambda_t}^{ndA}(\{s_t\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}})\}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})\} \\ = \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_p^{\mathcal{A}}) \mid r_t^{\mathcal{A}} \in S_{\alpha_t}^{ndA}(S_{\lambda_t}^{ndA}(\{s_t\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})\} \\ = \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_p^{\mathcal{A}}) \mid r_t^{\mathcal{A}} \in S_{\lambda_t}^{ndA}(\{s_t\}^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}))\} \\ = S_\beta^{ndA}((\{(f_\gamma)_k(s_1, \ldots, s_p)\})^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})) \\ = S_\beta^{ndA}(S^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})).$$

3) If S is an arbitrary subset of $W(i)$, then

$$
\begin{aligned}
& S_{\alpha}^{ndA}(S_{\beta}^{ndA}(S^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\
&= S_{\alpha}^{ndA}(\bigcup_{s\in S} S_{\beta}^{ndA}(\{s\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\
&= \bigcup_{s\in S} S_{\alpha}^{ndA}(S_{\beta}^{ndA}(\{s\}^{\mathcal{A}}, L_1^{\mathcal{A}}, \ldots, L_n^{\mathcal{A}}), T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}) \\
&= \bigcup_{s\in S} S_{\beta}^{ndA}(\{s\}^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})) \\
&= S_{\beta}^{ndA}(S^{\mathcal{A}}, S_{\beta_1}^{ndA}(L_1^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}}), \ldots, S_{\beta_n}^{ndA}(L_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_m^{\mathcal{A}})).
\end{aligned}
$$

□

Proposition 1.9 *Let $\mathcal{A}$ be a Σ-algebra, and let $\alpha = (k_1, \ldots, k_n; i) \in \Lambda$. For $T \in \mathcal{P}(W(i))$ and for any $x_{k_j j} \in X_{k_j}, 1 \le j \le n, n \in \mathbb{N}$ we have*

$$S_{\alpha}^{ndA}(T^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) = T^{\mathcal{A}}.$$

Proof If T is empty, then all is clear. If T is non-empty, then we will give a proof by induction on the complexity of the Σ-term which is the only element of the one-element set T.

1) If $T = \{x_{ij}\}$ where $x_{ij} \in X_i$, then

1.1) for $i \neq k_j$,

$$
\begin{aligned}
& S_{\alpha}^{ndA}(T^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) \\
&= S_{\alpha}^{ndA}(\{x_{ij}\}^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) \\
&= \{x_{ij}\}^{\mathcal{A}} \\
&= T^{\mathcal{A}},
\end{aligned}
$$

1.2) for $i = k_j$,

$$
\begin{aligned}
& S_{\alpha}^{ndA}(T^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) \\
&= S_{\alpha}^{ndA}(\{x_{ij}\}^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) \\
&= \{x_{k_j j}\}^{\mathcal{A}} \\
&= \{x_{ij}\}^{\mathcal{A}} \\
&= T^{\mathcal{A}}.
\end{aligned}
$$

2) If $T = \{(f_\gamma)_k(s_1, \ldots, s_m)\} \in \mathcal{P}(W(i))$ where $k \in K_\gamma, \gamma = (i_1, \ldots, i_m; i) \in \Sigma, s_q \in W(i_q), 1 \le q \le m, m \in \mathbb{N}$ and if we assume that the equations

$$S_{\alpha_q}^{ndA}(\{s_q\}^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) = \{s_q\}^{\mathcal{A}}$$

with $\alpha_q = (k_1, \ldots, k_n; i_q) \in \Lambda$, are satisfied, then

$$
\begin{aligned}
& S_{\alpha}^{ndA}(T^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) \\
&= S_{\alpha}^{ndA}((\{(f_\gamma)_k(s_1, \ldots, s_m)\})^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}}) \\
&= \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_m^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in S_{\alpha_q}^{ndA}(\{s_q\}^{\mathcal{A}}, \{x_{k_1 1}\}^{\mathcal{A}}, \ldots, \{x_{k_n n}\}^{\mathcal{A}})\} \\
&= \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_m^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in \{s_q\}^{\mathcal{A}}\} \\
&= \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_m^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in \{s_q^{\mathcal{A}}\}\} \\
&= \{((f_\gamma)_k)^{\mathcal{A}}(s_1^{\mathcal{A}}, \ldots, s_m^{\mathcal{A}})\} \\
&= \{((f_\gamma)_k(s_1, \ldots, s_m))^{\mathcal{A}}\} \\
&= \{((f_\gamma)_k(s_1, \ldots, s_m))\}^{\mathcal{A}} \\
&= T^{\mathcal{A}}.
\end{aligned}
$$

3) If T is an arbitrary subset of $W(i)$, then

$$\begin{aligned}&S_\alpha^{ndA}(T^{\mathcal{A}},\{x_{k_1 1}\}^{\mathcal{A}},\ldots,\{x_{k_n n}\}^{\mathcal{A}})\\ &=\bigcup_{t\in T} S_\alpha^{ndA}(\{t\}^{\mathcal{A}},\{x_{k_1 1}\}^{\mathcal{A}},\ldots,\{x_{k_n n}\}^{\mathcal{A}})\\ &=\bigcup_{t\in T}\{t\}^{\mathcal{A}}\\ &=T^{\mathcal{A}}.\end{aligned}$$

□

Lemma 1.10 *Let $\mathcal{A}$ be a Σ-algebra, and $\alpha=(k_1,\ldots,k_n,i)\in\Lambda$. Let $T\in\mathcal{P}(W(i))$, $T_j\in\mathcal{P}(W(k_j))$, $1\le j\le n, n\in\mathbb{N}$ such that T, T_j are non-empty. Then*

$$\bigcup_{t\in T}(S_\alpha^{nd}(\{t\},T_1,\ldots,T_n))^{\mathcal{A}}=(\bigcup_{t\in T}S_\alpha^{nd}(\{t\},T_1,\ldots,T_n))^{\mathcal{A}}.$$

Proof Let $s\in W(i)$. Then

$$\begin{aligned}s^{\mathcal{A}}\in\bigcup_{t\in T}(S_\alpha^{nd}(\{t\},T_1,\ldots,T_n))^{\mathcal{A}} &\Leftrightarrow s^{\mathcal{A}}\in(S_\alpha^{nd}(\{t\},T_1,\ldots,T_n))^{\mathcal{A}} \text{ for some } t\in T\\ &\Leftrightarrow s\in S_\alpha^{nd}(\{t\},T_1,\ldots,T_n) \text{ for some } t\in T\\ &\Leftrightarrow s\in\bigcup_{t\in T}S_\alpha^{nd}(\{t\},T_1,\ldots,T_n)\\ &\Leftrightarrow s^{\mathcal{A}}\in(\bigcup_{t\in T}S_\alpha^{nd}(\{t\},T_1,\ldots,T_n))^{\mathcal{A}}.\end{aligned}$$

□

Lemma 1.11 *Let $\mathcal{A}$ be a Σ-algebra, and let $\alpha=(k_1,\ldots,k_n;i)\in\Lambda$. Let $T\in\mathcal{P}(W(i)), T_j\in\mathcal{P}(W(k_j))$, $1\le j\le n, n\in\mathbb{N}$ such that T_j is non-empty. Then we have*

$$(S_\alpha^{nd}(T,T_1,\ldots,T_n))^{\mathcal{A}}=S_\alpha^{ndA}(T^{\mathcal{A}},T_1^{\mathcal{A}},\ldots,T_n^{\mathcal{A}}).$$

Proof If T is empty, then all is clear. If T is non-empty, then we will give a proof by induction on the complexity of the Σ-term which is the only element of the one-element set T.

1) If $T=\{x_{ij}\}$ where $x_{ij}\in X_i$, then

1.1) for $i\neq k_j$,

$$\begin{aligned}(S_\alpha^{nd}(T,T_1,\ldots,T_n))^{\mathcal{A}} &= (S_\alpha^{nd}(\{x_{ij}\},T_1,\ldots,T_n))^{\mathcal{A}}\\ &= \{x_{ij}\}^{\mathcal{A}}\\ &= S_\alpha^{ndA}(\{x_{ij}\}^{\mathcal{A}},T_1^{\mathcal{A}},\ldots,T_n^{\mathcal{A}})\\ &= S_\alpha^{ndA}(T^{\mathcal{A}},T_1^{\mathcal{A}},\ldots,T_n^{\mathcal{A}}),\end{aligned}$$

1.2) for $i=k_j$,

$$\begin{aligned}(S_\alpha^{nd}(T,T_1,\ldots,T_n))^{\mathcal{A}} &= (S_\alpha^{nd}(\{x_{ij}\},T_1,\ldots,T_n))^{\mathcal{A}}\\ &= T_j^{\mathcal{A}}\\ &= S_\alpha^{ndA}(\{x_{ij}\}^{\mathcal{A}},T_1^{\mathcal{A}},\ldots,T_n^{\mathcal{A}})\\ &= S_\alpha^{ndA}(T^{\mathcal{A}},T_1^{\mathcal{A}},\ldots,T_n^{\mathcal{A}}).\end{aligned}$$

2) If $T = \{(f_\gamma)_k(s_1, \ldots, s_m)\} \in \mathcal{P}(W(i))$ where $k \in K_\gamma, \gamma = (i_1, \ldots, i_m; i) \in \Sigma,\ s_q \in W(i_q), 1 \leq q \leq m, m \in \mathbb{N}^+$ and if we assume that the equations

$$(S^{nd}_{\alpha_q}(\{s_q\}, T_1, \ldots, T_n))^{\mathcal{A}} = S^{ndA}_{\alpha_q}(\{s_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})$$

with $\alpha_q = (k_1, \ldots, k_n; i_q) \in \Lambda$, are satisfied, then

$$\begin{aligned}
&(S^{nd}_{\alpha}(T, T_1, \ldots, T_n))^{\mathcal{A}} \\
&= (S^{nd}_{\alpha}(\{(f_\gamma)_k(s_1, \ldots, s_m)\}, T_1, \ldots, T_n))^{\mathcal{A}} \\
&= (\{(f_\gamma)_k(r_1, \ldots, r_m) \mid r_q \in S^{nd}_{\alpha_q}(\{s_q\}, T_1, \ldots, T_n)\})^{\mathcal{A}} \\
&= \{((f_\gamma)_k(r_1, \ldots, r_m))^{\mathcal{A}} \mid r_q^{\mathcal{A}} \in (S^{nd}_{\alpha_q}(\{s_q\}, T_1, \ldots, T_n))^{\mathcal{A}}\} \\
&= \{((f_\gamma)_k)^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_m^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in S^{ndA}_{\alpha_q}(\{s_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})\} \\
&= S^{ndA}_{\alpha}(\{(f_\gamma)_k(s_1, \ldots, s_m)\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) \\
&= S^{ndA}_{\alpha}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}).
\end{aligned}$$

3) If T is an arbitrary subset of $W(i)$, then

$$\begin{aligned}
(S^{nd}_{\alpha}(T, T_1, \ldots, T_n))^{\mathcal{A}} &= (\bigcup_{t \in T} S^{nd}_{\alpha}(\{t\}, T_1, \ldots, T_n))^{\mathcal{A}} \\
&= \bigcup_{t \in T} (S^{nd}_{\alpha}(\{t\}, T_1, \ldots, T_n))^{\mathcal{A}} \\
&= \bigcup_{t \in T} S^{ndA}_{\alpha}(\{t\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) \\
&= S^{ndA}_{\alpha}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}).
\end{aligned}$$

□

2 I-Sorted Nd-Identities and Nd-Model Classes

Let K be a subset of $Alg(\Sigma)$ and we set $\mathcal{P}(X) := (\mathcal{P}(X_i))_{i \in I}$.

Definition 2.1 A non-deterministic Σ-equation of sort i in $\mathcal{P}(X)$ is a pair $((B_1)_i, (B_2)_i)$ of elements from $\mathcal{P}(W(i)), i \in I$: Such pairs are more commonly written as $(B_1)_i \approx_i^{nd} (B_2)_i$. The non-deterministic Σ-equation $(B_1)_i \approx_i^{nd} (B_2)_i$ of sort i is said to be a non-deterministic Σ-identity of sort i in Σ-algebra $\mathcal{A}$ if $(B_1)_i^{\mathcal{A}} = (B_2)_i^{\mathcal{A}}$.

In this case we also say that the non-deterministic Σ-equation $(B_1)_i \approx_i^{nd} (B_2)_i$ is satisfied or modelled by the Σ-algebra $\mathcal{A}$, and write $\mathcal{A} \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$. If the non-deterministic Σ-equation $(B_1)_i \approx_i^{nd} (B_2)_i$ is satisfied by every Σ-algebra in K, we write $K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$, that is,

$$K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i :\Leftrightarrow \forall \mathcal{A} \in K(\mathcal{A} \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i).$$

Let $\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma))$. Then if the non-deterministic Σ-equation $(B_1)_i \approx_i^{nd} (B_2)_i$ is satisfied by every class in $\mathcal{K}$, we write $\mathcal{K} \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$, that is,

$$\mathcal{K} \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i :\Leftrightarrow \forall K \in \mathcal{K}(K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i).$$

For a set $\mathcal{PL}(i)$ of non-deterministic Σ-equations of sort i we write $K \models_i^{nd} \mathcal{PL}(i)$ if $K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$ for all $((B_1)_i, (B_2)_i) \in \mathcal{PL}(i)$. We write $\mathcal{K} \models_i^{nd} \mathcal{PL}(i)$ if $K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$ for all $K \in \mathcal{K}$, and $((B_1)_i, (B_2)_i) \in \mathcal{PL}(i)$.

For illustration we consider the following example.

Example 2.2 Let $I = \{1, 2\}$, $X^{(2)} = (X_i^{(2)})_{i\in I}$, and let $\Sigma = \{(1, 1, 1), (2, 1, 1)\}$. Let $\mathcal{V}$ be a real vector space, $A_1 = \{f_{(2,1,1)}(x_{21}, f_{(1,1,1)}(x_{11}, x_{12}))\}$, $B_1 = \{f_{(1,1,1)}$
$(f_{(2,1,1)}(x_{21}, x_{11}), f_{(2,1,1)}(x_{21}, x_{12}))\}$. Then the non-deterministic Σ-equation $A_1 \approx_1^{nd} B_1$ of sort 1 is a non-deterministic Σ-identity of sort 1 in $\mathcal{V}$, that is, $\mathcal{V} \models_1^{nd} A_1 \approx_1^{nd} B_1$. Then

$$\begin{array}{rcl} A_1^{\mathcal{V}} & = & \{f_{(2,1,1)}(x_{21}, f_{(1,1,1)}(x_{11}, x_{12}))\}^{\mathcal{V}} \\ & = & \{(f_{(2,1,1)}(x_{21}, f_{(1,1,1)}(x_{11}, x_{12})))^{\mathcal{V}}\} \\ & = & \{(f_{(1,1,1)}(f_{(2,1,1)}(x_{21}, x_{11}), f_{(2,1,1)}(x_{21}, x_{12})))^{\mathcal{V}}\} \\ & = & \{f_{(1,1,1)}(f_{(2,1,1)}(x_{21}, x_{11}), f_{(2,1,1)}(x_{21}, x_{12}))\}^{\mathcal{V}} \\ & = & B_1^{\mathcal{V}}. \end{array}$$

Therefore $\mathcal{V} \models_1^{nd} A_1 \approx_1^{nd} B_1$.

Let $\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma))$ and $\mathcal{PL}(i) \subseteq \mathcal{P}(W(i))^2$. Then we define a mapping

$$nd\Sigma(i)\text{-}Id : \mathcal{P}(\mathcal{P}(Alg(\Sigma))) \to \mathcal{P}(\mathcal{P}(W(i))^2)$$

by

$$nd\Sigma(i)\text{-}Id\mathcal{K} := \{((B_1)_i, (B_2)_i) \in \mathcal{P}(W(i))^2 \mid \forall K \in \mathcal{K}(K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i)\},$$

and a mapping

$$nd\Sigma(i)\text{-}Mod : \mathcal{P}(\mathcal{P}(W(i))^2) \to \mathcal{P}(\mathcal{P}(Alg(\Sigma)))$$

by

$$nd\Sigma(i)\text{-}Mod\mathcal{PL}(i) := \{K \in \mathcal{P}(Alg(\Sigma)) \mid \forall((B_1)_i, (B_2)_i) \in \mathcal{PL}(i)(K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i)\}.$$

In the next propositions we will show that these two mappings satisfy the Galois-connection properties.

Proposition 2.3 *Let $i \in I$, and let $\mathcal{P}(Alg(\Sigma))$ be the set of all subsets of $Alg(\Sigma)$ and let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{P}(Alg(\Sigma))$. Then*

(1) *If $\mathcal{K}_1 \subseteq \mathcal{K}_2$, then $nd\Sigma(i)\text{-}Id\mathcal{K}_2 \subseteq nd\Sigma(i)\text{-}Id\mathcal{K}_1$,*

(2) *$\mathcal{K} \subseteq nd\Sigma(i)\text{-}Modnd\Sigma(i)\text{-}Id\mathcal{K}$.*

Proof (1) Assume that $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and let $(B_1)_i \approx_i^{nd} (B_2)_i \in nd\Sigma(i)\text{-}Id\mathcal{K}_2$. Then for all $K \in \mathcal{K}_2, K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$, but we have $\mathcal{K}_1 \subseteq \mathcal{K}_2$, so that $K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$ for all $K \in \mathcal{K}_1$. It follows that $(B_1)_i \approx_i^{nd} (B_2)_i \in nd\Sigma(i)\text{-}Id\mathcal{K}_1$, and then $nd\Sigma(i)\text{-}Id\mathcal{K}_2 \subseteq nd\Sigma(i)\text{-}Id\mathcal{K}_1$.

(2) Let $K \in \mathcal{K}$. Then $K \models_i^{nd} nd\Sigma(i)\text{-}Id\mathcal{K}$, means that $K \in nd\Sigma(i)\text{-}Modnd\Sigma(i)\text{-}Id\mathcal{K}$, and then $\mathcal{K} \subseteq nd\Sigma(i)\text{-}Modnd\Sigma(i)\text{-}Id\mathcal{K}$. □

Proposition 2.4 *Let $i \in I$, and let $\mathcal{P}(W(i))$ be the set of all subsets of $W(i)$ and let $\mathcal{PL}(i)$, $\mathcal{PL}_1(i)$,$\mathcal{PL}_2(i) \subseteq \mathcal{P}(W(i))^2$. Then*

(1) *If $\mathcal{PL}_1(i) \subseteq \mathcal{PL}_2(i)$, then $nd\Sigma(i)$-$Mod\mathcal{PL}_2(i) \subseteq nd\Sigma(i)$-$Mod\mathcal{PL}_1(i)$,*

(2) *$\mathcal{PL}(i) \subseteq nd\Sigma(i)$-$Idnd\Sigma(i)$-$Mod\mathcal{PL}(i)$.*

Proof (1) Assume that $\mathcal{PL}_1(i) \subseteq \mathcal{PL}_2(i)$ and let $K \in nd\Sigma(i)$-$Mod\mathcal{PL}_2(i)$. Then for all $((B_1)_i, (B_2)_i) \in \mathcal{PL}_2(i), K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$, but we have $\mathcal{PL}_1(i) \subseteq \mathcal{PL}_2(i)$, so that $K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$ for all $((B_1)_i, (B_2)_i) \in \mathcal{PL}_1(i)$, which means $K \in nd\Sigma(i)$-$Mod\mathcal{PL}_1(i)$, and then $nd\Sigma(i)$-$Mod\mathcal{PL}_2(i) \subseteq nd\Sigma(i)$-$Mod\mathcal{PL}_1(i)$.

(2) Let $((B_1)_i, (B_2)_i) \in \mathcal{PL}(i)$. Then $nd\Sigma(i)$-$Mod\mathcal{PL}(i) \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i$, means that $((B_1)_i, (B_2)_i) \in nd\Sigma(i)$-$Idnd\Sigma(i)$-$Mod\mathcal{PL}(i)$, and then $\mathcal{PL}(i) \subseteq nd\Sigma(i)$-$Idnd\Sigma(i)$-$Mod\mathcal{PL}(i)$. □

From both propositions we have that $(nd\Sigma(i)\text{-}Mod, nd\Sigma(i)\text{-}Id)$ is a Galois connection between $\mathcal{P}(Alg(\Sigma))$ and $\mathcal{P}(W(i))^2$ with respect to the relation

$$\models_i^{nd} := \{(K, ((B_1)_i, (B_2)_i)) \in \mathcal{P}(Alg(\Sigma)) \times \mathcal{P}(W(i))^2 \mid K \models_i^{nd} (B_1)_i \approx_i^{nd} (B_2)_i\}.$$

The fixed points with respect to the closure operator $nd\Sigma(i)$-$Modnd\Sigma(i)$-Id are called non-deterministic Σ-varieties of sort i and the fixed points with respect to the closure operator $nd\Sigma(i)$-$Idnd\Sigma(i)$-Mod are called non-deterministic Σ-equational theories of sort i.

3 Application of Nd-Hypersubstitutions

Now we apply non-deterministic Σ-hypersubstitutions to many-sorted algebras and to many-sorted equations.

Definition 3.1 ([1])Let $((f_\gamma)_k)_{k \in K_\gamma, \gamma \in \Sigma}$ be an indexed set of Σ-sorted operation symbols. Any mapping

$$\sigma_i : \{(f_\gamma))_k \mid k \in K_\gamma, \gamma \in \Sigma(i)\} \to W(i), i \in I$$

which preserves the arity, is said to be a Σ-hypersubstitution of sort i. Let $\Sigma(i)$-Hyp be the set of all Σ-hypersubstitutions of sort i. The I-sorted mapping $\sigma := (\sigma_i)_{i \in I}$ is called an I-sorted Σ-hypersubstitution. Let Σ-Hyp be the set of all I-sorted Σ-hypersubstitutions. Any I-sorted Σ-hypersubstitution can inductively be extended to an I-sorted mapping $\hat{\sigma} := (\hat{\sigma}_i)_{i \in I}$. The I-sorted mapping

$$\hat{\sigma} : W_\Sigma(X) \to W_\Sigma(X)$$

is inductively defined in the following way: For all $i \in I$, for every $t \in W(i)$,

(1) if $t = x_{ij} \in X_i$ with $1 \le j \le n$, then $\hat{\sigma}_i[t] := x_{ij}$,

(2) if $t = (f_\gamma)_k(t_1, \ldots, t_m)$ with $k \in K_\gamma, \gamma \in \Sigma_m(i)$ and $t_q \in W(k_q), 1 \leq q \leq m$ whenever $\gamma = (k_1, \ldots, k_m, i)$ and if we assume that $\hat{\sigma}_{k_q}[t_q]$ are already defined, then

$$\hat{\sigma}_i[t] := S_\gamma(\sigma_i((f_\gamma)_k), \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_m}[t_m]).$$

Definition 3.2 ([3]) Let $\mathcal{A} = (A; (((f_\gamma)_k)^{\mathcal{A}})_{k \in K_\gamma, \gamma \in \Sigma})$ be a Σ-algebra and $\sigma \in \Sigma$-Hyp. Then we define a Σ-algebra by

$$\sigma(\mathcal{A}) := (A; ((\sigma_i((f_\gamma)_k))^{\mathcal{A}})_{k \in K_\gamma, \gamma \in \Sigma(i), i \in I}),$$

This algebra is called derived Σ-algebra determined by σ and $\mathcal{A}$.

Definition 3.3 Let A be an I-sorted set, $\mathcal{A} = (A; (((f_\gamma)_k)^{\mathcal{A}})_{k \in K_\gamma, \gamma \in \Sigma})$ be a Σ-algebra and let $\sigma^{nd} \in nd\Sigma$-Hyp. Then we define a set of Σ-algebras by
$\sigma^{nd}(\mathcal{A}) := \{\rho(\mathcal{A}) \mid \rho \in \Sigma\text{-}Hyp,\ (\rho_i((f_\gamma)_k))^{\mathcal{A}} \in (\sigma_i^{nd}((f_\gamma)_k))^{\mathcal{A}}, k \in K_\gamma, \gamma \in \Sigma(i),$
$i \in I\}.$

Here ϱ is a many-sorted deterministic hypersubstitution (see [1]). This set of Σ-algebras is called the set of derived Σ-algebras determined by $\mathcal{A}$ and σ^{nd}.

For illustration we consider the following example.

Example 3.4 Let $I = \{1, 2\}$, $A = (A_i)_{i \in I}$. Let $\Sigma = \{(1, 2, 1), (2, 1, 2)\}$, $\mathcal{A} = (A; f^{\mathcal{A}}_{(1,2,1)}.f^{\mathcal{A}}_{(2,1,2)})$, and $\rho_1, \rho_2, \rho_3 \in \Sigma$-$Hyp$. Let $\sigma^{nd} \in nd\Sigma$-Hyp and assume that
$((\rho_2)_1(f_{(1,2,1)}))^{\mathcal{A}}, ((\rho_3)_1(f_{(1,2,1)}))^{\mathcal{A}} \in (\sigma_1^{nd}(f_{(1,2,1)}))^{\mathcal{A}}$, $((\rho_1)_2(f_{(2,1,2)}))^{\mathcal{A}}$, $((\rho_2)_2(f_{(2,1,2)}))^{\mathcal{A}}, ((\rho_3)_2(f_{(2,1,2)}))^{\mathcal{A}} \in (\sigma_2^{nd}(f_{(2,1,2)}))^{\mathcal{A}}$. Then we have $\rho_2(\mathcal{A})$, $\rho_3(\mathcal{A}) \in \sigma^{nd}(\mathcal{A})$ and $\rho_1(\mathcal{A}) \notin \sigma^{nd}(\mathcal{A})$, since $(\rho_1)_1(f_{(1,2,1)})^{\mathcal{A}} \notin \sigma_1^{nd}(f_{(1,2,1)})^{\mathcal{A}}$.

Definition 3.5 Let $B \in \mathcal{P}(W(i))$ and let $\mathcal{A}$ be a Σ-algebra. Let $\sigma^{nd} \in nd\Sigma$-Hyp, $\sigma^{nd}(\mathcal{A})$ be the set of derived algebras determined by $\mathcal{A}$ and σ^{nd}. Then we define the set $B^{\sigma^{nd}(\mathcal{A})}$ of Σ-term operations induced by the set $\sigma^{nd}(\mathcal{A})$ of derived algebras as follows:

(1) If $B = \{x_{ij}\}$ where $x_{ij} \in X_i$, then
$B^{\sigma^{nd}(\mathcal{A})} := \{x_{ij}^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma^{nd}(\mathcal{A})\}$.

(2) If $B = \{(f_\gamma)_k(s_1, \ldots, s_m)\}$ where $k \in K_\gamma, \gamma \in \Sigma_m(i)$ and $s_q \in W(i_q), 1 \leq q \leq m, m \in \mathbb{N}$ whenever $\gamma = (i_1, \ldots, i_m, i)$, and if we assume that $\{s_q\}^{\sigma^{nd}(\mathcal{A})}$ are already defined, then
$B^{\sigma^{nd}(\mathcal{A})} := S_\gamma^{ndA}(\{((f_\gamma)_k)^{\rho(\mathcal{A})} | \rho(\mathcal{A}) \in \sigma^{nd}(\mathcal{A})\}, \{s_1\}^{\sigma^{nd}(\mathcal{A})}, \ldots, \{s_m\}^{\sigma^{nd}(\mathcal{A})})$.

(3) If B is an arbitrary subset of $W(i)$, then $B^{\sigma^{nd}(\mathcal{A})} := \bigcup\limits_{b \in B} \{b\}^{\sigma^{nd}(\mathcal{A})}$.

If B is empty, then $B^{\sigma^{nd}(\mathcal{A})} := \emptyset$.

Theorem 3.6 *Let A be an I-sorted set and $\mathcal{A} = (A; ((f_\gamma)_k)^{\mathcal{A}})_{k\in K_\gamma,\gamma\in\Sigma})$ be a Σ-algebra. Let $\sigma^{nd} \in nd\Sigma$-Hyp and $B \in \mathcal{P}(W(i))$. Then $B^{\sigma^{nd}(\mathcal{A})} = (\hat{\sigma}_i^{nd}[B])^{\mathcal{A}}$.*

Proof If B is empty, then all is clear. If B is non-empty, then we will give a proof by induction on the complexity of the Σ-term which is the only element of the one-element set B.

1) If $B = \{x_{ij}\}$ where $x_{ij} \in X_i$, then

$$\begin{aligned} B^{\sigma^{nd}(\mathcal{A})} &= \{x_{ij}\}^{\sigma^{nd}(\mathcal{A})} \\ &= \{x_{ij}^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma^{nd}(\mathcal{A})\} \\ &= \{(\hat{\rho}_i[x_{ij}])^{\mathcal{A}} \mid \rho(\mathcal{A}) \in \sigma^{nd}(\mathcal{A})\} \\ &= \{x_{ij}^{\mathcal{A}} \mid \rho(\mathcal{A}) \in \sigma^{nd}(\mathcal{A})\} \\ &= \{x_{ij}^{\mathcal{A}}\} \\ &= \{x_{ij}\}^{\mathcal{A}} \\ &= (\hat{\sigma}_i^{nd}[\{x_{ij}\}])^{\mathcal{A}} \\ &= (\hat{\sigma}_i^{nd}[B])^{\mathcal{A}}. \end{aligned}$$

2) If $B = \{(f_\gamma)_k(t_1, \ldots, t_m)\} \in \mathcal{P}(W(i))$ where $k \in K_\gamma, \gamma \in \Sigma_m(i)$ and $t_q \in W(i_q), 1 \le q \le m, m \in \mathbb{N}$ whenever $\gamma = (i_1, \ldots, i_m, i)$ and if we assume that the equations

$$\{t_q\}^{\sigma^{nd}(\mathcal{A})} = (\hat{\sigma}_{i_q}^{nd}[\{t_q\}])^{\mathcal{A}}$$

are satisfied, then

$$\begin{aligned} & B^{\sigma^{nd}(\mathcal{A})} \\ &= \{(f_\gamma)_k(t_1, \ldots, t_n)\}^{\sigma^{nd}(\mathcal{A})} \\ &= S_\gamma^{ndA}(\{((f_\gamma)_k)^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma^{nd}(\mathcal{A})\}, \{t_1\}^{\sigma^{nd}(\mathcal{A})}, \ldots, \{t_m\}^{\sigma^{nd}(\mathcal{A})}) \\ &= S_\gamma^{ndA}(\{(\rho_i((f_\gamma)_k))^{\mathcal{A}} \mid (\rho_i((f_\gamma)_k))^{\mathcal{A}} \in (\sigma_i^{nd}((f_\gamma)_k))^{\mathcal{A}}\}, \{t_1\}^{\sigma^{nd}(\mathcal{A})}, \ldots, \\ &\quad \{t_m\}^{\sigma^{nd}(\mathcal{A})}) \\ &= S_\gamma^{ndA}(\{(\rho_i((f_\gamma)_k))^{\mathcal{A}} \mid (\rho_i((f_\gamma)_k))^{\mathcal{A}} \in (\sigma_i^{nd}((f_\gamma)_k))^{\mathcal{A}}\}, (\hat{\sigma}_{i_1}^{nd}[\{t_1\}])^{\mathcal{A}}, \ldots, \\ &\quad (\hat{\sigma}_{i_m}^{nd}[\{t_m\}])^{\mathcal{A}}) \\ &= S_\gamma^{ndA}((\sigma_i^{nd}((f_\gamma)_k))^{\mathcal{A}}, (\hat{\sigma}_{i_1}^{nd}[\{t_1\}])^{\mathcal{A}}, \ldots, (\hat{\sigma}_{i_m}^{nd}[\{t_m\}])^{\mathcal{A}}) \\ &= (S_\gamma^{nd}(\sigma_i^{nd}((f_\gamma)_k), (\hat{\sigma}_{i_1}^{nd}[\{t_1\}]), \ldots, (\hat{\sigma}_{i_m}^{nd}[\{t_m\}])))^{\mathcal{A}} \\ &= (\hat{\sigma}_i^{nd}[\{(f_\gamma)_k(t_1, \ldots, t_m)\}])^{\mathcal{A}} \\ &= (\hat{\sigma}_i^{nd}[B])^{\mathcal{A}}. \end{aligned}$$

3) If B is an arbitrary subset of $W(i)$, then

$$\begin{aligned} B^{\sigma^{nd}(\mathcal{A})} &= \bigcup_{b\in B} \{b\}^{\sigma^{nd}(\mathcal{A})} \\ &= \bigcup_{b\in B} (\hat{\sigma}_i^{nd}[\{b\}])^{\mathcal{A}} \\ &= (\bigcup_{b\in B} \hat{\sigma}_i^{nd}[\{b\}])^{\mathcal{A}} \\ &= (\hat{\sigma}_i^{nd}[B])^{\mathcal{A}}. \end{aligned}$$

□

Let $K \subseteq Alg(\Sigma)$. Then we set $\sigma^{nd}(K) := \bigcup\limits_{\mathcal{A}\in K} \sigma^{nd}(\mathcal{A})$. Let $\sigma_1, \sigma_2 \in \Sigma\text{-}Hyp$. Then we define $\sigma_1 \diamond \sigma_2 := ((\sigma_1)_i \circ_i (\sigma_2)_i)_{i\in I}$.

Lemma 3.7 *Let $\mathcal{A}$ be a Σ-algebra and $\sigma_1^{nd}, \sigma_2^{nd}$ be elements in $nd\Sigma\text{-}Hyp$. Then*

$$\sigma_1^{nd}(\sigma_2^{nd}(\mathcal{A})) = (\sigma_2^{nd} \circ^{nd} \sigma_1^{nd})(\mathcal{A}).$$

Proof Assume that $k \in K_\gamma, \gamma \in \Sigma(i), i \in I$. Then we have

$$\begin{aligned}
\sigma_1^{nd}(\sigma_2^{nd}(\mathcal{A})) &= \bigcup_{\rho(\mathcal{A})\in\sigma_2^{nd}(\mathcal{A})} \sigma_1^{nd}(\rho(\mathcal{A}))\\
&= \bigcup_{\rho(\mathcal{A})\in\sigma_2^{nd}(\mathcal{A})} \{\lambda(\rho(\mathcal{A})) \mid (\lambda_i((f_\gamma)_k))^{\rho(\mathcal{A})} \in ((\sigma_1^{nd})_i((f_\gamma)_k))^{\rho(\mathcal{A})}\}\\
&= \bigcup_{\rho(\mathcal{A})\in\sigma_2^{nd}(\mathcal{A})} \{(\rho\diamond\lambda)(\mathcal{A}) \mid (\hat{\rho}_i[\lambda_i((f_\gamma)_k)])^{\mathcal{A}} \in ((\sigma_1^{nd})_i((f_\gamma)_k))^{\rho(\mathcal{A})}\}\\
&= \{(\rho\diamond\lambda)(\mathcal{A}) \mid (\hat{\rho}_i[\lambda_i((f_\gamma)_k)])^{\mathcal{A}} \in ((\sigma_1^{nd})_i((f_\gamma)_k))^{\sigma_2^{nd}(\mathcal{A})}\}\\
&= \{(\rho\diamond\lambda)(\mathcal{A}) \mid (\hat{\rho}_i[\lambda_i((f_\gamma)_k)])^{\mathcal{A}} \in ((\hat{\sigma}_2^{nd})_i[(\sigma_1^{nd})_i((f_\gamma)_k)])^{\mathcal{A}}\}\\
&= \{(\rho\diamond\lambda)(\mathcal{A}) \mid ((\hat{\rho}_i \circ \lambda_i)((f_\gamma)_k))^{\mathcal{A}} \in (((\hat{\sigma}_2^{nd})_i \circ (\sigma_1^{nd})_i)((f_\gamma)_k))^{\mathcal{A}}\}\\
&= \{(\rho\diamond\lambda)(\mathcal{A}) \mid ((\rho_i \circ_i \lambda_i)((f_\gamma)_k))^{\mathcal{A}} \in (((\sigma_2^{nd})_i \circ_i^{nd} (\sigma_1^{nd})_i)((f_\gamma)_k))^{\mathcal{A}}\}\\
&= (\sigma_2^{nd} \circ^{nd} \sigma_1^{nd})(\mathcal{A}).
\end{aligned}$$

□

Remark 3.8 Let K be a non-empty subset of $Alg(\Sigma)$. Then

$$\begin{aligned}
\sigma_{id}^{nd}(K) &= \bigcup_{\mathcal{A}\in K} \sigma_{id}^{nd}(\mathcal{A})\\
&= \bigcup_{\mathcal{A}\in K} \{\rho_{id}(\mathcal{A}) \mid \rho_{id}(\mathcal{A}) \in \sigma_{id}^{nd}(\mathcal{A})\}\\
&= \bigcup_{\mathcal{A}\in K} \{\mathcal{A} \mid \mathcal{A} \in \sigma_{id}^{nd}(\mathcal{A})\}\\
&= K.
\end{aligned}$$

Definition 3.9 Let $\mathcal{A} \in Alg(\Sigma)$ said to hypersatisfy the non-deterministic Σ-identity $(B_1)_i \approx_i^{nd} (B_2)_i$ of sort $i \in I$ if for every $\sigma_i^{nd} \in nd\Sigma(i)\text{-}Hyp$, the non-deterministic Σ-identities $\hat{\sigma}_i^{nd}[(B_1)_i] \approx_i^{nd} \hat{\sigma}_i^{nd}[(B_2)_i]$ hold in $\mathcal{A}$.

In this case we say that the non-deterministic Σ-identity $(B_1)_i \approx_i^{nd} (B_2)_i$ of sort i is satisfied as a non-deterministic Σ-hyperidentity of sort i in $\mathcal{A}$ and write $\mathcal{A} \underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} (B_1)_i \approx_i^{nd} (B_2)_i$, that is,

$\mathcal{A} \underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} (B_1)_i \approx_i^{nd} (B_2)_i :\Leftrightarrow \forall \sigma_i^{nd} \in nd\Sigma(i)\text{-}Hyp\ (\mathcal{A} \models_i^{nd} \hat{\sigma}_i^{nd}[(B_1)_i] \approx_i^{nd} \hat{\sigma}_i^{nd}[(B_2)_i])$.

For illustration we consider the following example.

Example 3.10 Let $I = \{1, 2\}$, $X^{(2)} = (X_i^{(2)})_{i\in I}$, $\Sigma = \{(1, 1, 1)\}$. Let $(B_1; \circ_1)$, $(B_2; \circ_2)$ be a band, and let $\mathcal{DB}$ be double bands where $\mathcal{DB} := ((B_i)_{i\in I}; (\circ_i)_{i\in I})$. Let $A_1 = \{f_{(1,1,1)}(x_{1j}, x_{1j})\}$, $B_1 = \{x_{1j}\}, 1 \leq j \leq 2$. Then $\mathcal{DB} \underset{nd\Sigma\text{-}hyp}{\models_1^{nd}} A_1 \approx_1^{nd}$

B_1. It is easy to see that $\mathcal{DB} \models_1^{nd} A_1 \approx_1^{nd} B_1$. Let $\sigma_1^{nd} \in nd\Sigma(1)\text{-}Hyp$ and $\rho(\mathcal{DB}) \in \sigma^{nd}(\mathcal{DB})$ where $\rho \in \Sigma\text{-}Hyp$. Then

$$\begin{array}{rcl} \{f_{(1,1,1)}(x_{1j}, x_{1j})\}^{\rho(\mathcal{DB})} & = & \{(f_{(1,1,1)}(x_{1j}, x_{1j}))^{\rho(\mathcal{DB})}\} \\ & = & \{(\hat{\rho}_1[f_{(1,1,1)}(x_{1j}, x_{1j})])^{\mathcal{DB}}\} \\ & = & \{(\hat{\rho}_1[x_{1j}])^{\mathcal{DB}}\} \\ & = & \{x_{1j}^{\rho(\mathcal{DB})}\} \\ & = & \{x_{1j}\}^{\rho(\mathcal{DB})}, \end{array}$$

this means, $\sigma^{nd}(\mathcal{DB}) \models_1^{nd} A_1 \approx_1^{nd} B_1$, that is, $A_1^{\sigma^{nd}(\mathcal{DB})} = B_1^{\sigma^{nd}(\mathcal{DB})}$. It follows that $(\hat{\sigma}_1^{nd}[A_1])^{\mathcal{DB}} = (\hat{\sigma}_1^{nd}[B_1])^{\mathcal{DB}}$. Thus $\mathcal{DB} \underset{nd\Sigma\text{-}hyp}{\models_1^{nd}} A_1 \approx_1^{nd} B_1$.

Now we define two mappings which give a second Galois connection.

Definition 3.11 Let $\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma))$ and $\mathcal{PL}(i) \subseteq \mathcal{P}(W(i))^2$. Then we define a mapping

$$ndH\Sigma(i)\text{-}Id : \mathcal{P}(\mathcal{P}(Alg(\Sigma))) \to \mathcal{P}(\mathcal{P}(W(i))^2)$$

by

$ndH\Sigma(i)\text{-}Id\mathcal{K} := \{(B_1)_i \approx_i^{nd} (B_2)_i \in \mathcal{P}(W(i))^2 \mid \forall\ K \in \mathcal{K}(K \underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} (B_1)_i \approx_i^{nd} (B_2)_i)\}$,

and define a mapping

$$ndH\Sigma(i)\text{-}Mod : \mathcal{P}(\mathcal{P}(W(i))^2) \to \mathcal{P}(\mathcal{P}(Alg(\Sigma)))$$

by

$ndH\Sigma(i)\text{-}Mod\mathcal{PL}(i) := \{K \in \mathcal{P}(Alg(\Sigma)) \mid \forall((B_1)_i, (B_2)_i) \in \mathcal{PL}(i)(K \underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} (B_1)_i \approx_i^{nd} (B_2)_i)\}$.

We see that $(ndH\Sigma(i)\text{-}Mod, ndH\Sigma(i)\text{-}Id)$ is a Galois connection between $\mathcal{P}(Alg(\Sigma))$ and $\mathcal{P}(W(i))^2$ with respect to the relation

$\underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} := \{(K, (B_1)_i \approx_i^{nd} (B_2)_i) \in \mathcal{P}(Alg(\Sigma)) \times \mathcal{P}(W(i))^2 \mid K \underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} (B_1)_i \approx_i^{nd} (B_2)_i\}$.

Definition 3.12 Let $\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma))$ and $\mathcal{PL}(i) \subseteq \mathcal{P}(W(i))^2$. Then we set

$$\chi^{nd\Sigma\text{-}E(i)}[(B_1)_i \approx_i^{nd} (B_2)_i] := \{\hat{\sigma}_i^{nd}[(B_1)_i] \approx_i^{nd} \hat{\sigma}_i^{nd}[(B_2)_i] \mid \sigma_i^{nd} \in nd\Sigma(i)\text{-}Hyp\}$$

and

$$\chi^{nd\Sigma\text{-}A}[K] := \{\sigma^{nd}(K) \mid \sigma^{nd} \in nd\Sigma\text{-}Hyp\}.$$

We define two operators in the following way:

$$\chi^{nd\Sigma\text{-}E(i)} : \mathcal{P}(\mathcal{P}(W(i))^2) \to \mathcal{P}(\mathcal{P}(W(i))^2) \text{ by}$$

$$\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)] := \bigcup_{(B_1)_i \approx_i^{nd} (B_2)_i \in \mathcal{PL}(i)} \chi^{nd\Sigma\text{-}E(i)}[(B_1)_i \approx_i^{nd} (B_2)_i]$$

and

$$\chi^{nd\Sigma\text{-}A} : \mathcal{P}(\mathcal{P}(Alg(\Sigma))) \to \mathcal{P}(\mathcal{P}(Alg(\Sigma))) \text{ by}$$

$$\chi^{nd\Sigma\text{-}A}[\mathcal{K}] := \bigcup_{K \in \mathcal{K}} \chi^{nd\Sigma\text{-}A}[K].$$

In the next propositions we will show that the both operators are closure operators.

Proposition 3.13 *Let $\mathcal{PL}(i), \mathcal{PL}_1(i), \mathcal{PL}_2(i)$ be subsets of $\mathcal{P}(W(i))^2$. Then*

(i) $\mathcal{PL}(i) \subseteq \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$,

(ii) $\mathcal{PL}_1(i) \subseteq \mathcal{PL}_2(i) \Rightarrow \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}_1(i)] \subseteq \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}_2(i)]$,

(iii) $\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)] = \chi^{nd\Sigma\text{-}E(i)}[\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]]$.

Proof (i) Let $(B_1)_i \approx_i^{nd} (B_2)_i \in \mathcal{PL}(i)$. Then, since $(B_1)_i = (\hat{\sigma}_i^{nd})_{id}[(B_1)_i]$ and $(B_2)_i = (\hat{\sigma}_i^{nd})_{id}[(B_2)_i]$, we have $(\hat{\sigma}_i^{nd})_{id}[(B_1)_i] = (B_1)_i \approx_i^{nd} (B_2)_i = (\hat{\sigma}_i^{nd})_{id}[(B_2)_i] \in \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$ and this means $\mathcal{PL}(i) \subseteq \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$.

(ii) Assume that $\mathcal{PL}_1(i) \subseteq \mathcal{PL}_2(i)$ and let $\hat{\sigma}_i^{nd}[(B_1)_i] \approx_i^{nd} \hat{\sigma}_i^{nd}[(B_2)_i] \in \chi^{nd\Sigma\text{-}E(i)}$ $[\mathcal{PL}_1(i)]$. Then $(B_1)_i \approx_i^{nd} (B_2)_i \in \mathcal{PL}_1(i)$, but $\mathcal{PL}_1(i) \subseteq \mathcal{PL}_2(i)$, so that $(B_1)_i \approx_i^{nd} (B_2)_i \in \mathcal{PL}_2(i)$ and $\hat{\sigma}_i^{nd}[(B_1)_i] \approx_i^{nd} \hat{\sigma}_i^{nd}[(B_2)_i] \in \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}_2(i)]$. We have $\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}_1(i)] \subseteq \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}_2(i)]$.

(iii) By (i) we have $\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)] \subseteq \chi^{nd\Sigma\text{-}E(i)}[\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]]$. On the other hand, let $\hat{\sigma}_i^{nd}[(B_1)_i] \approx_i^{nd} \hat{\sigma}_i^{nd}[(B_2)_i] \in \chi^{nd\Sigma-E(i)}[\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]]$. Then $(B_1)_i \approx_i^{nd} (B_2)_i \in \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$, and there exists $\rho_i^{nd} \in nd\Sigma(i)\text{-}Hyp$ and $(C_1)_i \approx_i^{nd} (C_2)_i \in \mathcal{PL}(i)$ such that $(B_1)_i = \hat{\rho}_i^{nd}[(C_1)_i]$ and $(B_2)_i = \hat{\rho}_i^{nd}[(C_2)_i]$, and we have

$$\begin{aligned}
\hat{\sigma}_i^{nd}[(B_1)_i] &= \hat{\sigma}_i^{nd}[\hat{\rho}_i^{nd}[(C_1)_i]] \\
&= \hat{\sigma}_i^{nd} \circ \hat{\rho}_i^{nd}[(C_1)_i] \\
&= (\sigma_i^{nd} \circ_i^{nd} \rho_i^{nd})\hat{}[(C_1)_i] \\
&= \hat{\lambda}_i^{nd}[(C_1)_i], \text{ where } \lambda_i^{nd} = \sigma_i^{nd} \circ_i^{nd} \rho_i^{nd} \in nd\Sigma(i)\text{-}Hyp, \text{and} \\
\hat{\sigma}_i^{nd}[(B_2)_i] &= \hat{\sigma}_i^{nd}[\hat{\rho}_i^{nd}[(C_2)_i]] \\
&= \hat{\sigma}_i^{nd} \circ \hat{\rho}_i^{nd}[(C_2)_i] \\
&= (\sigma_i^{nd} \circ_i^{nd} \rho_i^{nd})\hat{}[(C_2)_i] \\
&= \hat{\lambda}_i^{nd}[(C_2)_i].
\end{aligned}$$

Then we set $\hat{\lambda}_i^{nd}[(C_1)_i] = \hat{\sigma}_i^{nd}[(B_1)_i] \approx_i^{nd} \hat{\sigma}_i^{nd}[(B_2)_i] = \hat{\lambda}_i^{nd}[(C_2)_i] \in \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$, and then $\chi^{nd\Sigma\text{-}E(i)}[\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]] \subseteq \chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$. □

Proposition 3.14 *Let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{P}(Alg(\Sigma))$. Then*

(i) $\mathcal{K} \subseteq \chi^{nd\Sigma\text{-}A}[\mathcal{K}]$,

(ii) $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \chi^{nd\Sigma\text{-}A}[\mathcal{K}_1] \subseteq \chi^{nd\Sigma\text{-}A}[\mathcal{K}_2]$,

(iii) $\chi^{nd\Sigma\text{-}A}[\mathcal{K}] = \chi^{nd\Sigma\text{-}A}[\chi^{nd\Sigma\text{-}A}[\mathcal{K}]]$.

Proof (i) Let $K \in \mathcal{K}$. Then, since $K = \sigma^{nd}_{id}(K) \in \chi^{nd\Sigma\text{-}A}[\mathcal{K}]$ we have $\mathcal{K} \subseteq \chi^{nd\Sigma\text{-}A}[\mathcal{K}]$.

(ii) Assume that $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and let $\sigma^{nd}(K) \in \chi^{nd\Sigma\text{-}A}[\mathcal{K}_1]$. Then $K \in \mathcal{K}_1$ by our assumption that we have $K \in \mathcal{K}_2$, with $\sigma^{nd}(K) \in \chi^{nd\Sigma\text{-}A}[\mathcal{K}_2]$, and then $\chi^{nd\Sigma\text{-}A}[\mathcal{K}_1] \subseteq \chi^{nd\Sigma\text{-}A}[\mathcal{K}_2]$.

(iii) By (i), we have $\chi^{nd\Sigma\text{-}A}[\mathcal{K}] \subseteq \chi^{nd\Sigma\text{-}A}[\chi^{nd\Sigma\text{-}A}[\mathcal{K}]]$. We will show that $\chi^{nd\Sigma\text{-}A}$
$[\chi^{nd\Sigma\text{-}A}[\mathcal{K}]] \subseteq \chi^{nd\Sigma\text{-}A}[\mathcal{K}]$. Let $\sigma^{nd}(K) \in \chi^{nd\Sigma\text{-}A}[\chi^{nd\Sigma\text{-}A}[\mathcal{K}]]$. Then $K \in \chi^{nd\Sigma\text{-}A}[\mathcal{K}]$, and there exists $\rho^{nd} \in nd\Sigma\text{-}Hyp$ and $K_1 \in \mathcal{K}$ such that $K = \rho^{nd}(K_1)$ and we have

$$\begin{aligned}\sigma^{nd}(K) &= \sigma^{nd}(\rho^{nd}(K_1))\\ &= (\rho^{nd} \circ^{nd} \sigma^{nd})(K_1)\\ &= \lambda^{nd}(K_1),\ \text{where } \lambda^{nd} = \rho^{nd} \circ^{nd} \sigma^{nd} \in nd\Sigma\text{-}Hyp.\end{aligned}$$

Thus we have $\sigma^{nd}(K) = \lambda^{nd}(K_1) \in \chi^{nd\Sigma\text{-}A}[\mathcal{K}]$, and is $\chi^{nd\Sigma\text{-}A}[\chi^{nd\Sigma\text{-}A}[\mathcal{K}]] \subseteq \chi^{nd\Sigma\text{-}A}[\mathcal{K}]$. □

Definition 3.15 Let K be a subset of $Alg(\Sigma)$, and $(B_1)_i, (B_2)_i$ be subsets of $W(i), i \in I$. Let $\sigma^{nd} \in nd\Sigma\text{-}Hyp$. Then we define

$$\sigma^{nd}(K) \models^{nd}_i (B_1)_i \approx^{nd}_i (B_2)_i :\Leftrightarrow \forall \mathcal{A} \in K(\sigma^{nd}(\mathcal{A}) \models^{nd}_i (B_1)_i \approx^{nd}_i (B_2)_i).$$

Theorem 3.16 *Let K be a subset of $Alg(\Sigma)$ and $(B_1)_i \approx^{nd}_i (B_2)_i \in \mathcal{P}(W(i))^2$, $\sigma^{nd} \in nd\Sigma\text{-}Hyp$. Then we have*

$$\sigma^{nd}(K) \models^{nd}_i (B_1)_i \approx^{nd}_i (B_2)_i \Longleftrightarrow K \models^{nd}_i \hat{\sigma}^{nd}_i[(B_1)_i] \approx^{nd}_i \hat{\sigma}^{nd}_i[(B_2)_i].$$

Proof We obtain

$$\begin{aligned}\sigma^{nd}(K) \models^{nd}_i (B_1)_i &\approx^{nd}_i (B_2)_i \Longleftrightarrow \forall \mathcal{A} \in K(\sigma^{nd}(\mathcal{A}) \models^{nd}_i (B_1)_i \approx^{nd}_i (B_2)_i)\\ &\Longleftrightarrow \forall \mathcal{A} \in K((B_1)_i^{\sigma^{nd}(\mathcal{A})} = (B_2)_i^{\sigma^{nd}(\mathcal{A})})\\ &\Longleftrightarrow \forall \mathcal{A} \in K((\hat{\sigma}^{nd}_i[(B_1)_i])^{\mathcal{A}} = (\hat{\sigma}^{nd}_i[(B_2)_i])^{\mathcal{A}})\\ &\Longleftrightarrow \forall \mathcal{A} \in K(\mathcal{A} \models^{nd}_i \hat{\sigma}^{nd}_i[(B_1)_i] \approx^{nd}_i \hat{\sigma}^{nd}_i[(B_2)_i])\\ &\Longleftrightarrow K \models^{nd}_i \hat{\sigma}^{nd}_i[(B_1)_i] \approx^{nd}_i \hat{\sigma}^{nd}_i[(B_2)_i].\end{aligned}$$

□

Theorem 3.17 *The pair $(\chi^{nd\Sigma\text{-}A}, \chi^{nd\Sigma\text{-}E(i)})$ is a conjugate pair of completely additive closure operators with respect to the relation $\models^{nd}_i$*

Proof By Definition 3.12, Propositions 3.13-3.14, and Theorem 3.16. □

Now we may apply the theory of conjugate pairs of additive closure operators (see e.g. [7]) and obtain the following propositions:

Lemma 3.18 ([7]) *For all $\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma))$ and for all $\mathcal{PL}(i) \subseteq \mathcal{P}(W(i))^2$ the following properties hold:*

(i) $ndH\Sigma(i)\text{-}Mod\mathcal{PL}(i) = nd\Sigma(i)\text{-}Mod\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$,

(ii) $ndH\Sigma(i)\text{-}Mod\mathcal{PL}(i) \subseteq nd\Sigma(i)\text{-}Mod\mathcal{PL}(i)$,

(iii) $\chi^{nd\Sigma\text{-}A}[ndH\Sigma(i)\text{-}Mod\mathcal{PL}(i)] = ndH\Sigma(i)\text{-}Mod\mathcal{PL}(i)$,

(iv) $\chi^{nd\Sigma\text{-}E(i)}[nd\Sigma(i)\text{-}IdndH\Sigma(i)\text{-}Mod\mathcal{PL}(i)] = nd\Sigma(i)\text{-}IdndH\Sigma(i)\text{-}Mod\mathcal{PL}(i)$,

(v) $ndH\Sigma(i)\text{-}ModndH\Sigma(i)\text{-}Id\mathcal{K} = nd\Sigma(i)\text{-}Modnd\Sigma(i)\text{-}Id\chi^{nd\Sigma\text{-}A}[\mathcal{K}]$, *and*

(i)$'$ $ndH\Sigma(i)\text{-}Id\mathcal{K} = nd\Sigma(i)\text{-}Id\chi^{nd\Sigma\text{-}A}[\mathcal{K}]$,

(ii)$'$ $ndH\Sigma(i)\text{-}Id\mathcal{K} \subseteq nd\Sigma(i)\text{-}Id\mathcal{K}$,

(iii)$'$ $\chi^{nd\Sigma\text{-}E(i)}[ndH\Sigma(i)\text{-}Id\mathcal{K}] = ndH\Sigma(i)\text{-}Id\mathcal{K}$,

(iv)$'$ $\chi^{nd\Sigma\text{-}A}[nd\Sigma(i)\text{-}ModndH\Sigma(i)\text{-}Id\mathcal{K}] = nd\Sigma(i)\text{-}ModndH\Sigma(i)\text{-}Id\mathcal{K}$,

(v)$'$ $ndH\Sigma(i)\text{-}IdndH\Sigma(i)\text{-}Mod\mathcal{PL}(i) = nd\Sigma(i)\text{-}Idnd\Sigma(i)\text{-}Mod\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)]$.

4 I-Sorted Nd-Solid Varieties

Definition 4.1 Let $\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma))$ be a subclass of the set of all subsets of $Alg(\Sigma)$ and let $\mathcal{PL}(i) \subseteq \mathcal{P}(W(i))^2$ be a subset of the set of all non-deterministic Σ-equations of sort i. Then $\mathcal{K}$ is called a non-determistic solid model class of sort i or is called a non-deterministic solid Σ-variety of sort i if every non-deterministic Σ-identity of sort i is satisfied as a non-deterministic Σ-hyperidentity of sort i :

$$\mathcal{K} \underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} nd\Sigma(i)\text{-}Id\mathcal{K}.$$

$\mathcal{K}$ is called I-sorted non-deterministic solid model class if every non-deterministic Σ-identity of sort i is satisfied as a non-deterministic Σ-hyperidentity of sort i for all $i \in I$, that is,

$$\mathcal{K} \underset{nd\Sigma\text{-}hyp}{\models_i^{nd}} nd\Sigma(i)\text{-}Id\mathcal{K}, \text{ for all } i \in I.$$

$\mathcal{PL}(i), i \in I$ is said to be a non-deterministic Σ-equational theory of sort i if there exists a class $\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma))$ such that $\mathcal{PL}(i) = nd\Sigma(i)\text{-}Id\mathcal{K}$. Then we set $\mathcal{PL} := (\mathcal{PL}(i))_{i\in I}$. This I-sorted set is called I-sorted non-deterministic Σ-equational theory.

Using the propositions of Lemma 3.18 one obtains the following characterization of non-deterministic solid Σ-varieties.

Theorem 4.2 ([7]) *Let $\mathcal{K}$ be a non-deterministic Σ-variety of sort $i \in I$. Then the following properties are equivalent:*

(i) $\mathcal{K} = ndH\Sigma(i)\text{-}ModndH\Sigma(i)\text{-}Id\mathcal{K}$,

(ii) $\chi^{nd\Sigma\text{-}A}[\mathcal{K}] = \mathcal{K}$,

(iii) $nd\Sigma(i)\text{-}Id\mathcal{K} = ndH\Sigma(i)\text{-}Id\mathcal{K}$,

(iv) $\chi^{nd\Sigma\text{-}E(i)}[nd\Sigma(i)\text{-}Id\mathcal{K}] = nd\Sigma(i)\text{-}Id\mathcal{K}$.

Theorem 4.3 ([7]) *Let $\mathcal{PL}(i)$ be a non-deterministic Σ-equational theory of sort $i \in I$. Then the following properties are equivalent:*

(i) $\mathcal{PL}(i) = ndH\Sigma(i)\text{-}IdndH\Sigma(i)\text{-}Mod\mathcal{PL}(i)$,

(ii) $\chi^{nd\Sigma\text{-}E(i)}[\mathcal{PL}(i)] = \mathcal{PL}(i)$,

(iii) $nd\Sigma(i)\text{-}Mod\mathcal{PL}(i) = ndH\Sigma(i)\text{-}Mod\mathcal{PL}(i)$,

(iv) $\chi^{nd\Sigma\text{-}A}[nd\Sigma(i)\text{-}Mod\mathcal{PL}(i)] = nd\Sigma(i)\text{-}Mod\mathcal{PL}(i)$.

5 I-sorted Nd-Complete Lattices

Let $\mathcal{PH}(i)$ be the class of all fixed points with respect to the closure operator $nd\Sigma(i)\text{-}Modnd\Sigma(i)\text{-}Id$:

$$\mathcal{PH}(i) := \{\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma)) \mid \mathcal{K} = nd\Sigma(i)\text{-}Modnd\Sigma(i)\text{-}Id\mathcal{K}\},$$

that is, $\mathcal{PH}(i)$ is the class of all non-deterministic Σ-varieties of sort i. Then $\mathcal{PH}(i)$ forms a non-deterministic complete lattice of non-deterministic Σ-varieties of sort i. Let $\mathcal{PH}y(i)$ be the class of all fixed points with respect to the closure operator $ndH\Sigma(i)\text{-}ModndH\Sigma(i)\text{-}Id$:

$$\mathcal{PH}y(i) := \{\mathcal{K} \subseteq \mathcal{P}(Alg(\Sigma)) \mid \mathcal{K} = ndH\Sigma(i)\text{-}ModndH\Sigma(i)\text{-}Id\mathcal{K}\},$$

that is, $\mathcal{PH}y(i)$ is the class of all non-deterministic solid Σ-varieties of sort i. Then $\mathcal{PH}y(i)$ forms a non-deterministic complete lattice of non-deterministic solid Σ-varieties of sort i and $\mathcal{PH}y(i)$ is a non-deterministic complete sublattice of $\mathcal{PH}(i)$. We set $\mathcal{PH} := (\mathcal{PH}(i))_{i\in I}$ and $\mathcal{PH}y := (\mathcal{PH}y(i))_{i\in I}$. $\mathcal{PH}$ is called an I-sorted non-deterministic complete lattice. $\mathcal{PH}y$ is called an I-sorted non-deterministic complete sublattice of $\mathcal{PH}$, since for every $i \in I, \mathcal{PH}y(i)$ is a non-deterministic complete sublattice of $\mathcal{PH}(i)$. Dually, let $\mathcal{PL}(i)$ be the class of all fixed points with respect to the closure operator $nd\Sigma(i)\text{-}Idnd\Sigma(i)\text{-}Mod$:

$$\mathcal{PPL}(i) := \{\mathcal{PL}(i) \subseteq \mathcal{P}(W(i))^2 \mid \mathcal{PL}(i) = nd\Sigma(i)\text{-}Idnd\Sigma(i)\text{-}Mod\mathcal{PL}(i)\},$$

that is, $\mathcal{PPL}(i)$ is the class of all non-deterministic Σ-equational theories of sort i. Then $\mathcal{PPL}(i)$ forms a nondeterministic complete lattice of Σ-equational theories of sort i. Let $\mathcal{PPL}y(i)$ be the class of all fixed points with respect to the closure operator $ndH\Sigma(i)$-$IdndH\Sigma(i)$-Mod:

$$\mathcal{PPL}y(i) := \{\mathcal{PL}(i) \subseteq \mathcal{P}(W(i))^2 \mid \mathcal{PL}(i) = ndH\Sigma(i)\text{-}IdndH\Sigma(i)\text{-}Mod\mathcal{PL}(i)\},$$

that is, $\mathcal{PPL}y(i)$ is the class of all non-deterministic solid Σ-equational theories of sort i. Then $\mathcal{PPL}y(i)$ forms a non-deterministic complete lattice of non-deterministic solid Σ-equational theories of sort i and $\mathcal{PPL}y(i)$ is a non-deterministic complete sublattice of $\mathcal{PPL}(i)$. We set $\mathcal{PPL} := (\mathcal{PPL}(i))_{i\in I}$ and $\mathcal{PPL}y := (\mathcal{PPL}y(i))_{i\in I}$. $\mathcal{PPL}$ is called an I-sorted non-deterministic complete lattice. $\mathcal{PPL}y$ is called an I-sorted non-deterministic complete sublattice of $\mathcal{PPL}$, since for every $i \in I, \mathcal{PPL}y(i)$ is a non-deterministic complete sublattice of $\mathcal{PPL}(i)$.

Our results show that the most results of [4] are valid also in the many-sorted case if the superposition of many-sorted tree languages and of sets of many-sorted terms are defined in the way in which we did.

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