East-West J. of Mathematics: Vol. 10, No 2 (2008) pp. 179-186

REDUCED RINGS, MORITA CONTEXTS AND DERIVATIONS

N. M. Muthana and **S. K. Nauman**

Department of Mathematics
King Abdulaziz University, Jeddah
Kingdom of Saudi Arabia
najat_muthana@hotmail.com, synakhaled@hotmail.com

Abstract

If two rings are ingredients of a semi-projective Morita context, in which one ring is commutative and the other is reduced, then it is shown here that the reduced ring will also become commutative. Some consequences are studied and effects of various types of derivations on these rings are listed.

A classical problem in ring theory is to study and generalize conditions under which a ring becomes commutative. So far the best tools found for this purpose are the derivations on rings and also on their modules. One can also achieve this goal by comparing two rings and impose conditions on them. If one of the rings is appeared to be commutative, in a compatible way, the other ring will also become commutative. In order to explore these ideas Morita theory is found to be a suitable tool.

It is proved here that if two rings are ingredients of a semi-projective Morita context, in which one is commutative and the other is reduced, then the reduced ring will also become commutative. Some consequences related to domains and division rings are stated and proved. We have listed several results from the theory of derivations in which some prime and semiprime rings are proved to be commutative. If such a ring is an ingredient of a semi-projective Morita context then the other ring will also become commutative and if the context is strict then both rings become isomorphic fields.

Key words: (Semi-) projective Morita contexts, Morita similar rings, reduced rings, prime rings, semiprime rings, and derivations.
2000 AMS Mathematics Subject Classification: 16D90; 16W25

1. Preliminaries

The term *rings* is used here for *associative rings that must possess the multiplicative identity.* While the term *rings* is reserved for *associative rings may or may not possess the multiplicative identity.* Modules over rings are considered to be unital and ring homomorphisms are identity preserving. Unless otherwise stated, we assume throughout that the lower case letters x, x' *or* x_i are elements of the upper case letter X.

A ring is reduced if and only if it has no non-zero nilpotent elements while semiprime if and only if it has no non-zero nilpotent ideals. Clearly, every reduced ring is semiprime and every commutative semiprime ring is reduced. In the following we have given an example of a semiprime ring which is not reduced. Division rings and domains are both semiprime and reduced, while a prime ring which is also reduced is a domain. By a domain we mean a ring without non-zero zero divisors. So a commutative domain with identity is an integral domain.

By a *derivation* on a ring A we mean the most *natural derivation* $d : A \longrightarrow A$ which is additive as well as satisfying the relation $d(a_1a_2) = d(a_1)a_2 + a_1d(a_2)$. If A is a ring (with identity 1_A), then it is clear that $d(1_A) = 0$.

We assume throughout that the datum $K(A,B) = [A, B, M, N, \langle,\rangle_A, \langle,\rangle_B]$ is a Morita context (in short, "*mc*") in which A and B are rings, M and N are $(B,A)-$ and $(A,B)-$ bimodules, respectively, $\langle,\rangle_A : N \otimes_B M \longrightarrow A$ and $\langle,\rangle_B : M \otimes_A N \longrightarrow B$ are bimodule morphisms with the associativity (or compatibility) conditions

$$(i)\ m' \langle n,m\rangle_A = \langle m',n\rangle_B\, m \quad \text{and} \quad (ii)\ \langle n,m\rangle_A\, n' = n\, \langle m,n'\rangle_B\,,$$

where $\langle,\rangle_A$ and $\langle,\rangle_B$ are the Morita maps (in short, *mc maps*). The images $I = \langle N,M\rangle_A$ and $J = \langle M,N\rangle_B$ are the *trace ideals* of A and B, respectively.

A Morita context $K(A,B)$ is said to be a "projective Morita context", in short a "pmc" (or strict), if both mc maps, $\langle,\rangle_A$ and $\langle,\rangle_B$, are epimorphisms. $K(A,B)$ is said to be a "semi-projective Morita context", or a "semi-pmc", if one of the mc maps, $\langle,\rangle_A$ or $\langle,\rangle_B$, is an epimorphism.

Let A and B be rings. In case an mc $K(A,B)$ of rings is a pmc, i.e., if both mc maps $\langle,\rangle_A$ and $\langle,\rangle_B$ are epimorphisms, then they become isomorphisms. In this case, the category of right (respt. left) $A-$modules is equivalent to right (respt. left) $B-$modules and $Cent(A) \cong Cent(B)$. For details and more references see for instance [14], [15], [17], [9], [16], and [18].

If $K(A,B)$ is a pmc of rings, then the rings A and B are said to be Morita similar (or Morita equivalent). Common properties shared by Morita similar rings are termed as Morita invariant. For instance, being prime or semiprime are Morita invariant, while being reduced, commutative, domain, division rings or fields are not Morita invariant

2. A commutativity theorem for a semi-pmc.

In this section we will study effects of semi-pmc and pmc on reduced rings, semiprime rings, domains and division rings. We will prove first a commutativity theorem for reduced rings by using a semi-pmc.

Theorem 2.1 *Let A and B be rings of a semi-pmc $K(A,B)$ in which $\langle,\rangle_B$ is epic. If A is commutative and B is reduced, then B is also commutative.*

Proof Assume that the mc $K(A,B)=[A,B,M,N,\langle,\rangle_A,\langle,\rangle_B]$ is a semi-pmc in which A is commutative, B is reduced, and the Morita map $\langle,\rangle_B$ is epic. If we let $b_1=\sum_{i=1}^{\alpha}\langle m_{1i},n_{1i}\rangle_B$ and $b_2=\sum_{j=1}^{\beta}\langle m_{2j},n_{2j}\rangle_B$, and if

$$\langle m_{1i},n_{1i}\rangle_B\langle m_{2j},n_{2j}\rangle_B=\langle m_{2j},n_{2j}\rangle_B\langle m_{1i},n_{1i}\rangle_B.$$

$\forall i=1\cdots\alpha,\ j=1\cdots\beta$, then $b_1b_2=b_2b_1$. Hence, in order to prove that B is commutative, it suffices to prove that an arbitrary commutator, say,

$$b=[\langle m,n\rangle_B,\langle m',n'\rangle_B]\in B \tag{(1)}$$

is zero.

In deed, if $\langle m,n\rangle_B=0$ or $\langle m',n'\rangle_B=0$, then $b=0$. So assume that $\langle m,n\rangle_B\neq 0$ and $\langle m',n'\rangle_B\neq 0$. Then

$$\begin{aligned}\langle n,bm\rangle_A &= \langle n,[\langle m,n\rangle_B\langle m',n'\rangle_B-\langle m',n'\rangle_B\langle m,n\rangle_B]m\rangle_A\\ &= \langle n,m\rangle_A\langle n,m'\rangle_A\langle n',m\rangle_A-\langle n,m'\rangle_A\langle n',m\rangle_A\langle n,m\rangle_A\\ &= 0.\end{aligned} \tag{(2)}$$

We further expand $\langle n,bm\rangle_A$ as

$$\langle m,\langle n,bm\rangle_A n\rangle_B=\langle m,n\rangle_B\, b\,\langle m,n\rangle_B=0. \tag{(3)}$$

Multiply either from left or from right the terms in (3) by b, we obtain

$$(\langle m,n\rangle_B\, b)^2=0=(b\,\langle m,n\rangle_B)^2.$$

which simply means that

$$\langle m,n\rangle_B\, b=0=b\,\langle m,n\rangle_B. \tag{(4)}$$

Then by (1) and (4) we get the relations

$$\begin{aligned}\langle m,n\rangle_B\langle m,n\rangle_B\langle m',n'\rangle_B &= \langle m,n\rangle_B\langle m',n'\rangle_B\langle m,n\rangle_B\\ &= \langle m',n'\rangle_B\langle m,n\rangle_B\langle m,n\rangle_B\end{aligned} \tag{(5)}$$

Let us repeat the entire phenomena by setting $\langle n', bm' \rangle_A = 0$, as in (2) above, we get relations

$$\begin{aligned} \langle m', n' \rangle_B \langle m, n \rangle_B \langle m', n' \rangle_B &= \langle m, n \rangle_B \langle m', n' \rangle_B \langle m', n' \rangle_B \\ &= \langle m', n' \rangle_B \langle m', n' \rangle_B \langle m, n \rangle_B . \qquad ((6)) \end{aligned}$$

Finally, by comparing various terms of equalities from (5) and (6) in b^2, we deduce that $b^2 = 0$. Hence $b = 0$. □

Corollary 2.2 *Let A and B be rings of a semi-pmc $K(A, B)$ in which $\langle , \rangle_B$ is epic.*

(1) *If A is commutative and B is a domain, then B is also commutative.*

(2) *If A is commutative and B is a division ring, then B becomes a field.*

Proof (1) holds because a domain is a reduced ring. The proof can also be followed directly from $Eq(3)$ in Theorem 2.1. Because $\langle m, n \rangle_B \, b \, \langle m, n \rangle_B = 0$ and as B is a domain, $b = 0$.

(2) holds because a division ring is a domain and a commutative division ring is a field.□

Corollary 2.3 *Let A and B be rings of an mc $K(A, B)$ and let $I \trianglelefteq A$ and $J \trianglelefteq B$ be the trace ideals.*

(1) *If $I \subseteq Cent(A)$ and B is reduced, then $J \subseteq Cent(B)$.*

(2) *If I is commutative and B is reduced, then J is commutative.*

Proof (1) and (2). Every element of J is of the form $\sum_{i=1}^{n} \langle m_i, n_i \rangle_B$ and I is commutative. The rest of the proof is same as the proof of Theorem 2.1. □

Corollary 2.4 *Let $K(A, B)$ be a pmc of rings in which A is commutative.*

(1) *If B is a reduced ring, then A is also reduced and $A \cong B$.*

(2) *If B is a domain, then both A and B become isomorphic integral domains.*

(3) *If B is a division ring, then both A and B become isomorphic fields.*

Proof (1) If A and B are rings with both mc maps $\langle , \rangle_A$ and $\langle , \rangle_B$ epimorphisms, then $Cent(A) \cong Cent(B)$. Being commutative, $A = Cent(A)$. If B is reduced, then by above theorem, $B = Cent(B)$. Hence $A \cong B$.

(2) and (3) are clear from (1) above and Theorem 2.1. □

Let $K(A,B)=[A,M,N,B,\langle,\rangle_A,\langle,\rangle_B]$ be an mc of rings and let its Morita ring be denoted by

$$R=\begin{bmatrix} A & N \\ M & B \end{bmatrix}.$$

Clearly, R is a ring if both A and B are rings. It is proved in [18; Theorem 2.1] that from an mc $K(A,B)$ of rings one can always get a semi-pmc and a pmc. In particular, if $\langle,\rangle_B$ is epic, then $K(R,B)$ is a pmc. Hence, in this case B and R are Morita similar.

In the following corollary we pose an example of a semiprime ring which is not reduced.

Corollary 2.5 *Let $K(A,B)$ be a semi-pmc of rings (with $1\neq 0$) in which $\langle,\rangle_B$ is epic. If A is commutative and B is reduced, then the Morita ring R is semiprime but not reduced.*

Proof For the mc $K(A,B)$, if the mc map $\langle,\rangle_B$ is epic, then by the above remark, the mc $K(R,B)$ is a pmc, hence R and B are Morita similar rings.

If A is commutative and B is reduced then by Theorem 2.1 above, B also becomes commutative. Because every reduced ring is semiprime and as semiprimeness is a Morita invariant property, so B is also semiprime. But R is Morita similar to B, so it is also semiprime.

Next, on the contrary assume that R is reduced. Then by Theorem 2.1, R becomes commutative. Let

$$r=\begin{bmatrix} a & n \\ m & b \end{bmatrix},\ r'=\begin{bmatrix} a' & n' \\ m' & b' \end{bmatrix}\in R.$$

Then

$$rr'=rr'\Longrightarrow$$

$$\begin{bmatrix} aa'+\langle n,m'\rangle_A & an'+nb' \\ ma'+bm' & \langle m,n'\rangle_B+bb' \end{bmatrix}=\begin{bmatrix} a'a+\langle n',m\rangle_A & a'n+n'b \\ m'a+b'm & \langle m',n\rangle_B+b'b \end{bmatrix}$$

Because B is commutative,

$$\langle m,n'\rangle_B=\langle m',n\rangle_B\,,\quad \forall m,m'\in M \text{ and } n,n'\in N.$$

This leads to the fact that

$$\begin{aligned}\langle m,n\rangle_B &= \langle 2m,n\rangle_B=2\langle m,n\rangle_B\\ &\Longrightarrow \langle m,n\rangle_B=0,\forall m\in M \text{ and } n\in N.\end{aligned}$$

Since every element of B is of the form $\sum_{i=1}^{n}\langle m_i,n_i\rangle_B$. Hence $B=0$, a contradiction. $\square$

3. Some remarks on derivations

A classical problem in ring theory is to pose conditions on a ring such that the ring appears to be commutative. Theorem 2.1 can be conveniently used to determine the commutativity of a ring (with some restrictions) of a semi-pmc if the other ring is commutative. For instance:

3.1 . Let $K(A, B)$ be a semi-pmc in which $\langle , \rangle_B$ is epic. If A is prime and has a reverse derivation, which Herstein in [10] defined as: $d : A \longrightarrow A$ is a reverse derivation in case $d(aa') = d(a')a + a'd(a)$, then by [10; Theorem 2.1], A becomes a commutative domain. If B is reduced then by Theorem 2.1 above, B also becomes commutative. In case $K(A, B)$ is a pmc, i.e., when A and B are Morita similar rings and B is a division ring, then A and B become isomorphic fields.

3.2 If $d : A \to A$ is a derivation, namely, $d(aa') = d(a)a' + ad(a')$, then Posner in [19; Lemma 3] proved that if A is prime and $d \neq 0$ such that the commutator $[a, d(a)] = 0, \forall a \in A$, then A is commutative. Hence, for a semi-pmc $K(A, B)$, if B is reduced, then B also becomes commutative and in case of a pmc if B is a division ring then both A and B become isomorphic fields.

3.3 An $n-$centralizing mapping on a subset S of a ring A is an additive map $d : A \to A$ such that $[S^n, d(s)] \in Cent(S), \forall s \in S$. Let $K(A, B)$ be a semi-pmc in which $\langle , \rangle_B$ is epic. If A is prime with $Char(A) \geq 0$, and $d \neq 0$ is a non-zero $n-$ centralizing derivation on a left ideal $U \neq 0$ of A, then it is proved by Deng [8; Theorem 2] that A is commutative. Hence if B is reduced then by Theorem 2.1, B is commutative.

In the following we list more conditions under which A and B become commutative rings.

Corollary 3.4 *Let $K(A, B)$ be a semi-pmc in which $\langle , \rangle_B$ is epic. Let A be prime, $d : A \longrightarrow A$ is a non-zero derivation and $U \neq (0)$ an ideal of A. If B is reduced then B becomes commutative if any one of the following conditions is satisfied. Moreover, if A and B are Morita similar rings, and B is a division ring, then A and B become isomorphic fields.*

(*i*) $[a, d(a)] \in Cent(A), \forall a \in A.$
(*ii*) $Char(A) \neq 2$ and $[d(a), d(a')] \in Cent(A)$, or $[d(a), d(a')] = 0, \forall a \in A.$
(*iii*) $Char(A) \neq 2$ and $d^2(A) \subseteq Cent(A).\forall a \in A$
(*iv*) $d(a^n) \in Cent(A),\ n > 0.$
(*v*) $Char(A) \neq 2,\ d[a, a'] \in Cent(A).$
(*vi*) $Char(A) \neq 2,\ d^3 \neq 0$, and $d(a - a^n) \in CentA,\ n > 1$
(*vii*) $Char(A) \neq 2, 3$, and $[[d(a), a], a] \in Cent(A).$
(*viii*) $[d(u), d(u')] = d[u, u']$ or $[d(u), d(u')] = d[, u', u]\ \forall u, u' \in U$
(*ix*) $d[u, u'] = 0$,or $[d(u), u] \in CentA, \forall u, u'^{.}$

(*x*) $d(u \circ u') = u \circ u'$ or $d(u \circ u') + u \circ u' = 0$,and if $Char(A) \neq 2$ such that $d(u) \circ d(u') = u \circ u'$ or $d(u) \circ d(u') = 0$ or $d(u) \circ d(u') + u \circ u' = 0$, where $u \circ u' = uu' + u'u, \forall u, u' \in U$

(*xi*) $d(uu') - uu' \in Cent(A)$ or $d(uu') + uu' \in Cent(A)$ or $d(u)d(u') - uu' \in Cent(A)$ or $d(u)d(u') + uu' \in Cent(A). \forall u, u' \in U$

(*xii*) $d(aa^{'}) + aa^{'} \in Cent(A)$ or $d(aa^{'}) - aa^{'} \in CentA$ or $d(a)d(a^{'}) + aa^{'} \in Cent(A)$ or $d(a)d(a^{'}) - aa^{'} \in CentA. \forall a, a^{'} \in A$.

(*xiii*) $g : A \to A$ another derivation on A, such that $d(u)u - ug(u) \in Cent(A), \forall u \in A$.

Proof The proof can be followed from [4; Corollary 2], [[1] , [2; Theorems 4.1, 4.2, 4.3, 4.4 and 4.5],.[20; Theorem 2], [19; Theorem 2],[12; Theorems 2 and 3], [8; Theorem 2], [11; Theorem], [6; Theorem 4.1], [3; Theorem 4] along with Theorem 2.1 and Corollary 2.4. □

Corollary 3.5 *Let $K(A, B)$ be a semi-pmc in which $\langle , \rangle_B$ is epic. Let A be semiprime and $d : A \longrightarrow A$ a non-zero derivation. If B is reduced, then B becomes commutative if any one of the following conditions is satisfied. Moreover, if A and B are Morita similar rings, and B is a division ring, then A and B become isomorphic fields.*

(*i*) $d([a, a']) + [a, a'] = 0$ *or* $d([a, a']) - [a, a'] = 0$.

(*ii*) $[d(a), d(a')] = [a, a']$

(*iii*) *A is 2-torsion free such that* $d([a, a']) + [a, a'] \in Cent(A)$ *or* $d[a, a'] - [a, a'] \in Cent(A)$.

Proof Proof can be followed from [5; Corollary 1], [7; Theorem 2], [13; Corollary 1] and Theorem 2.1 and Corollary 2.4. □

References

[1] M. Ashraf, N. Ur-Rehman, *On derivations and commutativity of in prime rings,* East-West J. Math. 3(1), (2001), 87-91.

[2] M. Ashraf, N. Ur-Rehman, *On commutativity of rings with derivations,* Result. Math. 42 (2002), 3-8.

[3] H.E. Bell, W. S. Martindale, *Centralizing mappings of semiprime rings,* Canad. Math. Bull 30 (1987), 92-101.

[4] H. E. Bell, M. N. Daif, *On derivations and commutativity in prime rings,* Acta. Math. Hungar. 66 (1995), 337-343.

[5] H. E. Bell, M. N. Daif, *On commutativity and strong commutativity preserving maps,* Canad. Math. Bull. Vol. 37 (4), 1994, 443-447.

[6] M. Bresar, *Centralizing mapping and derivations in prime rings,* J. Algebra 156 (1993), 385-394.

[7] M. N. Daif, and H. E. Bell, *Remarks on derivations on semiprime ring,* Int. J. Math. & Math. Sci. 15 (1992), 205-206.

[8] Deng, Q., *On n-centralizing mappings of prime rings,* Proc. R. Ir. Acad. 93A (1993), 171-176.

[9] J.L.García, J.J. Simón, *Morita equivalence for idempotent rings,* J.Pure & Appl. Algebra, 76 (1991), 39-56.

[10] I.N.Herstein, *Jordan derivation of prime rings,* proc. AMS, 8(1957), 1104-1110.

[11] I. N. Herstein, *A note on derivations II,* Canad. Math. Bull. Vol. 21 , 1978, 369-370.

[12] P. H. Lee, T. K. Lee, *On derivations of prime rings,* Chinese J. of Math. Vol. 9 (2), 1981, 107-110.

[13] M. Hongan, *A note on semiprime rings with derivation,* Int. J. Math. Sci. 20 (1997), 413-415.

[14] N. Jacobson, "Basic Algebra-II", W. H. Freeman & Company, San Francisco, 1980.

[15] T.Y.Lam, "Lectures on Modules and Rings", Vol. 189, Graduate Texts in Math., Springer-Verlag, New-York, 1999.

[16] L. Marín, *Morita equivalence based on context for various categories of modules over associative rings,* J.Pure & Appl. Algebra, 133 (1998), 219-232.

[17] J. C. McConnell, J. C. Robson, "Noncommutative Noetherian Rings", AMS, 2001.

[18] S.K. Nauman, *Morita similar matrix rings and their Grothendieck groups,* The Alig. Bull. Math., Vol. 23 (2004) 49-60.

[19] E.C.Posner, *Derivations in prime rings,* Proc. AMS 8(1957) 1093-1100.

[20] J. Vukman, *Commuting and centralizing mappings in prime rings,* Proc. Amer. Math. Soc. Vol. 109, No.(1) (1990)