***East-West J. of Mathematics: Vol. 10, No 2*** (2008) pp. 161-167

# TOTAL COLORINGS OF SOME CLASSES OF GLUED GRAPHS

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**Abstract**

The total chromatic number $\chi''(G)$ of a graph $G$ is the minimum number of colors needed to color the elements (vertices and edges) of $G$ such that no incident or adjacent pairs of elements receive the same color. The Total Coloring Conjecture states that for every simple graph $G$, $\chi''(G) \leq \Delta(G) + 2$. A graph $G$ is of *type 1* if $\chi''(G) = \Delta(G) + 1$ and of *type 2* if $\chi''(G) = \Delta(G) + 2$. A *glued graph* results from combining two vertex-disjoint graphs by identifying connected isomorphic subgraphs of both graphs.

We prove that the glued graphs of cycles, bipartite graphs and complete graphs satisfy the Total Coloring Conjecture. Moreover, we investigate necessary and sufficient conditions for being either of type 1 or type 2 of the glued graphs of cycles, trees and complete graphs.

## 1 Introduction

A *k-total coloring* of a graph $G$ is a coloring $f : V(G) \cup E(G) \rightarrow S$ where $S = \{1, 2, \ldots, k\}$. A $k$-total coloring is *proper* if incident edges have different colors, adjacent vertices have different colors and edges and its endpoints have different colors. A graph is *k-total colorable* if it has a proper $k$-total coloring. The *total chromatic number* $\chi''(G)$ of a graph $G$ is the least positive integer $k$ such that $G$ is $k$-total colorable.

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**Key words:** total coloring, the total chromatic number, glued graph
2000 AMS Mathematics Subject Classification: 05C15

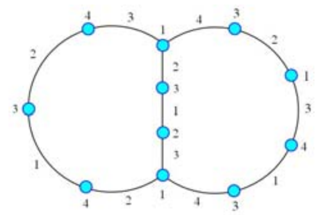

Figure 1.1: A 4-total colorable graph

Since a vertex and all edges incident to it cannot be assigned the same color, $\chi''(G) \geq \Delta(G) + 1$ for any graph $G$. The Total Coloring Conjecture, introduced independently by Behzad[1] and Vizing[8], states that for every graph $G$, $\chi''(G) \leq \Delta(G) + 2$. A graph $G$ is of *type 1* if $\chi''(G) = \Delta(G) + 1$ and *type 2* if $\chi''(G) = \Delta(G) + 2$.

Let $G_1$ and $G_2$ be any two graphs with distinct vertex sets. Let $H_1$ and $H_2$ be nontrivial connected subgraphs of $G_1$ and $G_2$, respectively, such that $H_1 \cong H_2$ with isomorphism $f$, then *the glued graph of* $G_1$ *and* $G_2$ *at* $H_1$ *and* $H_2$ *with respect to* $f$, denoted by $\underset{H_1 \cong_f H_2}{G_1 \diamond G_2}$, is the graph that results from combining $G_1$ with $G_2$ by identifying $H_1$ and $H_2$ with respect to the isomorphism $f$ between $H_1$ and $H_2$. Let $H$ be the copy of $H_1$ and $H_2$ in the glued graph. We refer to $H$ as the *clone* of the glued graph.

The *glued graph of* $G_1$ *and* $G_2$ *at the clone* $H$, written $\underset{H}{G_1 \diamond G_2}$, means that there exist a subgraph $H_1$ of $G_1$ and a subgraph $H_2$ of $G_2$ and an isomorphism $f$ between $H_1$ and $H_2$ such that $\underset{H_1 \cong_f H_2}{G_1 \diamond G_2}$ and $H$ is the copy of $H_1$ and $H_2$ in the resulting graph. We denote $G_1 \diamond G_2$ an arbitrary graph resulting from gluing graphs $G_1$ and $G_2$ at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism.

The vertex and edge colorings of glued graphs were investigated in [5] and [6]. More background regarding glued graphs can be explored in Promsakon's thesis [7]. Here we study the total colorings of glued graphs. In general, a glued graph of simple graphs is not necessary to be a simple graph. In this paper, we consider only simple connected glued graphs. We focus on four classes of graphs, namely, cycles, bipartite graphs, trees and complete graphs. We prove that the glued graphs of cycles, bipartite graphs and complete graphs satisfy the Total Colorings Conjecture. Moreover, we obtain necessary and sufficient conditions for being either of type 1 or type 2 of glued graphs of cycles, trees and complete graphs.

# 2 Main Results

## 2.1 The glued graphs of cycles

The first result in this section is in Theorem 2.3. We determine the total chromatic number of a glued graph of cycles $C_m \triangleleft\!\triangleright C_n$ by using Theorem 2.1 and Theorem 2.2.

**Theorem 2.1.** [10] $\chi''(C_n) = \begin{cases} 3 & \textit{if} \;\; n \equiv 0 \pmod 3, \\ 4 & \textit{otherwise}. \end{cases}$

**Theorem 2.2.** [4] *For a graph $G$, $\chi''(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$.*

**Theorem 2.3.** *For a glued graph $C_m \triangleleft\!\triangleright C_n$,*

$$\chi''(C_m \triangleleft\!\triangleright C_n) = \begin{cases} 3 & \textit{if } C_m \triangleleft\!\triangleright C_n \textit{ is a cycle and } \quad m = n \equiv 0 \pmod 3, \\ 4 & \textit{otherwise.} \end{cases}$$

*Proof.* A glued graph of cycles $C_m \triangleleft\!\triangleright C_n$ is a cycle only when $C_m \triangleleft\!\triangleright C_n \cong C_m \cong C_n$. By Theorem 2.1, if $m = n \equiv 0 \pmod 3$, we have $\chi''(C_m \triangleleft\!\triangleright C_n) = 3$. Otherwise, $\chi''(C_m \triangleleft\!\triangleright C_n) = 4$.

Assume that $C_m \triangleleft\!\triangleright C_n$ is not a cycle. Then $\Delta(C_m \triangleleft\!\triangleright C_n) = 3$. Thus $\chi''(C_m \triangleleft\!\triangleright C_n) \geq \Delta(C_m \triangleleft\!\triangleright C_n) + 1 = 4$. By Theorem 2.2, $\chi''(C_m \triangleleft\!\triangleright C_n) \leq \lfloor \frac{3}{2}\Delta(C_m \triangleleft\!\triangleright C_n) \rfloor \leq \lfloor \frac{3}{2} \times 3 \rfloor = 4$. Hence $\chi''(C_m \triangleleft\!\triangleright C_n) = 4$. ☐

**Corollary 2.4.** *Any glued graph of cycles satisfies the Total Coloring Conjecture.*

*Proof.* By Theorem 2.3, $\chi''(C_m \triangleleft\!\triangleright C_n) \leq 4$. Since $\Delta(C_m \triangleleft\!\triangleright C_n) + 2 \geq 4$, we get $\chi''(C_m \triangleleft\!\triangleright C_n) \leq \Delta(C_m \triangleleft\!\triangleright C_n) + 2$. ☐

**Theorem 2.5.** *If the glued graph $C_m \triangleleft\!\triangleright C_n$ is a cycle and $m = n \equiv 1, 2 \pmod 3$ then $C_m \triangleleft\!\triangleright C_n$ is of type 2. Otherwise, $C_m \triangleleft\!\triangleright C_n$ is of type 1.*

*Proof. Case 1.* $C_m \triangleleft\!\triangleright C_n$ is not a cycle. Then $\Delta(C_m \triangleleft\!\triangleright C_n) = 3$. By Theorem 2.3, $\chi''(C_m \triangleleft\!\triangleright C_n) = 4 = \Delta(C_m \triangleleft\!\triangleright C_n) + 1$. Hence, $C_m \triangleleft\!\triangleright C_n$ is of type 1.
*Case 2.* $C_m \triangleleft\!\triangleright C_n$ is a cycle. Then $m = n$ and $C_m \triangleleft\!\triangleright C_n \cong C_m \cong C_n$. If $m = n \equiv 0 \pmod 3$, by Theorem 2.1, we get $\chi''(C_m \triangleleft\!\triangleright C_n) = 3 = \Delta(C_m \triangleleft\!\triangleright C_n) + 1$. Thus $C_m \triangleleft\!\triangleright C_n$ is of type 1. If $m = n \equiv 1, 2 \pmod 3$, by Theorem 2.1, $\chi''(C_m \triangleleft\!\triangleright C_n) = 4 = \Delta(C_m \triangleleft\!\triangleright C_n) + 2$. ☐

## 2.2 The glued graphs of bipartite graphs and trees

**Proposition 2.6.** [1] *A bipartite graph satisfies the Total Coloring Conjecture.*

**Proposition 2.7.** [5] *If $G_1$ and $G_2$ are graphs, then*
(a) *$G_1 \triangleleft\!\triangleright G_2$ is bipartite if and only if $G_1$ and $G_2$ are bipartite,*
(b) *$G_1 \triangleleft\!\triangleright G_2$ is tree if and only if $G_1$ and $G_2$ are trees.*

**Corollary 2.8.** *Any glued graph of bipartite graphs satisfies the Total Coloring Conjecture.*

*Proof.* It follows immediately from Proposition 2.6 and Proposition 2.7. □

We know by Proposition 2.7 that the glued graph of trees is a tree. Since a tree is bipartite, any glued graph of trees satisfies the Total Coloring Conjecture. A necessary and sufficient condition to be either of type 1 or type 2 of the glued graph of trees is obtained next.

**Theorem 2.9.** *For a tree $T \neq P_2$, $\chi''(T) = \Delta(T) + 1$.*

*Proof.* If $T$ has only one vertex, then $\chi''(T) = 1 = \Delta(T) + 1$. If $T$ is $P_2$, then $\chi''(T) = 3 = \Delta(T) + 2$. Assume that $T$ is a tree with $n$ vertices, where $n \geq 3$. Then $\Delta(T) \geq 2$. We will proceed by induction on $n$.
When $n = 3$, we get $T \cong P_3$. It is easy to see that $\chi''(T) = 3 = \Delta(T) + 1$.
Assume that $\chi''(T) = \Delta(T)+1$ for all $T$ with $k$ vertices where $k \geq 3$. Let $T$ be a tree with $k+1$ vertices where $k \geq 3$ and $m = \Delta(T) + 1$. It suffices to show that there is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$. Since $T$ is a tree, $T$ has a vertex with degree 1, say $v$. Let $u$ be a vertex which is adjacent to $v$.
*Case 1.* $u$ is a vertex with maximum degree in $T - v$. Then $\Delta(T - v) + 1 = \Delta(T) = m - 1$. Since $T - v$ is a tree with $k$ vertices where $k \geq 3$, by induction hypothesis, $\chi''(T - v) \leq \Delta(T - v) + 1 = m - 1$. Thus we have a proper total coloring $f : V(T-v) \cup E(T-v) \to \{1, 2, ..., m-1\}$. Since $m-1 = \Delta(T) \geq 2$, there is a color $r$ which differs from $f(u)$. Let $f' : V(T) \cup E(T) \to \{1, 2, ..., m\}$ be a total coloring of $T$ defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(T-v) \cup E(T-v), \\ m & \text{if } x = uv, \\ r & \text{if } x = v. \end{cases}$$

Then $f'$ is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$.
*Case 2.* $u$ is not a vertex with maximum degree in $T - v$. Consequently $\Delta(T - v) + 1 = \Delta(T) + 1 = m$. Since $T - v$ is a tree with $k$ vertices where $k \geq 3$, by induction hypothesis, $\chi''(T - v) \leq \Delta(T - v) + 1 = m$. Then there is a proper total coloring $f : V(T - v) \cup E(T - v) \to \{1, 2, ..., m\}$. Since $d_{T-v}(u) + 1 \leq \Delta(T - v) = \Delta(T) = m - 1$, at most $m - 1$ colors are used to color $u$ and edges incident to $u$ in $T - v$, so we have a remaining color in $\{1, 2, ..., m\}$, say $r$. Since $m = \Delta(T) + 1 \geq 2 + 1 = 3$, there is a color which differs from $f(u)$ and $r$, say $r'$. Let $f' : V(T) \cup E(T) \to \{1, 2, ..., m\}$ be a total coloring of $T$ defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(T-v) \cup E(T-v), \\ r & \text{if } x = uv, \\ r' & \text{if } x = v. \end{cases}$$

Then $f'$ is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$. Hence $\chi''(T) \leq m = \Delta(T)+1$. Since $\chi''(T) \geq \Delta(T)+1$, we get $\chi''(T) = \Delta(T)+1$. □

**Corollary 2.10.** *If $T_1$ and $T_2$ are trees, then $T_1 \triangleleft\!\triangleright T_2$ is of type 1 unless $T_1 \cong T_2 \cong P_2$.*

*Proof.* By Proposition 2.7, $T_1 \triangleleft\!\triangleright T_2$ is a tree. By Theorem 2.9, we obtain the desired result. □

Corollary 2.8 guarantees that any glued graph of bipartite graphs satisfies the Total Coloring Conjecture. It is an open problem to find a necessary and sufficient condition to be of type 1 or type 2 of any glued graph of bipartite graphs.

## 2.3 The glued graphs of complete graphs

The total chromatic number and the maximum degree of complete graphs in Theorem 2.11 and Lemma 2.12 yield a proof in Theorem 2.13 that any glued graph of complete graphs satisfies the Total Coloring Conjecture.

**Theorem 2.11.** [2] $\chi''(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$

**Lemma 2.12.** *If a glued graph of complete graphs $K_m \triangleleft\!\triangleright K_n$ is a simple graph, then $\Delta(K_m \triangleleft\!\triangleright K_n) = n(K_m \triangleleft\!\triangleright K_n) - 1$.*

*Proof.* Assume that $K_m \triangleleft\!\triangleright K_n$ is a simple graph. Then the clone of $K_m \triangleleft\!\triangleright K_n$ is a complete graph, say $K_r$ for some $r$. Each vertex in the clone of $K_m \underset{K_r}{\triangleleft\!\triangleright} K_n$ gives the maximum degree. Hence $\Delta(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) = (m-1)+(n-1)-(r-1) = m+n-r-1$. Moreover, $n(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) = n(K_m)+n(K_n)-n(K_r) = m+n-r$. Therefore $\Delta(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) = n(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) - 1$. □

**Theorem 2.13.** *Any glued graph of complete graphs satisfies the Total Coloring Conjecture.*

*Proof.* Let $k = n(K_m \triangleleft\!\triangleright K_n)$. Then

$$\begin{aligned} \chi''(K_m \triangleleft\!\triangleright K_n) &\leq \chi''(K_k), && \text{(since } K_m \triangleleft\!\triangleright K_n \text{ is a subgraph of } K_k) \\ &\leq \Delta(K_k) + 2, && \text{(by Theorem 2.11)} \\ &= n(K_m \triangleleft\!\triangleright K_n) - 1 + 2, && \\ &= \Delta(K_m \triangleleft\!\triangleright K_n) + 2. && \text{(by Lemma 2.12)} \end{aligned}$$

□

Now we look for a necessary and sufficient condition to be either of type 1 or type 2 for any glued graph of complete graphs. Theorem 2.16 gives this result by using Theorem 2.14 and Lemma 2.15.

A *matching* in a graph $G$ is a set of edges with no shared endpoints. The maximum size of matching of a graph $G$ is denoted by $\alpha'(G)$. The *complement* $\overline{G}$ of a graph $G$ is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

**Theorem 2.14.** [3] *Suppose that $G$ is a graph of order $2k$ and $\Delta(G) = 2k-1$. We have $\chi''(G) = 2k$ if and only if $e(\overline{G}) + \alpha'(\overline{G}) \geq k$.*

**Lemma 2.15.** *For $m, n, r \in \mathbb{N}$ such that $n > r$*
*$m < r + \frac{2r-n}{2n-2r-1}$ if and only if $(m-r)(n-r) + (n-r) < \frac{m+n-r}{2}$.*

*Proof.* Let $m, n, r \in \mathbb{N}$ such that $n > r$. Then

$$\begin{aligned}
m < r + \frac{2r-n}{2n-2r-1} &\Leftrightarrow m < \frac{r(2n-2r-1)+(2r-n)}{2n-2r-1} \\
&\Leftrightarrow m < \frac{2r(n-r)-(n-r)}{2n-2r-1} \\
&\Leftrightarrow m < \frac{n-r}{2n-2r-1}(2r-1) \\
&\Leftrightarrow (2n-2r-1)m < (n-r)(2r-1) \\
&\Leftrightarrow 2m(n-r) - m < (n-r)(2r-1) \\
&\Leftrightarrow (n-r)(2m-2r+1) < m \\
&\Leftrightarrow (n-r)(2m-2r+1) + n - r < m + n - r \\
&\Leftrightarrow 2(n-r)(m-r+1) < m + n - r \\
&\Leftrightarrow (n-r)(m-r) + (n-r) < \frac{m+n-r}{2}.
\end{aligned}$$

$\square$

**Theorem 2.16.** *Let $m \geq n > r$. If $m+n-r$ is even and $m < r + \frac{2r-n}{2n-2r-1}$, then $K_m \underset{K_r}{\Diamond} K_n$ is of type 2. Otherwise, $K_m \underset{K_r}{\Diamond} K_n$ is of type 1.*

*Proof.* Let $m \geq n > r$ and $G = K_m \underset{K_r}{\Diamond} K_n$. *Case 1.* $m+n-r$ is odd. By Theorem 2.11, $\chi''(K_{m+n-r}) = m+n-r = \Delta(K_{m+n-r})+1$. Since $G$ is a subgraph of $K_{m+n-r}$ and $\Delta(G) = \Delta(K_{m+n-r})$, we get $\chi''(G) \leq \chi''(K_{m+n-r}) = \Delta(K_{m+n-r}) + 1 = \Delta(G) + 1$. Thus $G$ is of type 1.

*Case 2.* $m+n-r$ is even. By Lemma 2.12, $\Delta(G) = n(G) - 1 = m+n-r-1$. The complement of $G$, $\overline{K_m \underset{K_r}{\Diamond} K_n}$, has only one nontrivial component, $K_{m-r,n-r}$. Then $e(\overline{G}) = (m-r)(n-r)$. Since $m \geq n$, we get $\alpha'(\overline{G}) = n-r$. Thus

$e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r)$. If $m \geq r+\frac{2r-n}{2n-2r-1}$, by Lemma 2.15, $e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r) \geq \frac{m+n-r}{2}$. Consequently, by Theorem 2.14, $G$ is of type 1. If $m < r+\frac{2r-n}{2n-2r-1}$, by Lemma 2.15, $e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r) < \frac{m+n-r}{2}$. Hence, by Theorem 2.14, $\chi''(G) \neq n+m-r$. Since $n+m-r=n(G)=\Delta(G)+1$, we have $\chi''(G) \neq \Delta(G)+1$. By Theorem 2.13, $\chi''(G) \leq \Delta(G)+2$. Therefore $\chi''(G)=\Delta(G)+2$, and so, $G$ is of type 2. $\square$

**Corollary 2.17.** *For $m \geq n > r$,*
(a) $K_m \underset{K_r}{\Phi} K_n$ *is of type* 1 *if and only if $m+n-r$ is odd or $m \geq r+\frac{2r-n}{2n-2r-1}$,*
(b) $K_m \underset{K_r}{\Phi} K_n$ *is of type* 2 *if and only if $m+n-r$ is even and $m < r+\frac{2r-n}{2n-2r-1}$.*

*Proof.* It follows immediately from Theorem 2.13 and Theorem 2.16. $\square$

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