East-West J. of Mathematics: Vol. 10, No 2 (2008) pp. 161-167

TOTAL COLORINGS OF SOME CLASSES OF GLUED GRAPHS

W. Charoenpanitseri[†], **W. Hemakul**[‡]
and **C. Uiyyasathian**[*]

Department of Mathematics, Faculty of Science,
Chulalongkorn University, Bangkok 10330, Thailand
[†]*idolhere@hotmail.com*, [‡]*wanida.H@chula.ac.th*, [*]*Chariya.U@chula.ac.th*

Abstract

The total chromatic number $\chi''(G)$ of a graph G is the minimum number of colors needed to color the elements (vertices and edges) of G such that no incident or adjacent pairs of elements receive the same color. The Total Coloring Conjecture states that for every simple graph G, $\chi''(G) \leq \Delta(G) + 2$. A graph G is of *type 1* if $\chi''(G) = \Delta(G) + 1$ and of *type 2* if $\chi''(G) = \Delta(G) + 2$. A *glued graph* results from combining two vertex-disjoint graphs by identifying connected isomorphic subgraphs of both graphs.

We prove that the glued graphs of cycles, bipartite graphs and complete graphs satisfy the Total Coloring Conjecture. Moreover, we investigate necessary and sufficient conditions for being either of type 1 or type 2 of the glued graphs of cycles, trees and complete graphs.

1 Introduction

A *k-total coloring* of a graph G is a coloring $f : V(G) \cup E(G) \to S$ where $S = \{1, 2, \ldots, k\}$. A k-total coloring is *proper* if incident edges have different colors, adjacent vertices have different colors and edges and its endpoints have different colors. A graph is *k-total colorable* if it has a proper k-total coloring. The *total chromatic number* $\chi''(G)$ of a graph G is the least positive integer k such that G is k-total colorable.

[*] corresponding author
Key words: total coloring, the total chromatic number, glued graph
2000 AMS Mathematics Subject Classification: 05C15

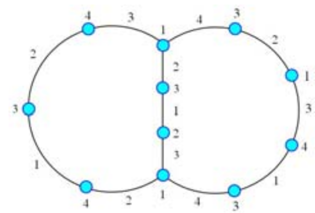

Figure 1.1: A 4-total colorable graph

Since a vertex and all edges incident to it cannot be assigned the same color, $\chi''(G) \geq \Delta(G) + 1$ for any graph G. The Total Coloring Conjecture, introduced independently by Behzad[1] and Vizing[8], states that for every graph G, $\chi''(G) \leq \Delta(G) + 2$. A graph G is of *type 1* if $\chi''(G) = \Delta(G) + 1$ and *type 2* if $\chi''(G) = \Delta(G) + 2$.

Let G_1 and G_2 be any two graphs with distinct vertex sets. Let H_1 and H_2 be nontrivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with isomorphism f, then *the glued graph of G_1 and G_2 at H_1 and H_2 with respect to f*, denoted by $\underset{H_1 \cong_f H_2}{G_1 \diamondplus G_2}$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer to H as the *clone* of the glued graph.

The *glued graph of G_1 and G_2 at the clone H*, written $\underset{H}{G_1 \diamondplus G_2}$, means that there exist a subgraph H_1 of G_1 and a subgraph H_2 of G_2 and an isomorphism f between H_1 and H_2 such that $\underset{H_1 \cong_f H_2}{G_1 \diamondplus G_2}$ and H is the copy of H_1 and H_2 in the resulting graph. We denote $G_1 \diamondplus G_2$ an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism.

The vertex and edge colorings of glued graphs were investigated in [5] and [6]. More background regarding glued graphs can be explored in Promsakon's thesis [7]. Here we study the total colorings of glued graphs. In general, a glued graph of simple graphs is not necessary to be a simple graph. In this paper, we consider only simple connected glued graphs. We focus on four classes of graphs, namely, cycles, bipartite graphs, trees and complete graphs. We prove that the glued graphs of cycles, bipartite graphs and complete graphs satisfy the Total Colorings Conjecture. Moreover, we obtain necessary and sufficient conditions for being either of type 1 or type 2 of glued graphs of cycles, trees and complete graphs.

2 Main Results

2.1 The glued graphs of cycles

The first result in this section is in Theorem 2.3. We determine the total chromatic number of a glued graph of cycles $C_m \oplus C_n$ by using Theorem 2.1 and Theorem 2.2.

Theorem 2.1. [10] $\chi''(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$

Theorem 2.2. [4] *For a graph G, $\chi''(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$.*

Theorem 2.3. *For a glued graph $C_m \oplus C_n$,*
$$\chi''(C_m \oplus C_n) = \begin{cases} 3 & \text{if } C_m \oplus C_n \text{ is a cycle and } \quad m = n \equiv 0 \pmod 3, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. A glued graph of cycles $C_m \oplus C_n$ is a cycle only when $C_m \oplus C_n \cong C_m \cong C_n$. By Theorem 2.1, if $m = n \equiv 0 \pmod 3$, we have $\chi''(C_m \oplus C_n) = 3$. Otherwise, $\chi''(C_m \oplus C_n) = 4$.

Assume that $C_m \oplus C_n$ is not a cycle. Then $\Delta(C_m \oplus C_n) = 3$. Thus $\chi''(C_m \oplus C_n) \geq \Delta(C_m \oplus C_n) + 1 = 4$. By Theorem 2.2, $\chi''(C_m \oplus C_n) \leq \lfloor \frac{3}{2}\Delta(C_m \oplus C_n) \rfloor \leq \lfloor \frac{3}{2} \times 3 \rfloor = 4$. Hence $\chi''(C_m \oplus C_n) = 4$. ☐

Corollary 2.4. *Any glued graph of cycles satisfies the Total Coloring Conjecture.*

Proof. By Theorem 2.3, $\chi''(C_m \oplus C_n) \leq 4$. Since $\Delta(C_m \oplus C_n) + 2 \geq 4$, we get $\chi''(C_m \oplus C_n) \leq \Delta(C_m \oplus C_n) + 2$. ☐

Theorem 2.5. *If the glued graph $C_m \oplus C_n$ is a cycle and $m = n \equiv 1, 2 \pmod 3$ then $C_m \oplus C_n$ is of type 2. Otherwise, $C_m \oplus C_n$ is of type 1.*

Proof. Case 1. $C_m \oplus C_n$ is not a cycle. Then $\Delta(C_m \oplus C_n) = 3$. By Theorem 2.3, $\chi''(C_m \oplus C_n) = 4 = \Delta(C_m \oplus C_n) + 1$. Hence, $C_m \oplus C_n$ is of type 1.
Case 2. $C_m \oplus C_n$ is a cycle. Then $m = n$ and $C_m \oplus C_n \cong C_m \cong C_n$. If $m = n \equiv 0 \pmod 3$, by Theorem 2.1, we get $\chi''(C_m \oplus C_n) = 3 = \Delta(C_m \oplus C_n) + 1$. Thus $C_m \oplus C_n$ is of type 1. If $m = n \equiv 1, 2 \pmod 3$, by Theorem 2.1, $\chi''(C_m \oplus C_n) = 4 = \Delta(C_m \oplus C_n) + 2$. ☐

2.2 The glued graphs of bipartite graphs and trees

Proposition 2.6. [1] *A bipartite graph satisfies the Total Coloring Conjecture.*

Proposition 2.7. [5] *If G_1 and G_2 are graphs, then*
(a) *$G_1 \oplus G_2$ is bipartite if and only if G_1 and G_2 are bipartite,*
(b) *$G_1 \oplus G_2$ is tree if and only if G_1 and G_2 are trees.*

Corollary 2.8. *Any glued graph of bipartite graphs satisfies the Total Coloring Conjecture.*

Proof. It follows immediately from Proposition 2.6 and Proposition 2.7. $\square$

We know by Proposition 2.7 that the glued graph of trees is a tree. Since a tree is bipartite, any glued graph of trees satisfies the Total Coloring Conjecture. A necessary and sufficient condition to be either of type 1 or type 2 of the glued graph of trees is obtained next.

Theorem 2.9. *For a tree $T \neq P_2$, $\chi''(T) = \Delta(T) + 1$.*

Proof. If T has only one vertex, then $\chi''(T) = 1 = \Delta(T) + 1$. If T is P_2, then $\chi''(T) = 3 = \Delta(T) + 2$. Assume that T is a tree with n vertices, where $n \geq 3$. Then $\Delta(T) \geq 2$. We will proceed by induction on n.
When $n = 3$, we get $T \cong P_3$. It is easy to see that $\chi''(T) = 3 = \Delta(T) + 1$.
Assume that $\chi''(T) = \Delta(T) + 1$ for all T with k vertices where $k \geq 3$. Let T be a tree with $k + 1$ vertices where $k \geq 3$ and $m = \Delta(T) + 1$. It suffices to show that there is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$. Since T is a tree, T has a vertex with degree 1, say v. Let u be a vertex which is adjacent to v.
Case 1. u is a vertex with maximum degree in $T - v$. Then $\Delta(T - v) + 1 = \Delta(T) = m - 1$. Since $T - v$ is a tree with k vertices where $k \geq 3$, by induction hypothesis, $\chi''(T - v) \leq \Delta(T - v) + 1 = m - 1$. Thus we have a proper total coloring $f : V(T - v) \cup E(T - v) \to \{1, 2, ..., m - 1\}$. Since $m - 1 = \Delta(T) \geq 2$, there is a color r which differs from $f(u)$. Let $f' : V(T) \cup E(T) \to \{1, 2, ..., m\}$ be a total coloring of T defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(T - v) \cup E(T - v), \\ m & \text{if } x = uv, \\ r & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$.
Case 2. u is not a vertex with maximum degree in $T - v$. Consequently $\Delta(T - v) + 1 = \Delta(T) + 1 = m$. Since $T - v$ is a tree with k vertices where $k \geq 3$, by induction hypothesis, $\chi''(T - v) \leq \Delta(T - v) + 1 = m$. Then there is a proper total coloring $f : V(T - v) \cup E(T - v) \to \{1, 2, ..., m\}$. Since $d_{T-v}(u) + 1 \leq \Delta(T - v) = \Delta(T) = m - 1$, at most $m - 1$ colors are used to color u and edges incident to u in $T - v$, so we have a remaining color in $\{1, 2, ..., m\}$, say r. Since $m = \Delta(T) + 1 \geq 2 + 1 = 3$, there is a color which differs from $f(u)$ and r, say r'. Let $f' : V(T) \cup E(T) \to \{1, 2, ..., m\}$ be a total coloring of T defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(T - v) \cup E(T - v), \\ r & \text{if } x = uv, \\ r' & \text{if } x = v. \end{cases}$$

Then f' is a proper total coloring from $V(T) \cup E(T)$ to $\{1, 2, ..., m\}$. Hence $\chi''(T) \leq m = \Delta(T)+1$. Since $\chi''(T) \geq \Delta(T)+1$, we get $\chi''(T) = \Delta(T)+1$. □

Corollary 2.10. *If T_1 and T_2 are trees, then $T_1 \triangleleft\!\triangleright T_2$ is of type 1 unless $T_1 \cong T_2 \cong P_2$.*

Proof. By Proposition 2.7, $T_1 \triangleleft\!\triangleright T_2$ is a tree. By Theorem 2.9, we obtain the desired result. □

Corollary 2.8 guarantees that any glued graph of bipartite graphs satisfies the Total Coloring Conjecture. It is an open problem to find a necessary and sufficient condition to be of type 1 or type 2 of any glued graph of bipartite graphs.

2.3 The glued graphs of complete graphs

The total chromatic number and the maximum degree of complete graphs in Theorem 2.11 and Lemma 2.12 yield a proof in Theorem 2.13 that any glued graph of complete graphs satisfies the Total Coloring Conjecture.

Theorem 2.11. [2] $\chi''(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$

Lemma 2.12. *If a glued graph of complete graphs $K_m \triangleleft\!\triangleright K_n$ is a simple graph, then $\Delta(K_m \triangleleft\!\triangleright K_n) = n(K_m \triangleleft\!\triangleright K_n) - 1$.*

Proof. Assume that $K_m \triangleleft\!\triangleright K_n$ is a simple graph. Then the clone of $K_m \triangleleft\!\triangleright K_n$ is a complete graph, say K_r for some r. Each vertex in the clone of $K_m \underset{K_r}{\triangleleft\!\triangleright} K_n$ gives the maximum degree. Hence $\Delta(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) = (m-1)+(n-1)-(r-1) = m+n-r-1$. Moreover, $n(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) = n(K_m)+n(K_n)-n(K_r) = m+n-r$. Therefore $\Delta(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) = n(K_m \underset{K_r}{\triangleleft\!\triangleright} K_n) - 1$. □

Theorem 2.13. *Any glued graph of complete graphs satisfies the Total Coloring Conjecture.*

Proof. Let $k = n(K_m \triangleleft\!\triangleright K_n)$. Then

$$\begin{aligned}
\chi''(K_m \triangleleft\!\triangleright K_n) &\leq \chi''(K_k), && \text{(since } K_m \triangleleft\!\triangleright K_n \text{ is a subgraph of } K_k\text{)} \\
&\leq \Delta(K_k) + 2, && \text{(by Theorem 2.11)} \\
&= n(K_m \triangleleft\!\triangleright K_n) - 1 + 2, && \\
&= \Delta(K_m \triangleleft\!\triangleright K_n) + 2. && \text{(by Lemma 2.12)}
\end{aligned}$$

□

Now we look for a necessary and sufficient condition to be either of type 1 or type 2 for any glued graph of complete graphs. Theorem 2.16 gives this result by using Theorem 2.14 and Lemma 2.15.

A *matching* in a graph G is a set of edges with no shared endpoints. The maximum size of matching of a graph G is denoted by $\alpha'(G)$. The *complement* $\overline{G}$ of a graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Theorem 2.14. [3] *Suppose that G is a graph of order $2k$ and $\Delta(G) = 2k-1$. We have $\chi''(G) = 2k$ if and only if $e(\overline{G}) + \alpha'(\overline{G}) \geq k$.*

Lemma 2.15. *For $m, n, r \in \mathbb{N}$ such that $n > r$*
$m < r + \frac{2r-n}{2n-2r-1}$ if and only if $(m-r)(n-r) + (n-r) < \frac{m+n-r}{2}$.

Proof. Let $m, n, r \in \mathbb{N}$ such that $n > r$. Then

$$\begin{aligned}
m < r + \frac{2r-n}{2n-2r-1} &\Leftrightarrow m < \frac{r(2n-2r-1)+(2r-n)}{2n-2r-1} \\
&\Leftrightarrow m < \frac{2r(n-r)-(n-r)}{2n-2r-1} \\
&\Leftrightarrow m < \frac{n-r}{2n-2r-1}(2r-1) \\
&\Leftrightarrow (2n-2r-1)m < (n-r)(2r-1) \\
&\Leftrightarrow 2m(n-r) - m < (n-r)(2r-1) \\
&\Leftrightarrow (n-r)(2m-2r+1) < m \\
&\Leftrightarrow (n-r)(2m-2r+1) + n - r < m + n - r \\
&\Leftrightarrow 2(n-r)(m-r+1) < m + n - r \\
&\Leftrightarrow (n-r)(m-r) + (n-r) < \frac{m+n-r}{2}.
\end{aligned}$$

□

Theorem 2.16. *Let $m \geq n > r$. If $m+n-r$ is even and $m < r + \frac{2r-n}{2n-2r-1}$, then $K_m \underset{K_r}{\Diamond} K_n$ is of type 2. Otherwise, $K_m \underset{K_r}{\Diamond} K_n$ is of type 1.*

Proof. Let $m \geq n > r$ and $G = K_m \underset{K_r}{\Diamond} K_n$. *Case 1.* $m+n-r$ is odd. By Theorem 2.11, $\chi''(K_{m+n-r}) = m+n-r = \Delta(K_{m+n-r})+1$. Since G is a subgraph of K_{m+n-r} and $\Delta(G) = \Delta(K_{m+n-r})$, we get $\chi''(G) \leq \chi''(K_{m+n-r}) = \Delta(K_{m+n-r}) + 1 = \Delta(G) + 1$. Thus G is of type 1.

Case 2. $m+n-r$ is even. By Lemma 2.12, $\Delta(G) = n(G) - 1 = m+n-r-1$. The complement of G, $\overline{K_m \underset{K_r}{\Diamond} K_n}$, has only one nontrivial component, $K_{m-r,n-r}$. Then $e(\overline{G}) = (m-r)(n-r)$. Since $m \geq n$, we get $\alpha'(\overline{G}) = n-r$. Thus

$e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r)$. If $m\geq r+\frac{2r-n}{2n-2r-1}$, by Lemma 2.15, $e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r)\geq\frac{m+n-r}{2}$. Consequently, by Theorem 2.14, G is of type 1. If $m<r+\frac{2r-n}{2n-2r-1}$, by Lemma 2.15, $e(\overline{G})+\alpha'(\overline{G})=(m-r)(n-r)+(n-r)<\frac{m+n-r}{2}$. Hence, by Theorem 2.14, $\chi''(G)\neq n+m-r$. Since $n+m-r=n(G)=\Delta(G)+1$, we have $\chi''(G)\neq\Delta(G)+1$. By Theorem 2.13, $\chi''(G)\leq\Delta(G)+2$. Therefore $\chi''(G)=\Delta(G)+2$, and so, G is of type 2. $\square$

Corollary 2.17. *For $m\geq n>r$,*
(a) $K_m \underset{K_r}{\oplus} K_n$ *is of type* 1 *if and only if* $m+n-r$ *is odd or* $m\geq r+\frac{2r-n}{2n-2r-1}$,
(b) $K_m \underset{K_r}{\oplus} K_n$ *is of type* 2 *if and only if* $m+n-r$ *is even and* $m<r+\frac{2r-n}{2n-2r-1}$.

Proof. It follows immediately from Theorem 2.13 and Theorem 2.16. $\square$

References

[1] M. Behzad, "The total chromatic number of a graph", Combinatorial Mathematics and its Applications, Proceedings of the Conference Oxford Academic Press N.Y. (1971), 1-9.

[2] M. Bezhad, G. Chartrand and J. K. Cooper, *The colors numbers of complete graphs*, J. London Math. Soc., **42** (1967), 225-228.

[3] A. J. W. Hilton, *A total chromatic number analogue of Plantholt's theorem*, Discrete Math., **79** (1989), 169-175.

[4] A. V. Kostochka and N. P. Mazurova, *An inequality in the theory of graph coloring* (in Russian), Metody Diskret. Analiz., **30** (1977), 23-29.

[5] C. Promsakon and C. Uiyyasathian, *Chromatic numbers of glued graphs*, Thai J. Math. (special issued), **4** (2006), 75-81.

[6] C. Promsakon and C. Uiyyasathian, *Edge-chromatic numbers of glued graphs*, Thai J. Math., **4** (2006), 395-401.

[7] C. Promsakon, "Colorability of Glued Graphs", Master Degree Thesis, Chulalongkorn University (2006).

[8] V. G. Vizing, *On evaluation of chromatic number of a p-graph* (in Russian), Discrete Analysis, Collection of works of Sobolev Institute of Mathematics SB RAS, **3** (1964), 3-24.

[9] D. West, "Introduction to Graph Theory", Prentice Hall, New Jersey (2001).

[10] H. P. Yap, *Total Coloring of Graphs*, Lecture Note in Mathematics, Vol. 1623, Springer, Berlin (1996).