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# ON THE SOLVABILITY OF A FUNCTIONAL INTEGRAL EQUATION

**Wagdy G. El-Sayed**

*Department of Mathematics - Faculty of Science*
*Alexandria University - Alexandria - Egypt*
*E-mail: wagdygoma@yahoo.com*

**Abstract**

The paper proves the existence theorem of a functional integral equation of volterra type in the class $L^1$ of all Lebesgue integrable functions on $[0, \infty)$, we used the Schauder fixed point theorem and the De-Blasi measure of weak noncompactness.

## 1. Introduction

Generally, the subject of nonlinear integral equations considered as an important branch of mathematics because it is used for solving of many problems such as physics, engineering and economics (cf. [9,11]).
In this paper, we will investigate the solvability of the functional Volterra integral equation

$$x(t) = f_1(t, \int_0^t k(t,s) f_2(s, x(\phi(s)))ds), \qquad t > 0 \tag{1}$$

which is the general form of other functional integral equations (cf. [3, 7]). In [6], this equation had been investigated where the existence of monotonic solutions were proved.

## 2. Notation, definitions and Auxiliary Facts

This section is devoted to recall what we will be needed later, for this let $R^+ = [0, \infty)$ and $L^1 = L^1(R^+)$ be the class of Lebesgue integrable functions

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on $R^+$, whose norm defined as

$$\| x \| = \int_0^\infty | x(t) | \, dt.$$

Assume that $f(t, x) = f : R^+ \times R \rightarrow R$ satisfies Caratheodory conditions, i.e. it is measurable in $t$ for any $x \in R$ and continuous in $x$ for almost all $t \in R^+$. Then for every function $x(t)$ being measurable on $R^+$, we may assign the function $(Fx)(t) = f(t, x(t)), t \in R^+$ which is called the superposition operator generated by $f$ and we have the following theorem.

**Theorem (1). [1]**

The superposition operator $F$ generated by a function $f$ maps continuously the space $L^1(R^+)$ into itself iff $| f(t, x) | \leq a(t) + b \mid x \mid$, for all $t \in R^+, x \in R$, where $a(t) \in L^1(R^+)$ and $b$ is a nonnegative constant.

Next, let $E$ be a Banach space whose zero vector is $\theta$ and $X$ a nonempty and bounded subsect of $E$. Suppose $B_r$ is a closed ball whose center is $\theta$ and its radius $r$. De-Blasi measure of weak noncompactness is defined [5] as:

$\beta(X) = \inf\{r > 0$: there exists a weakly compact subset $Y$ of $E$ such that $X \subset Y + B_r\}$.

The measure $\beta$ has many useful properties [2].

On other hand there is another measure of weak noncompactness in $L(R^+)$ which is related to the De-Blasi measure $\beta$. This measure was introduced by Banas and Knap [9] and defined as:

$$\gamma(X) = c(X) + d(X),$$

where

$$c(X) = \lim_{\epsilon \to 0} \{ \sup_{x \in X} \{ \sup[ \int_D | x(t) | \, dt, D \subset R^+, \text{meas} D \leq \epsilon]\}\},$$

$$d(X) = \lim_{T \to \infty} \{ \sup[ \int_T^\infty | x(t) | \, dt : x \in X]\},$$

for any nonempty and bounded subset $X$ of $L^1(R^+)$ and we have the following theorem.

**Theorem (2). [4]**

The function $\gamma$ is a regular measure of weak noncompactness in the space $L^1(R^+)$ such that

$$\beta(X) \leq \gamma(X) \leq 2\beta(X).$$

Also, we have the following theorem.

**Theorem (3). [4]**

A subset $X \subset L(R^+)$ is relatively compact iff

(i) for any $\epsilon > 0$, there is $\delta > 0$ such that if meas $D \leq \delta$, then $\int_D | x(t) | \, dt \leq \epsilon, \quad x \in X$,

(ii) for any $\epsilon > 0$, there is $T > 0$ such that $\int_T^\infty \mid x(t) \mid dt \leq \epsilon \quad x \in X$.
Finally, we will recall Dragoni theorem and Schauder fixed point theorem respectively.
**Theorem (4). [10]**
Let $A$ be a compact metric space, $B$ a separable metric space and $C$ a Banach space. If $H : A \times B \to C$ is a function satisfies caratheodory conditions, then for every $\epsilon > 0$, there exists a measurable closed subset $D_\epsilon$ of $A$ such that $m(A/D_\epsilon) < \epsilon$ and $H \mid_{D_\epsilon \times B}$ is continuous.
**Theorem (5). [8]**
Assume that $X$ is a nonempty, convex, closed and bounded subset of a Banach space $E$ and $G : E \to E$ is completely continuous mapping (i.e. $G$ is continuous and $G(Y)$ is relatively compact for every bounded subset $Y$ of $E$) such that $G : X \to X$. Then $G$ has at least a fixed point in $X$.

## 3. Existence Theorem

Now, we will assume the necessary conditions under which the existence theorem of our functional equation will be proved.
Assume that:
(i) $f_i : R^+ \times R \to R, i = 1, 2$ satisfy Caratheodory conditions and there are two functions $a_i \in L^1, i = 1, 2$ and two nonnegative constants $b_i, i = 1, 2$ such that $\mid f_i(t, x) \mid \leq a_i(t) + b_i \mid x \mid \quad , \quad i = 1, 2,$ for all $t \in R^+ \quad , \quad x \in R,$

(ii) $k : R^+ \times R^+ \to R$ satisfies Caratheodory conditions such that the linear operator $K$ defined as

$$(Kx)(t) = \int_0^t k(t, s)x(s)ds \qquad , \qquad t > 0 \tag{2}$$

maps the space $L^1$ into itself (note that due to this assumption and [10] the linear operator $K$ will be continuous and so it is bounded whose norm $\parallel K \parallel$).
(iii) $\phi : R^+ \to R^+$ is an increasing, absolutely continuous function and there is a positive constant $M$ such that

$$\phi'(t) \geq M, \qquad \text{for almost all } t \geq 0,$$

(iv) $\alpha = b_1 b_2 \parallel K \parallel M^{-1} < 1$.
Hence, we have the following theorem.
**Theorem (6)**
If the assumptions (i) - (iv) are satisfied, then the functional integral equation (1) has at least a solution in the space $L^1$.
**Proof**
Let

$$(Hx)(t) = f_1(t, \int_0^t k(t, s)f_2(s, x(\phi(s)))ds) \qquad , \qquad t > 0. \tag{3}$$

Due to the assumptions (i), (ii), (iii) and theorem (1), we deduce that the operator $H$ defined by (3) maps continuously the space $L^1$ into itself.
Moreover, we have

$$\begin{aligned}
\| (Hx)(t) \| &= \int_0^\infty f_1(t, \int_0^t k(t,s) f_2(s, x(\phi(s))) ds) dt \\
&\leq \int_0^\infty [a_1(t) + b_1 \mid \int_0^t k(t,s) f_2(s, x(\phi(s))) ds \mid] dt \\
&\leq \| a_1 \| + b_1 \| KF\phi \|
\end{aligned}$$

where $F$ is the superposition operator generated by $f_2$ as defined in section (2) and $K$ is the linear integral operator as defined by (2).
Thus, we have

$$\begin{aligned}
\| (Hx)(t) \| &\leq \| a_1 \| + b_1 \| K \| \int_0^\infty [a_2(s) + b_2 \mid x(\phi(s)) \mid] ds \\
&\leq \| a_1 \| + b_1 \| a_2 \| \| K \| + (b_1 b_2 \| K \| /M). \int_0^\infty \mid x(\phi(s)) \mid \phi'(s) ds \\
&\leq \| a_1 \| + b_1 \| a_2 \| \| K \| + (b_1 b_2 \| K \| /M) \int_0^\infty \mid x(u) \mid du,
\end{aligned}$$

where $u = \phi(s)$.
So, using (iv), we have

$$\| (Hx)(t) \| \leq \| a_1 \| + b_1 \| a_2 \| \| K \| + \alpha \| x \|,$$

which means that the operator $H$ transforms the ball $B_r$ into itself, where $r = \frac{\|a_1\| + b_1 \|a_2\| \|K\|}{(1-\alpha)}$.
Next, we will prove that the operator $H$ is contraction with respect to the De-Blasi measure $\beta$ of weak noncompactness. For this, let $X \subset B_r$ and $D \subset [0,T]$, $0 < T < t$, with meas $D < \epsilon$ (in the sense of Lebesgue measure), then for any $x \in X$, we have

$$\begin{aligned}
\int_D \mid (Hx)(t) \mid dt &= \int_D \mid f_1(t, \int_0^t k(t,s) f_2(s, x(\phi(s))) ds) \mid dt \\
&\leq \int_D [a_1(t) + b_1 \mid \int_0^t k(t,s) f_2(s, x(\phi(s)) ds \mid] dt \\
&\leq \| a_1 \|_{L^1(D)} + b_1 \| K \|_{L^1(D)} .[\| a_2 \|_{L^1(D)} + b_2 M^{-1} \int_{\phi(D)} \mid x(u) \mid du],
\end{aligned}$$

where $u = \phi(s)$.
Since the operator $K$ transform the space $L^1(D)$ into itself and is continuous, we get

$$\int_D \mid (Hx)(t) \mid dt \leq \| a_1 \|_{L^1(D)} + b_1 \| a_2 \|_{L^1(D)} \| K \|_D + (b_1 b_2 \| K \|_D /M) \int_{\phi(D)} \mid x(u) \mid du,$$

where $\| K \|_D$ denotes the norm of the operator $K : L^1(D) \to L^1(D)$.
Since

$$\lim_{\epsilon\to 0}[\sup[\int_D a_i(t)dt : D \subset R^+, \mathrm{meas} D \le \epsilon]] = 0, \qquad i = 1,2$$

then we get

$$c(HX) \le (b_1 b_2 \| K \| /M)c(X) = \alpha c(X) \tag{4}$$

where $c(X)$ was defined as before in section (2).
Furthermore, fixing $\tau > 0$, we get

$$\begin{aligned}
\int_\tau^\infty | (Hx)(t) | dt &= \int_\tau^\infty | f_1(t, \int_0^t k(t,s) f_2(s, x(\phi(s)))ds | dt \\
&\le \int_\tau^\infty [a_1(t) + b_1 | \int_0^t k(t,s) f_2(s, x(\phi(s)))ds |]dt \\
&\le \int_\tau^\infty a_1(t)dt + b_1 \| K \| \int_\tau^\infty [a_2(s) + (b_2/M) | x(\phi(s)) | \phi'(s)]ds \\
&\le \int_\tau^\infty a_1(t)dt + b_1 \| K \| \int_\tau^\infty a_2(s)ds + (b_1 b_2 \| K \| /M) \int_{\phi(\tau)}^\infty | x(u) | du
\end{aligned}$$

Since $\lim_{\tau\to\infty} \int_\tau^\infty a_i(t)dt = 0, \quad i = 1,2$ and $\lim_{\tau\to\infty} \phi(\tau) = \infty$ then, the above inequality becomes as $\tau \to \infty$

$$d(HX) \le \alpha d(X) \tag{5}$$

Combining (4) and (5) we get

$$\gamma(HX) \le \alpha\gamma(X)$$

Using Theorem (2), we see that

$$\beta(HX) \le \alpha\beta(X) \quad , \quad \alpha < 1.$$

Let $B_r^1 = Conv(HB_r)$, where $Conv(HB_r)$ denotes the closure of the convex hull of $HB_r$, since $HB_r \subset B_r$, then $B_r^1 \subset B_r$. Similarly, let $B_r^2 = Conv(HB_r^1)$, then $B_r^2 \subset B_r^1$, also $B_r^3 = Conv(HB_r^2) \subset B_r^2$ and so on we get a decreasing sequence $(B_r^n)$ of bounded, convex, closed subsets of $B_r$ such that

$$H(B_r^n) \subset B_r^n \quad , \quad n \in N.$$

Using the properties of the De-Blasi measure $\beta$ of weak noncompactness, we see that

$$\begin{aligned}
\beta(B_r^{n+1}) &= \beta(Conv HB_r^n) \\
&= \beta(HB_r^n) \\
&\le \alpha\beta(B_r^n) \quad , \quad n \ge N
\end{aligned}$$

and so on, we have

$$\beta(B_r^{n+1}) \leq \alpha^n \beta(B_r) \quad , \quad \alpha < 1 \quad , \quad n \geq N$$

Hence, as $n \to \infty$, we get $\lim_{n\to\infty} \beta(B_r^{n+1}) = 0$.
So, $Y = \cap_{n\in N} B_r^n$ is a nonempty, closed, bounded convex and relatively compact subset of $B_r$ and $H(Y) \subset Y$.
In the sequel, we will prove that $H(Y)$ is relatively compact in the space $L^1$. To do this, let $\{y_n\}$ be a sequence in $Y$ and $\epsilon > 0$, then by using Theorem (4) there exists a closed measureable subset $D_\epsilon$ of $[0,t]$ such that $m([0,t]/D_\epsilon) < \epsilon$ and $f_i \mid_{D_\epsilon \times R}, i = 1,2$ and $k \mid_{D_\epsilon \times R}$ are continuous.
Let

$$U_n(t) = \int_0^t k(t,s) f_2(s, y_n(\phi(s)))ds,$$

then for $t_1, t_2 \in D_\epsilon$, we have:

$$\begin{aligned} \mid U_n(t_1) - U_n(t_2) \mid &= \mid \int_0^{t_1} k(t_1,s) f_2(s, y_n(\phi(s)))ds - \int_0^{t_2} k(t_2,s) f_2(s, y_n(\phi(s)))ds \mid \\ &\leq \int_0^{t_1} \mid k(t_1,s) - k(t_2,s) \mid [a_2(s) + b_2 \mid y_n(\phi(s)) \mid ]ds \\ &+ \int_{t_2}^{t_1} \mid k(t_2,s) \mid [a_2(s) + b_2 \mid y_n(\phi(s)) \mid ]ds. \end{aligned}$$

Since $(y_n) \subset Y$ and $Y$ is bounded, then so is $(y_n)$. Hence $(U_n)$ is a sequence of a equicontinuous and uniformly bounded functions in $c(D_\epsilon)$ and so $(H(y_n))$ is a sequence of equicontinuous and uniformly bounded functions in $c(D_\epsilon)$. By using Ascoli-Arzela theorem, we deduce that $(H(y_n))$ is relatively compact in $c(D_\epsilon)$, from which, we deduce that $H(y_n)$ is Cauchy sequence in $c(D_\epsilon)$.
Next, we will use the last result to prove that $(H(y_n))$ is Cauchy sequence in $L^1$.
Using Theorem (3) and the fact that $H(Y)$ is relatively compact in $(CD_\epsilon)$ that proved before in our theorem, we deduce that for every $\sigma > 0$, there is $\delta > 0$ such that

$$\sup_y \int_{D_i\delta} \mid (Hy)(t) \mid dt < \frac{\sigma}{4.2^i},$$

for meas $D_{i\delta} < \delta \quad , \quad D_{i\delta} \subset [i-1, i] \quad , \quad i = 1,2,3,...,n.$
Choose, for each $i, i = 1,2,...,n, r_i^* \in N$ with $m([i-1,i]/D_{r_i^*}) < \delta$, then for $n_1, n_2 \in N$, we have

$$\int_0^\infty \mid (Hy_{n_1})(t) - (Hy_{n_2})(t) \mid dt \quad = \quad \lim_{n\to\infty} \sum_{i=1}^n \int_{i-1}^i \mid (Hy_{n_1})(t) - (Hy_{n_2})(t) \mid dt$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{n} \int_{[i-1,i]/D_{r_i^*}} \mid (Hy_{n_1})(t) - (Hy_{n_2}(t) \mid dt$$

$$+ \int_{D_{r_i^*}} \mid (Hy_{n_1}(t) - (Hy_{n_2})(t) \mid dt$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{2.\sigma}{4.2^i} + \parallel Hy_{n_1} - Hy_{n_2} \parallel_{c(D_{r_i^*}})$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{\infty} (\frac{\sigma}{2.2^i} + \frac{\sigma}{2.2^i}) = \sum_{i=1}^{\infty} \frac{\sigma}{2^i} = \sigma,$$

for large value of $n_1, n_2$, we deduce that $(Hy_n)$ is Cauchy sequence in $L^1$, since $L^1$ is complete space, then $(Hy_n)$ is relatively compact in $L^1$.
Finally, we can use Schauder fixed point theorem to get a fixed point for our operator $H$, so the functional integral equation (1) is solvable in $L^1$.

**Example (1):**

Take $f_2(t,u) = u$ and $f_1(t,u) = f(t,u)$ then our functional integral equation (1) becomes

$$x(t) = f(t, \int_0^t k(t,s)x(\phi(s))ds) \tag{6}$$

which is solvable in $L^1$ under the same conditions (ii) and (iii) of Theorem (6) and (i), (iv) are replaced by
$(i)' f : R^+ \times R \to R$ satisfies Caratheodory conditions and there are a function $a \in L^1$ and a nonnegative constant $b$ such that

$$\mid f(t,x) \mid \leq a(t) + b \mid x \mid \quad , \quad \text{for all} t \in R^+ \quad , \quad x \in R,$$

$(iv)' \alpha = b \parallel K \parallel M^{-1} < 1$.

**Example (2)**

Take $f_i(t,u(t)) = u(t), i = 1,2$, then our functional integral equation (1) becomes the volterra integral equation

$$x(t) = \int_0^t k(t,s)x(\phi(s))ds \tag{7}$$

According to Theorem (6), equation (7) has at least a solution $x \in L^1$ if the conditions (ii), (iii) of Theorem (6) are satisfied as well as $\parallel K \parallel \leq M$.

**Example (3)**

Combining equation (6) and (7) we get the functional equation

$$x(t) = f(t, x(t)) \quad , \quad t > 0 \tag{8}$$

Hence, by using examples (1) and (2) we get the following theorem.

**Theorem (7)**

The functional equation (8) has at least a solution $x \in L^1$ if the condition $(i)'$ of example (1) is satisfied as well as $b < 1$.

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