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# SOME HYPERBOLIC SINE-COSINE TYPE FUNCTIONAL EQUATIONS

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**Abstract**

Using a technique of Kannapan from 2003, four functional equations, resemble certain well-known hyperbolic sine-cosine identities and generalizing the classical d'Alembert functional equation, are solved and interrelations among the solution functions are investigated.

## 1 Introduction

The two trigonometric functions $g(x) = \cos x,\ f(x) = \sin x$ clearly satisfy the following four sine-cosine type functional equations over $\mathbb{R}$

$$g(x-y) = g(x)g(y) + f(x)f(y) \tag{1.1}$$
$$g(x+y) = g(x)g(y) - f(x)f(y) \tag{1.2}$$
$$f(x+y) = f(x)g(y) + g(x)f(y) \tag{1.3}$$
$$f(x-y) = f(x)g(y) - g(x)f(y). \tag{1.4}$$

In 1953, V.L. Klee posed the following problem in [6].

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Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation (1.1) with $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$. Prove that $f$ and $g$ satisfy the functional equations (1.2), (1.3) and (1.4).

A solution by T.S. Chihara appeared in [3], but it unfortunately had a gap. In 2003, Kannappan, [5], gave a general solution of (1.1) without any additional conditions by proving:

**Theorem 1.1.** *Let $(G, +)$ be a two-divisible abelian group (i.e., a group for which to each $x \in G$, there exists a unique $y \in G$ such that $x = 2y$). If the functions $f,\ g : G \to \mathbb{C}$ satisfy the functional equation* (1.1)*, then they also satisfy the equations* (1.2)*,* (1.3) *and* (1.4)*.*

*Moreover, the solution functions are given by*

$$g(x) = \frac{1}{2}\left(E(x) + E^*(x)\right), \quad f(x) = b_0\left(E(x) - E^*(x)\right), \tag{1.5}$$

*where $b_0^2 = -1/4$, and the function $E : G \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ satisfies the (exponential) Cauchy functional equation $E(x + y) = E(x)E(y)$ with $E^*(x) = 1/E(x)$.*

Theorem 1.1 leads to ([1], [4]):

**Corollary 1.2.** *If $f, g : \mathbb{R} \to \mathbb{C}$ are nonconstant solutions of* (1.1) *and $g$ is continuous, then $f$ is also continuous, $g(x) = \cos(k_0 x)$ and $f(x) = b_0 \sin(k_0 x)$, where $b_0^2 = -1/4,\ k_0 \in \mathbb{C}$.*

One of the main tools used in the proof of Theorem 1.1 is the following result about d'Alembert functional equation appeared in [4], see also [2].

**Theorem 1.3.** *Let $(G, *)$ be a group. Then every solution function $f : G \to \mathbb{C}$ of the d'Alembert's functional equation*

$$f(x * y) + f(x * y^{-1}) = 2f(x)f(y) \tag{1.6}$$

*also satisfies*

$$f(x * y * z) = f(x * z * y) \tag{1.7}$$

*and is of the form*

$$f(x) = \frac{h(x) + h^*(x)}{2}, \tag{1.8}$$

*where $h$ is a homomorphism on $G$ into $\mathbb{C}^*$ and $h^*(x) = 1/h(x)$.*

Since the hyperbolic sine and hyperbolic cosine functions $\mathcal{G}(x) = \cosh x,\ \mathcal{F}(x) = \sinh x$ satisfy the functional equation

$$\mathcal{G}(x - y) = \mathcal{G}(x)\mathcal{G}(y) - \mathcal{F}(x)\mathcal{F}(y), \tag{1.9}$$

motivated by the above results of Kannappan, we ask whether results analogous to Theorem 1.1 and Corollary 1.2 hold for the hyperbolic sine-cosine functions. Using a modification of Kannappan's technique, here we affirmatively answer this query by proving:

**Theorem 1.4.** *Let $(G,+)$ be a two-divisible abelian group and let $\mathcal{F},\mathcal{G} : G \to \mathbb{C}$ be solutions of the functional equation* (1.9).

*I. If one of the functions $\mathcal{F},\mathcal{G}$ is a constant function, then the other is also a constant function and the two constant functions are $\mathcal{F} \equiv d$, $\mathcal{G} \equiv c$ with $c = c^2 - d^2$.*

*II. If both $\mathcal{F},\mathcal{G}$ are nonconstant functions, then they also satisfy*

$$\mathcal{G}(x+y) = \mathcal{G}(x)\mathcal{G}(y) + \mathcal{F}(x)\mathcal{F}(y), \tag{1.10}$$
$$\mathcal{F}(x \pm y) = \mathcal{F}(x)\mathcal{G}(y) \pm \mathcal{G}(x)\mathcal{F}(y). \tag{1.11}$$

*and the solution functions are given by*

$$\mathcal{G}(x) = \frac{1}{2}\left(E(x) + E^*(x)\right), \quad \mathcal{F}(x) = b_1\left(E(x) - E^*(x)\right), \tag{1.12}$$

*where $b_1^2 = 1/4$ and the function $E : G \to \mathbb{C}^*$ satisfies the (exponential) Cauchy functional equation $E(x+y) = E(x)E(y)$.*

Immediate from Theorem 1.4 is:

**Corollary 1.5.** *If $\mathcal{F},\mathcal{G} : \mathbb{R} \to \mathbb{C}$ are nonconstant solutions of* (1.9) *and $\mathcal{G}$ is continuous, then $\mathcal{F}$ is also continuous, $\mathcal{G}(x) = \cosh(c_1 x)$ and $\mathcal{F}(x) = b_1 \sinh(c_1 x)$ where $b_1^2 = 1/4$, $c_1 \in \mathbb{C}$.*

## 2 Proof of Theorem 1.4

Note first that by symmetry, the functional equation (1.9) yields

$$\mathcal{G}(y-x) = \mathcal{G}(y)\mathcal{G}(x) - \mathcal{F}(y)\mathcal{F}(x) = \mathcal{G}(x)\mathcal{G}(y) - \mathcal{F}(x)\mathcal{F}(y) = \mathcal{G}(x-y),$$

implying that $\mathcal{G}$ is an even function.

I. Assume first that $\mathcal{G}(x) \equiv c$, a constant function. If $\mathcal{F}(x) \not\equiv 0$, there is $\alpha \in G$ such that $\mathcal{F}(\alpha) \neq 0$. Substituting into (1.9) yields

$$\mathcal{F}(x) = \frac{c^2 - c}{\mathcal{F}(\alpha)} =: d$$

and so $c = c^2 - d^2$. The same assertion trivially holds if $\mathcal{F}(x) \equiv 0$.

Next assume $\mathcal{F}(x) \equiv d$, a constant function. Replacing $y$ by $-y$ in (1.9) and using the evenness of $\mathcal{G}$, we obtain

$$\mathcal{G}(x+y) = \mathcal{G}(x-y). \tag{2.1}$$

Putting $x = \frac{u+v}{2}, y = \frac{u-v}{2}$ $(u, v \in G)$ in (2.1), we get

$$\mathcal{G}(u) = \mathcal{G}(x+y) = \mathcal{G}(x-y) = \mathcal{G}(v), \tag{2.2}$$

i.e., $\mathcal{G}$ is a constant function and the two constants are related as shown before.

II. Consider nonconstant solutions $\mathcal{F}$ and $\mathcal{G}$ of the equation (1.9). Using (1.9) and the evenness of $\mathcal{G}$, we obtain

$$\mathcal{G}(x+y) = \mathcal{G}(x-(-y)) = \mathcal{G}(x)\mathcal{G}(-y) - \mathcal{F}(x)\mathcal{F}(-y) = \mathcal{G}(x)\mathcal{G}(y) - \mathcal{F}(x)\mathcal{F}(-y). \tag{2.3}$$

Similarly,

$$\mathcal{G}(x+y) = \mathcal{G}(-x-y) = \mathcal{G}(-x)\mathcal{G}(y) - \mathcal{F}(-x)\mathcal{F}(y) = \mathcal{G}(x)\mathcal{G}(y) - \mathcal{F}(-x)\mathcal{F}(y). \tag{2.4}$$

The equations (2.3) and (2.4) together give

$$\mathcal{F}(x)\mathcal{F}(-y) = \mathcal{F}(-x)\mathcal{F}(y).$$

Since $\mathcal{F}(x) \not\equiv 0$, there is $\alpha \in G$ such that $\mathcal{F}(\alpha) \neq 0$. Thus,

$$\mathcal{F}(x) = \frac{\mathcal{F}(-\alpha)}{\mathcal{F}(\alpha)}\mathcal{F}(-x) = k\mathcal{F}(-x) = k \times k\mathcal{F}(x) = k^2\mathcal{F}(x),$$

where $k := \mathcal{F}(-\alpha)/\mathcal{F}(\alpha)$. Clearly, $k = \pm\, 1$.

If $k = 1$, then $\mathcal{F}(x) = \mathcal{F}(-x)$, i.e., $\mathcal{F}$ is an even function. This together with (1.9) and (2.3) show that $\mathcal{G}(x-y) = \mathcal{G}(x+y)$. By the same argument as that leading to (2.2), we conclude that $\mathcal{G}$ is a constant function, which is a contradiction. Hence, $k = -1$, and so $\mathcal{F}(x) = -\mathcal{F}(-x)$, i.e., $\mathcal{F}$ is an odd function. Using this and (2.3), we see that (1.10) holds. Using (1.10) twice, we get

$$\mathcal{G}((x+y)+z) = \mathcal{G}(x)\mathcal{G}(y)\mathcal{G}(z) + \mathcal{F}(x)\mathcal{F}(y)\mathcal{G}(z) + \mathcal{F}(x+y)\mathcal{F}(z), \tag{2.5}$$

and

$$\mathcal{G}\left(x+(y+z)\right) = \mathcal{G}(x)\mathcal{G}(y)\mathcal{G}(z) + \mathcal{G}(x)\mathcal{F}(y)\mathcal{F}(z) + \mathcal{F}(x)\mathcal{F}(y+z). \tag{2.6}$$

Equating (2.5) and (2.6) and simplifying, we have

$$\mathcal{F}(x)\left(\mathcal{F}(y)\mathcal{G}(z) - \mathcal{F}(y+z)\right) = \left(\mathcal{G}(x)\mathcal{F}(y) - \mathcal{F}(x+y)\right)\mathcal{F}(z).$$

Putting $z = \alpha$ and noting $\mathcal{F}(\alpha) \neq 0$, we have

$$\mathcal{G}(x)\mathcal{F}(y) - \mathcal{F}(x+y) = h(y)\mathcal{F}(x), \tag{2.7}$$

where $h(y) := \frac{1}{\mathcal{F}(\alpha)}\left(\mathcal{F}(y)\mathcal{G}(\alpha) - \mathcal{F}(y+\alpha)\right)$. Replacing $x$ by $-x$ in (2.7), using the oddness of $\mathcal{F}$ and the evenness of $\mathcal{G}$, we get

$$-\mathcal{F}(x-y) = \mathcal{F}(-x+y) = \mathcal{G}(-x)\mathcal{F}(y) - h(y)\mathcal{F}(-x) = \mathcal{G}(x)\mathcal{F}(y) + h(y)\mathcal{F}(x),$$

and so

$$\mathcal{F}(x-y) = -\mathcal{G}(x)\mathcal{F}(y) - h(y)\mathcal{F}(x). \tag{2.8}$$

Incorporating (2.7) and (2.8), we get

$$\mathcal{F}(x+y) + \mathcal{F}(x-y) = -2h(y)\mathcal{F}(x). \tag{2.9}$$

Interchanging $x$ and $y$ in (2.9) and using the oddness of $\mathcal{F}$, we have

$$\mathcal{F}(x+y) - \mathcal{F}(x-y) = -2h(x)\mathcal{F}(y). \tag{2.10}$$

Adding (2.9) to (2.10), we have

$$\mathcal{F}(x+y) = -h(y)\mathcal{F}(x) - h(x)\mathcal{F}(y). \tag{2.11}$$

Combining (2.7) and (2.11), we get

$$\mathcal{G}(x)\mathcal{F}(y) = -h(x)\mathcal{F}(y),$$

and so

$$\mathcal{G}(x) = -\frac{\mathcal{F}(\alpha)}{\mathcal{F}(\alpha)}h(x) = -h(x).$$

Putting this last relation back into the equation (2.7), we get one of the two relations in (1.11), namely,

$$\mathcal{F}(x+y) = \mathcal{G}(x)\mathcal{F}(y) + \mathcal{G}(y)\mathcal{F}(x). \tag{2.12}$$

Replacing $y$ by $-y$ in (2.12), using the oddness of $\mathcal{F}$ and the evenness of $\mathcal{G}$, we have

$$\mathcal{F}(x-y) = \mathcal{G}(x)\mathcal{F}(-y) + \mathcal{G}(-y)\mathcal{F}(x) = \mathcal{F}(x)\mathcal{G}(y) - \mathcal{G}(x)\mathcal{F}(y) \tag{2.13}$$

which is the other equation in (1.11).

There remains to find general shapes of the two solution functions. From (1.9) and (1.10), we have

$$\mathcal{G}(x+y) + \mathcal{G}(x-y) = 2\mathcal{G}(x)\mathcal{G}(y) \tag{2.14}$$

which is the d'Alembert functional equation and by Theorem 1.3, we have

$$\mathcal{G}(x) = \frac{E(x) + E^*(x)}{2}, \tag{2.15}$$

where $E : G \to \mathbb{C}^*$ satisfies $E(x+y) = E(x)E(y)$ and $E^*(x) = 1/E(x)$. To find $\mathcal{F}$, using (1.9), we have

$$\frac{E(x)E(-y) + E^*(x)E^*(-y)}{2} = \frac{E(x-y) + E^*(x-y)}{2} = \mathcal{G}(x-y)$$
$$= \mathcal{G}(x)\mathcal{G}(y) - \mathcal{F}(x)\mathcal{F}(y) = \left(\frac{E(x)+E^*(x)}{2}\right)\left(\frac{E(y)+E^*(y)}{2}\right) - \mathcal{F}(x)\mathcal{F}(y),$$

i.e.,

$$\mathcal{F}(x)\mathcal{F}(y) = \left(\frac{E(x)+E^*(x)}{2}\right)\left(\frac{E(y)+E^*(y)}{2}\right) - \frac{E(x)E(-y)+E^*(x)E^*(-y)}{2}$$
$$= \frac{1}{4}\left(E(x)-E^*(x)\right)\left(E(y)-E^*(y)\right).$$

Consequently,

$$\mathcal{F}(x) = \frac{1}{4}\frac{(E(\alpha)-E^*(\alpha))}{\mathcal{F}(\alpha)}\left(E(x)-E^*(x)\right) = b_1\left(E(x)-E^*(x)\right), \qquad (2.16)$$

where $b_1^2 = 1/4$.

## References

[1] J. Aczél, "Lectures on Functional Equations and Their Applications", Academic Press, New York, 1966.

[2] J. Aczél and J. Dhombres, "Functional Equations in Several Variables", Cambridge University Press, Cambridge, 1988.

[3] T. S. Chihara, *Solution to E1079*. Amer. Math. Monthly **61** (1954), 197.

[4] Pl. Kannappan, *The functional equations* $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ *for groups*. Proc. Amer. Math. Soc. **19** (1968), 69-74.

[5] Pl. Kannappan, *Klee's trigonometry problem*. Amer. Math. Monthly **110** (2003), 940-944.

[6] V. L. Klee, *Problem E1079*. Amer. Math. Monthly **60** (1953), 479.