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ON GENERALIZED PP-RINGS

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Abstract

In this paper, we characterize generalized PP-rings and their generalizations via P-injectivity, AP-injectivity or AGP-injectivity.

1. Introduction

Throughout this paper, R is an associative ring with identity $1 \neq 0$ and all modules are unitary modules. We write M_R (resp. ${}_RM$) to indicate that M is a right (resp. left) R-module. The category of right (resp. left) R-module is denoted by Mod-R (resp. R-Mod).

Let M be a right R-module, we denote the injective hull of M by $E(M)$. The notation $A \leq M$ (resp. $A < M$) stands for the fact that A is a submodule (resp. a proper submodule) of M. The right and left annihilators of a subset X of a ring R are denoted by $r(X)$ and $l(X)$, respectively.

Let M be an R-module and I a right ideal of R, and let f be an R-homomorphism of I to M. Consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "f"'] & R \arrow[dl, dotted, "\exists\, h"] \\ & M & \end{tikzcd}$$

Key words: P-ring, PP-ring, generalized PP-ring, GP-injective module, AP-injective module, AGP-module.
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If there exists $h \in Hom_R(R, M)$ for every principal right ideal I in R and any $f \in Hom_R(I, M)$, then we say that M is *principally injective*, or *P-injective* for short; or equivalently, f is the left multiplication by some element $m \in M$ with I. This is equivalent to saying that $l_M r_R(a) = Ma$ for all $a \in R$, where l and r are the left and right annihilators, respectively. If a ring R is P-injective as a right R-module, then R is called a right P-injective ring.

A ring is called a right PP-ring if all principal right ideals are projective.

For basic concepts and results that are not defined here we refer to the texts: Anderson and Fuller [1], Faith [3] and Wisbauer [11].

As is known, a ring R is right PP if and only if every factor module of an P-injective module is P-injective and if and only if every factor module of an injective module is P-injective. In this note, we will characterize certain classes of rings that are generalizations of PP-rings.

2. Hereditary, Semihereditary and PP-Rings

A right module M_R is called F-injective (FP-injective, resp.) if for any finitely generated right ideal K of R ($R^{(\mathbb{N})}$, resp.), any right R-homomorphism $g : K \to M$ can be extended to $R \to M$ ($R^{(\mathbb{N})} \to M$, resp.). It follows that FP-injectivity implies F-injectivity. A ring R is called right hereditary (semihereditary, resp.) if every (finitely generated, resp.) right ideal of R is projective. The following results give us some characterizations of a semihereditary or a hereditary ring via FP-, F- or injectivity.

In this section we mention some well-known results of the characterization of hereditary rings, semihereditary rings and PP-rings. In the next section we follow this line to characterize some rings that are generalizations of right PP-rings.

Theorm 2.1. *The following conditions are equivalent for a ring R.*

(i) R is a right semihereditary ring.

(ii) Every factor module of an FP-injective right R-module is FP-injective.

(iii) Every factor module of an injective envelope $E(R_R)$ is FP-injective.

Theorem 2.2. *For a ring R the following conditions are equivalent:*

(i) R is a right hereditary ring.

(ii) Every factor module of an injective right R-module is injective.

Theorem 2.3. *For a ring R the following conditions are equivalent:*

(i) R is a right PP-ring.

(ii) Every factor module of an P-injective right R-module is P-injective.

(iii) Every factor module of an injective right R-module is P-injective.

For the proofs of these theorems see, for example, [11, 39.13, 39.16] and [11, Exercise 4(i), p. 340], respectively.

Moreover, the following result from [4] is useful in our study of PP-rings.

Theorem 2.4. *For a ring R the following conditions are equivalent:*

(i) R is a right PP-ring.

(ii) For each element $a \in R$ and for the homomorphism $\varphi : R \longrightarrow aR$ defined by $\varphi(r) = ar$ splits, i.e., Ker φ is a direct summand of R.

(iii) The right annihilator of each element of R is generated by an idempotent.

3. Generalized PP-Rings

A ring R is called *generalized* right PP if for any $0 \neq x \in R$ and for some positive n, depending on x, the right nonzero ideal $x^n R$ is projective.

Lemma 3.1. *For a ring R the following conditions are equivalent:*

(i) R is a generalized right PP-ring.

(ii) For each element $x \in R$, the right annihilator of non-zero element x^n is generated by an idempotent for some positive n, depending on x.

Proof: Straightforward.

Following [5] and [7] a right R-module M is called GP-injective (= YJ-injective in [6] or in [12]) if for every $0 \neq a \in R$ there exists $n \in \mathbb{N}$ with $a^n \neq 0$ and every right R-homomorphism $a^n R \longrightarrow M$ extends to $R \longrightarrow M$.

Proposition 3.2. *For a right R-module M the following conditions are equivalent:*

(i) M is GP-injective.

(ii) For each element $0 \neq a \in R$, there exists $n \in \mathbb{N}$ with $a^n \neq 0$, $l_M(r_R(a^n)) = Ma^n$.

Proof. By Lemma 1.3 of [9].

A ring R is called right GP-injective if the right R-module R_R is GP-injective, or equivalently if for every $0 \neq a \in R$ there exists $n \in \mathbb{N}$ with $a^n \neq 0$ and $lr(a^n) = Ra^n$. The ring R in the following example was essentially given by Clark [2] proved that the GP-injectivity is a proper generalization of the P-injectivity.

Example 3.3. Let $\mathbb{Z}_2$ be the field of integers modulo 2 and A be the subring of $\mathbb{Z}_2^{\mathbb{N}}$ consisting of elements of the form

$$\{(a_1, a_2, ..., a_n, a, a, ...)| \;\; a_1, a_2, ..., a_n, a \in \mathbb{Z}_2\}.$$

Let

$$R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & A \end{pmatrix}$$

then R is right GP-injective but not P-injective. □

Characterizations of some classes of rings via GP-injectivity have been studied by many authors (e.g, [5], [6], [7], [12], ...). It is known that if R is a von Neumann regular ring, then every right (left) R-module is GP-injective ([7]). Wongwai [10, Theorem 2.6] prove Theorem 2.3 in the module case. We now obtain the following result.

Theorem 3.4. *For a ring R the following conditions are equivalent:*

(i) R is a generalized right PP-ring.

(ii) Every factor module of an P-injective right R-module is GP-injective.

(iii) Every factor module of an injective right R-module is GP-injective.

Proof. (i) $\Longrightarrow$ (ii). Let R be a generalized right PP-ring and N_R be an P-injective module. For every $X \leq N$,we will prove that N/X is also an GP - injective module. For every $0 \neq b \in R$, there exists $n \in \mathbb{N}$ such that $b^n \neq 0$ and b^nR is projective and then for any R-homomorphism $\varphi : b^nR \longrightarrow N/X$, there exists an R-homomorphism $\varphi' : b^nR \longrightarrow N$ such that $\eta_X\varphi' = \varphi$, i.e. the following diagram is commutative

$$\begin{tikzcd}
& b^nR \arrow[r] \arrow[dl, "\varphi'"'] \arrow[d, "\varphi"] & R \arrow[dl, "\widehat{\varphi}"] \\
N \arrow[r, "\eta_X"] & N/X \arrow[r] & 0
\end{tikzcd}$$

in which $\eta_X : N \longrightarrow N/X$ is the natural epimorphism. Since N is P-injective, φ' can be extended to $\widehat{\varphi}' : R \to N$ and then φ can be extended to $\widehat{\varphi} = \eta_X \widehat{\varphi}'$. So N/X is GP - injective.

(ii) $\Longrightarrow$ (iii) is clear.

(iii) $\Longrightarrow$ (i). For every $0 \neq x \in R$, we consider the epimorphism $h : A \to B$, in which A, B are any right R-modules. By (iii), $E(A)/Ker(h)$ is GP-injective, so there exists $n \in \mathbb{N}$ such that $x^n \neq 0$ and every R-homomorphism of x^nR to $E(A)/Ker(h)$ extends to R.

Since $B \cong A/Ker(h) \leq E(A)/Ker(h)$,

$$\begin{array}{rccc} \overline{\alpha} : & x^nR & \to & E(A)/Ker(h) \\ & a & \longmapsto & \overline{\alpha}(a) = \alpha(a) \end{array}$$

is an R-homomorphism and can be extended to $\widehat{\alpha} : R \to E(A)/Ker(h)$.

Since R_R is projective, there exists an R-homomorphism $g : R \longrightarrow E(A)$ such that $pg = \widehat{\alpha}$, i.e., the following diagram commutes

$$\begin{tikzcd}
& x^nR \arrow[r] \arrow[d, "\alpha"] \arrow[dl, "\varphi"'] & R \\
A \arrow[r, "h"] \arrow[d] & B \arrow[r] \arrow[d, "j"] & 0 \\
E(R) \arrow[r, "p"] & E(A)/Ker(h) \arrow[r] & 0
\end{tikzcd}$$

in which $p : E(A) \longrightarrow E(A)/Ker(h)$ is the natural epimorphism and $p|_A = h$.

It is easy to see that $g(x^nR) \leq A$, so there exists an R-homomorphism $\varphi : x^nR \longrightarrow A$ such that $\varphi = g|_{x^nR}$. Since $pg = \widehat{\alpha}$, it follows that $pg|_{x^nR} = \widehat{\alpha}|_{x^nR}$, i.e., $h\varphi = \alpha$. Hence x^nR is projective, proving (i). $\square$

4. Rings with condition (*)

Condition (*). A ring R is said to satisfy the condition (*) if in R every principal right ideal is a direct sum of a projective right ideal and a right ideal not containing a nonzero projective right ideal. It is clear that PP-ring implies condition (*).

A module M is said to be almost principally injective (or AP-injective for short) if, for any $a \in R$, there exists a S-submodule X of M such that $l_M r_R(a) = Ma \oplus X$, as a direct sum of $End_R(M)$-modules.

A ring R is called right AP-injective if R_R is AP-injective ([9]).

Lemma 4.1. ([9], Lemma 1.2) *Let M_R be a module, $S = End(M)$, and $a \in R$.*

(i) If $l_M(r_R(a)) = Ma \oplus X$ for some $X \subseteq M$ as a left S-module, then we have $Hom_R(aR, M) = Hom_R(R, M) \oplus \Gamma$ as a left S-module, where $\Gamma = \{f \in Hom_R(aR, M) : f(a) \in X\}$.

(ii) If $Hom_R(aR, M) = Hom_R(R, M) \oplus \Gamma$ as a left S-module, then $l_M(r_R(a)) = Ma \oplus X$ as a left S-module, where $X = \{f(a) : f \in \Gamma\}$.

(iii) Ma is a summand of $l_M(r_R(a))$ as a left S-module iff $Hom_R(R, M)$ is a summand of $Hom_R(aR, M)$ as a left S-module. $\square$

We obtain the following result that gives us the characterization of a ring satisfying (*) via AP-injective modules.

Theorem 4.2. *For a ring R the following conditions are equivalent:*

(i) R satisfies (*).

(ii) Every factor module of an AP-injective right R-module is AP-injective.

(iii) Every factor module of an P-injective right R-module is AP-injective.

(iv) Every factor module of an injective right R-module is AP-injective.

Proof. (i) $\Longrightarrow$ (ii). Assume (i). Let M_R be an AP-injective module, $N \leq M$ and $a \in R$. Then $l_M(r_R(a)) = Ma \oplus X$ for some $X \subseteq M$ as left S-module. By Lemma 4.1, we have $\mathrm{Hom}_R(aR, M) = \mathrm{Hom}_R(R, M) \oplus \Gamma$ as left S-module, where $\Gamma = \{f \in \mathrm{Hom}_R(aR, M) : f(a) \in X\}$. We will prove that $\mathrm{Hom}_R(aR, M/N) = \mathrm{Hom}_R(R, M/N) \oplus \Gamma'$ as left S-module. In fact, since $aR = P \oplus Y$ where P is projective and some $Y \leq aR$, $aR/Y \overset{\beta}{\simeq} P$. Let $f : aR \longrightarrow M/N$ be an R-homomorphism, there exists an R-homomorphism $\theta : aR/Y \longrightarrow M$ such that $\eta_N \theta = f j \beta$ where $j : P \longrightarrow aR$ is the inclusion and $\eta_N : M \longrightarrow M/N$ is the natural epimorphism.

$$\begin{tikzcd}
 & P \arrow[d, "j"] & \\
aR/Y \arrow[ur, "\beta"] \arrow[d, "\theta"'] & aR \arrow[dl, "\theta\eta_Y"] \arrow[d, "f"] & \\
M \arrow[r, "\eta_N"'] & M/N \arrow[r] & 0
\end{tikzcd}$$

Take $\theta' = \theta\eta_Y$ where $\eta_Y : aR \longrightarrow aR/Y$ is the natural epimorphism. Then $\theta' = \theta_1 \oplus \gamma$ where $\theta_1 \in \mathrm{Hom}_R(R, M)$ and $\gamma \in \Gamma$. From this $\eta_N\theta_1 \in \mathrm{Hom}_R(R, M/N)$ and $\eta_N\gamma \in \Gamma'$.

(ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are clear.

(iv) $\Longrightarrow$ (i). Assume (iv). For every $a \in R$, we take I the sum of all right ideal of aR not containing a nonzero projective right ideal. We prove that aR/I is projective. Let $h : A \to B$ be any R-epimorphism, in which A, B are any right R-modules and $\alpha : aR/I \longrightarrow B$ be any R-homomorphisms.

Since $B \cong A/Ker(h) \overset{j}{\hookrightarrow} E(A)/Ker(h)$,

$$\begin{array}{rccc} \alpha' : & aR/I & \to & E(A)/Ker(h) \\ & a & \longmapsto & \alpha'(a) = \alpha(a) \end{array}$$

is an R-homomorphism. We set $\overline{\alpha} = \alpha'\eta_I = j\alpha\eta_I$, in which η_I is the natural epimorphism. By (iv), $E(A)/Ker(h)$ is AP-injective, so there exist $f_1, f_2 \in Hom_R(aR, E(A)/Ker(h))$ such that f_1 can be extended to $\widehat{f_1} : R \to E(A)/Ker(h)$.

Since R_R is projective, there exists an R-homomorphism $g : R \longrightarrow E(A)$ such that $pg = \widehat{f_1}$, i.e., the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & aR \arrow[r] \arrow[d, "\eta_I"] \arrow[ddl, "\varphi'"'] & R \\
 & aR/I \arrow[d, "\alpha"] \arrow[dl, "\varphi"] & \\
A \arrow[r, "h"'] \arrow[d] & B \arrow[r] \arrow[d, "j"] & 0 \\
E(A) \arrow[r, "p"'] & E(A)/Kerh \arrow[r] & 0
\end{tikzcd}$$

in which $p : E(A) \longrightarrow E(A)/Ker(h)$ is the natural epimorphism and $p|_A = h$.

It is easy to see that $g(aR) \le A$, so there exists an R-homomorphism $\varphi' : aR \longrightarrow A$ such that $\varphi' = g|_{aR}$.

Now we prove that $I \le Ker(\varphi')$. In fact, for any $i \in I, j\alpha\eta_I(i) = 0 = f_1(i) + f_2(i)$ and then $f_1(i) = -f_2(i)$. By Lemma 4.1,

$$l_{E(A)/Ker(h)}(r_R(i)) = (E(A)/Ker(h))i \oplus X$$

as left S-module, where $X \le E(A)/Ker(h)$. So $f_2(i) = f_1(i) = 0$, i.e., $i \in Ker(f_1) \cap aR = Ker(\varphi')$. Then there exists $\varphi : aR/I \longrightarrow A$ such that $\varphi\eta_I = \varphi'$.

Since $pg = \widehat{f_1}$, it follows that $pg|_{aR} = \widehat{f_1}|_{aR}$, i.e., $h\varphi' = \alpha\eta_I$. It follows that $h\varphi\eta_I = \alpha\eta_I$ and since η_I is epimorphism, $h\varphi = \alpha$. Hence aR/I is projective. Set $P = aR/I$ we have $aR \cong P \oplus I$, proving (i). $\square$

5. Rings with condition (**)

Condition (**). A ring R in which for every $0 \neq a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \neq 0$ and $a^n R$ is a direct sum of a projective right ideal and a right ideal not containing a nonzero projective right ideal. It is clear that "generalized PP-ring" implies condition (**).

A module M is said to be almost general principally injective ([9]) (or AGP-injective for short) if, for any $0 \neq a \in R$, there exist a positive integer $n = n(a)$ and an S-submodule X of M such that $a^n \neq 0$ and $l_M r_R(a^n) = Ma^n \oplus X$ as a direct sum of $End_R(M)$-modules.

A ring R is called right AGP-injective if R_R is AGP-injective.

The following result provides a characterization of rings satisfying (**) via AGP-injective modules.

Theorem 5.1. *For a ring R the following conditions are equivalent:*

(i) R satisfies (**).

(ii) Every factor module of a AP-injective right R-module is AGP-injective.

(iii) Every factor module of a P-injective right R-module is AGP-injective.

(iv) Every factor module of an injective right R-module is AGP-injective.

Proof. By the same argument of proving Theorem 3.3 and 4.2. □

6. Remarks

6.1. In [4], a ring R is called a generalized right PP-ring if for any $x \in R$, the right ideal $x^n R$ is projective for some positive integer n, depending on x, or equivalently, if for any $x \in R$, the right annihilator of x^n is generated by an idempotent for some positive integer n, depending on x. They gave an example of a generalized PP non PP-ring as follows. Let $\mathbb{Z}_2$ be the field of integers modulo 2, and

$$R = \{a_0 + a_1 i + a_2 j + a_3 k | a_i \in \mathbb{Z}_2 \text{ for } i = 0, 1, 2, 3\}$$

be the Hamilton quaternions over $\mathbb{Z}_2$.

Let $R_{\mathbb{Z}}$ be the ring of quaternions over $\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers and $I = \{a_0 + a_1 i + a_2 j + a_3 k | a_i \in 2\mathbb{Z}, i = 0, 1, 2, 3\}$ be the ideal of $R_{\mathbb{Z}}$. Then

$$R \simeq R_{\mathbb{Z}}/I.$$

Note that by the definition of [4], if a ring R is generalized PP, we only need to find a positive integer n such that $x^n R$ is projective for each non-nilpotent $x \in R$. Therefore, the defenition of a generalized PP-ring in this paper is different from that in [4]. We have the following implications:

$$PP - rings \implies generalized\ PP - rings\ (in\ this\ paper) \implies$$

$$generalized\ PP - rings\ in\ the\ sense\ of\ [4]$$

but the converses are not true, in general.

Since $(a_0+a_1 i+a_2 j+a_3 k)^2 = a_0^2 - a_1^2 - a_2^2 - a_3^2 \in \mathbb{Z}_2$ with $a_0+a_1 i+a_2 j+a_3 k \in R$, so $(a_0 + a_1 i + a_2 j + a_3 k)^2 R = 0$ or R, i.e., R is generalized PP in the sense of [4]. But R is not a generalized PP-ring in the sense of this paper. In fact, all idempotents in R are 0 and 1 and $(1 + i)^2 = 0$, so $1 + i \in r_R(1 + i)$ and $1 \notin r_R(1+i)$ show that $r_R(1+i)$ cannot be generated by an idempotent in R. It is clear that R is not a PP-ring by our definition. $\square$

6.2. Many examples of a right AP-injective ring which is not right GP-injective were given by [9]. For example, let $R = \mathbb{Z}_4 \propto (\mathbb{Z}_4 \oplus \mathbb{Z}_4)$ be the trivial extension of $\mathbb{Z}_4$ by the $\mathbb{Z}_4$-module $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. Let $a = (\bar{0}, \bar{1}, \bar{0})$. Then $a^2 = 0$ and $lr(a) = 0 \propto (\mathbb{Z}_4 \oplus \mathbb{Z}_4) \neq 0 \propto (\mathbb{Z}_4 \oplus 0) = Ra$. Therefore, R is not GP-injective. $\square$

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